# Pacific Journal of Mathematics

# ON NONNEGATIVE MATRICES

MORDECHAI LEWIN

Vol. 36, No. 3 BadMonth 1971

# ON NONNEGATIVE MATRICES

#### M. LEWIN

The following characterisation of totally indecomposable nonnegative n-square matrices is introduced: A nonnegative n-square matrix is totally indecomposable if and only if it diminishes the number of zeros of every n-dimensional nonnegative vector which is neither positive nor zero. From this characterisation it follows quite easily that:

- I. The class of totally indecomposable nonnegative n-square matrices is closed with respect to matrix multiplication.
- II. The (n-1)-st power of a matrix of that class is positive.

A very short proof of two equivalent versions of the König-Frobenius duality theorem on (0,1)-matrices is supplied at the end.

A matrix is called nonnegative or positive according as all its elements are nonnegative or positive respectively. An n-square matrix A is said to be decomposable if there exists a permutation matrix P such that  $PAP^{T} = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$ , where B and D are square matrices; otherwise it is indecomposable. A is said to be  $partly\ decomposable$  if there exist permutation matrices P, Q such that

$$PAQ = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$$
, where  $B$  and  $D$  are square

matrices; otherwise it is totally indecomposable.

Whereas the notion of indecomposable matrices first appeared in 1912 in a paper by Frobenius [2] dealing with the spectral properties of nonnegative matrices, totally indecomposable matrices were introduced fairly recently apparently by Marcus and Minc [10]. Their properties have been studied in several papers on inequalities for the permanent function.

In [11] Minc gives the following characterisation of totally indecomposable matrices:

A nonnegative *n*-square matrix A,  $n \ge 2$ , is totally indecomposable if and only if every (n-1)-square submatrix of A has a positive permanent.

A well-known theorem states:

Theorem 1. If A is an indecomposable nonnegative n-square  $matrix\ then$ 

754 M. LEWIN

$$(A+I)^{n-1} > 0$$
 [3], [9].

An indecomposable matrix is primitive if its characteristic value of maximum modulus is unique.

Wielandt [15] states (without proof) that for primitive n-square matrices we have

$$A^{n^2-2n+2} > 0$$
 .

By using solely the properties of total indecomposability we establish a different characterisation for totally indecomposable matrices from the one given by Minc. Using part of the characterisation we show that if A is a totally indecomposable nonnegative n-square matrix then  $A^{n-1} > 0$ . This result is best possible as for every n there exist totally indecomposable n-square matrices A for which  $A^{n-2} > 0$ . Theorem 1 then follows as a corollary of the latter result.

We should like to point out that Theorem 2 is by no means essential for the proof of Theorem 3. Two independent proofs of Theorem 3 are given in § 4. It seems justified however to present Theorem 2 on its own merit.

We conclude with a very short proof of two equivalent versions of König's theorem on matrices.

- 2. Preliminaries. |S| denotes the number of elements of a given set S. Let  $M_n$  be the set of all nonnegative n-square matrices, let  $D_n$  be the subset of  $M_n$  of indecomposable matrices and let  $T_n$  be the subset of  $D_n$  of totally indecomposable matrices. Let  $A \in M_n$  and let p and q be nonempty subsets of  $N = \{1, \dots, n\}$ . Then A[p|q], A(p|q) is the  $|p| \times |q|$  submatrix of A consisting precisely of those elements  $a_{ij}$  of A for which  $i \in p$  and  $j \in q$ ,  $i \notin p$  and  $j \notin q$  respectively. A[p|q) and A(p|q] are defined accordingly. We can now formulate equivalent definitions for matrices in  $D_n$  and  $T_n$ :
  - D. 1.  $A \in D_n$  if  $A[p | N p] \neq 0$  for every nonempty  $p \subset N$ .
- D. 2.  $A \in T_n$  if  $A[p | q] \neq 0$  for any nonempty subsets p and q of N such that |p| + |q| = n.

Let us now establish some connections between indecomposable and totally indecomposable matrices.

LEMMA 1. If  $A \in (D_n - T_n)$  then A has a zero on its main diagonal.

*Proof.* Since  $A \notin T_n$  there exists a zero-submatrix  $A[p \mid q]$  with |p| + |q| = n; but since  $A \in D_n$ ,  $p \cap q \neq \emptyset$ , which means that A has

<sup>&</sup>lt;sup>1</sup> A proof is supplied in [5].

<sup>&</sup>lt;sup>2</sup> Lemma 1 is part of Lemma 2.3 in [1] but the shortness of our proof seems to justify its presentation.

a zero on its main diagonal.

COROLLARY 1. If  $A \in D_n$  then  $A + I \in T_n$ .

Proof obvious.

3. The main results. Let  $A = (a_{ij}) \in M_n$  and let v denote an n-dimensional vector with  $a_i(v)$  its ith entry.

Define:  $J_k = \{j: a_{kj} = 0\}, I_k = \{i: a_{ik} = 0\},\$ 

$$I_{\scriptscriptstyle 0}(v) = \{i \colon a_i(v) = 0\} \;, \quad I_{\scriptscriptstyle +}(v) = \{i \colon a_i(v) > 0\} \;.$$

Let  $R_n$  denote the space of *n*-tuples of real numbers.

Let  $X_n$  be the set of all nonnegative vectors in  $R_n$  which are neither positive nor zero. We then have the following

THEOREM 2. A nonnegative n-square matrix A is totally indecomposable if and only if  $|I_0(Ax)| < |I_0(x)|$  for every  $x \in X_n$ .

*Proof.* Let  $A \in T_n$  and  $x \in X_n$ . A necessary and sufficient condition for  $a_{i_0}(Ax) = 0$  for some  $i_0$  is

$$(1) I_{+}(x) \subseteq J_{i_0}.$$

If  $I_0(Ax) = \emptyset$ , then there is nothing to prove, so we may assume

$$I_0(Ax) 
eq \emptyset$$
.

 $x \in X_n$  implies

$$I_{+}(x) 
eq \varnothing$$
 .

(1), (2) and (3) imply that  $A[I_0(Ax) | I_+(x)]$  is a zero-submatrix of A. Since  $A \in T_n$  by assumption, we have (by D. 2.)

$$|I_0(Ax)| + |I_+(x)| < n = |I_0(x)| + |I_+(x)|$$

and hence  $|I_0(Ax)| < |I_0(x)|$  which proves the first part of the theorem. (It is not generally true however that  $I_0(Ax) \subseteq I_0(x)$  as it may happen that  $a_i(x) > 0$  and  $a_i(Ax) = 0$ , a situation which differs somewhat from that in the similar case for indecomposable matrices (5.2.2 in [9])).

Let now  $A \in T_n$ . Then A contains a zero-submatrix A[I|J] such that  $I, J \neq \emptyset$  and |I| + |J| = n. Choose now  $x \in R_n$  such that

$$I_{+}(x)=J.$$

Then clearly  $x \in X_n$ . We have  $I_0(x) = N - I_+(x) = N - J$ , and hence  $|I_0(x)| = |I|$ . For  $i \in I$  we have  $J_i \supseteq J$ , and hence by (4)  $I_+(x) \subseteq J_i$ ,

756 M. LEWIN

so that for  $i \in I$  according to (1)  $a_i(Ax) = 0$  and hence  $I_0(Ax) \supseteq I$ . Then  $|I_0(Ax)| \ge |I| = |I_0(x)|$ . This completes the proof.

 $X_n$  in Theorem 2 may of course be replaced by its subset  $Y_n$  consisting of the  $2^n - 2$  zero-one vectors.

Theorem 2 admits of two simple corollaries which we present as Theorems 3 and 4.

Theorem 3. If A is a totally indecomposable nonnegative n-square matrix then

$$A^{n-1}>0$$
 .

*Proof.* If for some  $j_0$  we had  $|I_{j_0}| \ge n-1$  then A would be partly decomposable and hence  $|I_{j_0}| \le n-2$  for  $j \in N$  and the rest follows.

Theorem 1 follows from Theorem 3 as an immediate consequence of Corollary 1. For A = I + P where P is the n-square permutation matrix with ones in the superdiagonal, so that  $a_{ij} = 1$  if i = j or i = j - 1,  $a_{n1} = 1$  and  $a_{ij} = 0$  otherwise, it is easy to show that  $A^{n-2} \geqslant 0$ , which shows that our result is best possible.

THEOREM 4. The product of any finite number of totally indecomposable nonnegative n-square matrices is totally indecomposable.

*Proof.* It is clearly sufficient to prove the statement for two matrices. Let therefore  $A, B \in T_n$ . Choose an arbitrary element x of  $X_n$ . We then have

$$|I_0(ABx)| \leq |I_0(Bx)| < |I_0(x)|$$

by Theorem 2. Since x was arbitrary, (5) applies to all elements of  $X_n$ . Again by Theorem 2 it follows that AB is totally indecomposable, which proves the theorem.

4. Independent proofs of Theorem 3. A lemma of Gantmacher [3] states that if  $A \in D_n$  and  $x \in X_n$ , then  $I_0[(A+I)x] \subset I_0(x)$ .

The following proof of Theorem 3 assuming the lemma has been suggested by London<sup>3</sup>: Let  $A \in T_n$ . Using the fact that a matrix in  $T_n$  possesses a positive diagonal d, put

$$A_{\scriptscriptstyle 1} = \frac{1}{\alpha} P^{\scriptscriptstyle T} (A - \alpha P) = \frac{1}{\alpha} \quad P^{\scriptscriptstyle T} A - I \text{ where} \quad 0 < \alpha < \min \, a_{ij} (a_{ij} \in d)$$

<sup>&</sup>lt;sup>3</sup> D. London, oral communication.

and  $P=(p_{ij})$  is an *n*-square permutation matrix such that  $p_{ij}=1$  if and only if  $a_{ij} \in d$ . Then  $A \in T_n$  implies  $A_1 \in T_n$ .

We have  $A = \alpha P(A_1 + I)$ ; since  $A_1 \in D_n$  we obtain

$$I_{\scriptscriptstyle 0}(Ax) = I_{\scriptscriptstyle 0}[P(A_{\scriptscriptstyle 1}+I)x] = I_{\scriptscriptstyle 0}[(A_{\scriptscriptstyle 1}+I)x] \subset I_{\scriptscriptstyle 0}(x)$$
 ,

for  $x \in X_n$ . Then  $I_0(A^{n-1}x) = \emptyset$ , and  $A^{n-1} > 0$ .

Another proof has been kindly suggested by the referee of this paper: We show that if A is totally indecomposable, then if  $x \in X_n$ , then

$$|I_0(Ax)| < |I_0(x)|$$
.

The theorem then follows immediately.

Suppose  $|I_0(Ay)| \ge |I_0(y)|$  for some  $y \in X_n$ .

Put  $|I_0(y)|=s$ . There are permutation matrices P and Q such that

$$PAy = \left[egin{array}{c} 0 \ u \end{array}
ight] \quad ext{and} \quad Q^{\scriptscriptstyle T}y = \left[egin{array}{c} 0 \ v \end{array}
ight]$$

where u is an (n-s)-dimensional nonnegative victor and v is an (n-s)-dimensional positive vector: The 0's represent s zero components in each case.

We now write  $PAQ = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$  where  $A_1$  is  $s \times s$ ,  $A_2$  is  $s \times (n-s)$ ,  $A_3$  is  $(n-s) \times s$  and  $A_4$  is  $(n-s) \times (n-s)$ . Then  $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 0 \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix}$  and so  $A_2V = 0$ . Thus  $A_2 = 0$  and hence  $A \notin T_n$ , a contradiction.

5. König's Theorem. Let A be an  $m \times n$  matrix. A covering of A is a set of lines (rows or columns) containing all the positive elements of A. A covering of A is a minimal covering of A if there does not exist a covering of A consisting of fewer lines. Let M(A) denote the number of lines in a minimal covering of A. A basis of A is a positive subdiagonal of A of maximal length. m(A) denotes the length of a basis of A. The jth column of A is essential to A if  $M(A(\emptyset J)) < M(A)$ .

We now give the two versions of König's Theorem and their proofs:

- K. T. 1. If A is an  $m \times n$  matrix, then m(A) = M(A).
- K. T. 2. If A is an n-square matrix, then A has k zeros on every diagonal (k > 0) if and only if A contains an  $s \times t$  zero-submatrix with s + t = n + k.

This is a generalized version of a theorem of Frobenius. The following theorem appears in [8] (we reproduce it here in a hypothetical form).

758 M. LEWIN

E. T.: If A is an  $m \times n$  matrix and K.T.I. holds for A, then there exists a minimal covering of A (called essential covering) containing precisely the essential columns of A (and may be some rows).

Proof of K. T. 1.  $m(A) \leq M(A)$  holds trivially. The theorem is clearly true for  $1 \times n$  matrices for all n. Assume that the theorem is true for all  $\mu \times n$  matrices,  $\mu < m$  and all n. Let A be an  $m \times n$  matrix. Consider  $A' = A(\{m\} \mid N]$ . A' is an  $(m-1) \times n$  matrix so that K. T. 1, holds for A' and hence E. T. holds for A'. Let Q be the essential covering of A'.

Case 1. Q is a covering of A. Then  $m(A) \ge m(A') = M(A') \ge M(A)$ .

Case 2. Q is not a covering of A. Then there exists  $j_0 \in N$  for which  $a_{mj_0} > 0$  which is not covered by Q and hence the  $j_0$ th column is not essential to A'. Then clearly there exists a basis b' of A' without elements in the  $j_0$ th column. Then  $b = b' \cup \{a_{mj_0}\}$  is a subdiagonal of A and hence  $M(A) \leq M(A') + 1 = m(A') + 1 \leq m(A)$ . This proves K. T. 1.

*Proof of K. T. 2. Necessity.* If A has k zeros on every diagonal then  $m(A) \leq n - k$ . By K. T. 1,  $M(A) \leq n - k$ . Apply a minimal covering to A. Then there remains an  $s \times t$  zero-matrix of A which is not covered, with  $s + t \geq 2n - M(A) \geq n + k$ .

Sufficiency. Let A contain an  $s \times t$  zero-submatrix with s + t = n + k. Then there are positive elements on at most 2n - (n + k) = n - k lines, meaning that there are at least k zero-rows, which proves the sufficiency.

### REFERENCES

- 1. R. A. Brualdi, S. V. Parter and H. Schneider, The diagonal equivalence of a nonnegative matrix to a stochastic matrix, J. Math. Anal. Appl. 16 (1966) 31-50.
- 2. G. Frobenius, Über Matrizen aus nichtnegativen Elementen, Sitzb. d. Preuss. Akad.
- d. Wiss, (1912), 456-477.
- 3. F. R. Gantmacher, The Theory of Matrices, vol. 2 Chelsea, New York (1959).
- 4. D. J. Hartfiel, A simplified form for nearly reducible and nearly decomposable matrices, (To appear in the Proc. Amer. Math. Soc.).
- 5. J. C. Holladay and R. S. Varga, On powers of nonnegative matrices, Proc. Amer. Math. Soc., 9 (1958), 631-634.
- 6. D. König, Theorie der endlichen und unendlichen Graphen, New York, Chelsea (1950).
- 7. R. Sinkhorn and P. Knopp, Concerning nonnegative matrices and doubly stochastic matrices, Pacific J. Math., 21 (1967), 343-348.

- 8. M. Lewin, Essential coverings of matrices, Proc. Camb. Phil. Soc., 67 (1970). 263-267.
- 9. M. Marcus and H. Minc, A Survey of Matrix Theory and Matrix Inequalities, Boston (1964).
- 10. M. Marcus and H. Minc, Disjoint pairs of sets and incidence matrices, Illinois J. Math., 7 (1963), 137-147.
- 11. H. Minc, On lower bounds for permanents of (0, 1)-matrices. Proc. Amer. Math. Soc. 22 (1969), 233-237.
- 12. H. Minc. Nearly decomposable matrices, (To appear).
- 13. H. J. Ryser, Combinatorial Mathematics, The Carus Mathematical Monographs (1963).
- 14. R. Sinkhorn, Concerning a conjecture of Marshall Hall. Proc. Amer. Math. Soc., 21 (1969), 197-201.
- 15. H. Wielandt, Unzerlegbare nicht negative Matrizen, Math. Z., 52 (1950), 642-648.

Received January 27, 1970.

TECHNION. ISRAEL INSTITUTE OF TECHNOLOGY HAIFA, ISRAEL.

# PACIFIC JOURNAL OF MATHEMATICS

### **EDITORS**

H. SAMELSON Stanford University Stanford, California 94305

C. R. Hobby University of Washington Seattle, Washington 98105 J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD ARENS
University of California
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. Wole

K. Yoshida

#### SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON

AMERICAN MATHEMATICAL SOCIETY CHEVRON RESEARCH CORPORATION NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

# **Pacific Journal of Mathematics**

Vol. 36, No. 3 BadMonth, 1971

| E. M. Alfsen and B. Hirsberg, <i>On dominated extensions in linear subspaces of</i> $\mathscr{C}_{C}(X)$ | 567 |  |  |  |
|--|-----|--|--|--|
| Joby Milo Anthony, Topologies for quotient fields of commutative integral                                | 201 |  |  |  |
| domains  | 585 |  |  |  |
| V. Balakrishnan, G. Sankaranarayanan and C. Suyambulingom, <i>Ordered cycle</i>                          |     |  |  |  |
| lengths in a random permutation  | 603 |  |  |  |
| Victor Allen Belfi, Nontangential homotopy equivalences  | 615 |  |  |  |
| Jane Maxwell Day, Compact semigroups with square roots   | 623 |  |  |  |
| Norman Henry Eggert, Jr., Quasi regular groups of finite commutative nilpotent                           |     |  |  |  |
| algebras   | 631 |  |  |  |
| Paul Erdős and Ernst Gabor Straus, Some number theoretic results   | 635 |  |  |  |
| George Rudolph Gordh, Jr., Monotone decompositions of irreducible Hausdorff                              |     |  |  |  |
| continua   | 647 |  |  |  |
| Darald Joe Hartfiel, <i>The matrix equation</i> $AXB = X$  |     |  |  |  |
| James Howard Hedlund, Expansive automorphisms of Banach spaces. II                                       | 671 |  |  |  |
| I. Martin (Irving) Isaacs, The p-parts of character degrees in p-solvable                                |     |  |  |  |
| groups   | 677 |  |  |  |
| Donald Glen Johnson, Rings of quotients of Φ-algebras  | 693 |  |  |  |
| Norman Lloyd Johnson, Transition planes constructed from semifield                                       |     |  |  |  |
| planes   | 701 |  |  |  |
| Anne Bramble Searle Koehler, <i>Quasi-projective and quasi-injective</i>                                 |     |  |  |  |
| modules  | 713 |  |  |  |
| James J. Kuzmanovich, Completions of Dedekind prime rings as second                                      |     |  |  |  |
| endomorphism rings   | 721 |  |  |  |
| B. T. Y. Kwee, On generalized translated quasi-Cesàro summability  | 731 |  |  |  |
| Yves A. Lequain, <i>Differential simplicity and complete integr<mark>al closure</mark></i>               | 741 |  |  |  |
| Mordechai Lewin, On nonnegative matrices   | 753 |  |  |  |
| Kevin Mor McCrimmon, Speciality of quadratic Jordan algebras   | 761 |  |  |  |
| Hussain Sayid Nur, Singular perturbations of differential equations in abstract                          |     |  |  |  |
| spaces   | 775 |  |  |  |
| D. K. Oates, A non-compact Krein-Milman theorem  | 781 |  |  |  |
| Lavon Barry Page, <i>Operators that commute with a unilateral shift on an</i>                            |     |  |  |  |
| invariant subspace   | 787 |  |  |  |
| Helga Schirmer, <i>Properties of fixed point sets on dendrites</i>                                       | 795 |  |  |  |
| Saharon Shelah, On the number of non-almost isomorphic models of T in a                                  |     |  |  |  |
| power  | 811 |  |  |  |
| Robert Moffatt Stephenson Jr., <i>Minimal first countable Hausdorff spaces</i>                           | 819 |  |  |  |
| Masamichi Takesaki, The quotient algebra of a finite von Neumann   |     |  |  |  |
| algebra  | 827 |  |  |  |
| Benjamin Baxter Wells, Jr., Interpolation in $C(\Omega)$   | 833 |  |  |  |