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SINGULAR PERTURBATIONS OF DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

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SINGULAR PERTURBATIONS OF DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

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In a recent paper, Kisynski studied the solutions of the abstract Cauchy problem $\varepsilon x^{\cdot \cdot}(t) + x^{\cdot}(t) + Ax(t) = 0$, $x(0) = x_0$ and $x^{\cdot}(0) = x_1$ where $0 \le t \le T$, $\varepsilon > 0$ is small parameter and A is a nonnegative self-adjoint operator in a Hilbert space H. With the aid of the functional calculus of the operator A, he has showed that as $\varepsilon \to 0$ the solution of this problem converges to the solution of the unperturbed Cauchy problem $x^{\cdot}(t) + Ax(t) = 0$, $x(0) = x_0$. Smoller has proved the same result for equation of higher order,

The purpose of this paper is to study the solution of a similar problem and allowing the operator A to depend on t.

To be precise, we shall show that if the initial data is taken from a suitable dense subset of H, then the solution of the Cauchy problem:

(1.1)
$$\varepsilon x^{*}(t) + x^{*}(t) + A(t)x(t) = 0, x(0) = x_{0}, x^{*}(0) = x_{1}$$

converges to the solution of the unperturbed Cauchy problem

$$(1.2) x^{\bullet}(t) + A(t)x(t) = 0, x(0) = x_0$$

as $\varepsilon \to 0$ where $0 \le t \le T$, $\varepsilon > 0$ is a small parameter, A(t) is a continuous semi-group of nonnegative self-adjoint operators in H with infinitesimal generator A.

2. The problem (1.1) when $H=R_1$. Before considering (1.1) in the general case, it is necessary to consider (1.1) in the case when $H=R_1$ (i.e., the real line). Thus we consider the Cauchy problem:

(2.1)
$$\varepsilon u^{\bullet \bullet}(t) + u^{\bullet}(t) + e^{\mu t}u(t) = 0$$
. $u(0) = x_0, u^{\bullet}(0) = x_1$

when $t \geq 0$, $\mu \geq 0$. $\varepsilon > 0$.

According to theorem (1) in [2], equation (2.1) has two linearly independent solutions:

$$egin{aligned} u_1 &= \sum\limits_0^{m-1} u_{1,m{t}}(t)arepsilon^j + arepsilon^m E_0 \;, \qquad u_1 &= \sum\limits_0^{m-1} u_{1j}(t)arepsilon^j + arepsilon^{m-1} E_1 \ u_2 &= \sum\limits_0^{m-1} u_{2j}(t)arepsilon^j e^{-t/arepsilon} + arepsilon^m E_0 \;, \quad u_2 &= \sum\limits_0^{m-1} (d/dt)[u_{2j}(t)e^{-tarepsilon}]arepsilon^j + arepsilon^{m-1} E_1 \end{aligned}$$

where $u_{ij}(t)$ (i=1,2) are C^{∞} functions on [0,T] and $u_{i0}(t)$ (i=1,2) does not vanish at any point of [0,T] and E_0 , E_1 are functions of ε and others, but bounded for small $\varepsilon \geq 0$.

Hence the general solution of equation (2.1) is $u = c_1u_1 + c_2u_2$. Solving for c_1 and c_2 by using the initial condition we obtain $u = x_0s_{00} + x_1s_{01}$ and $u^* = x_0s_{10} + x_1s_{11}$ where

$$egin{align} s_{\scriptscriptstyle 00} &= H^{-\!1}(arepsilon)[u_{\scriptscriptstyle 2}(0)u_{\scriptscriptstyle 1}(t) - u_{\scriptscriptstyle 1}(0)u_{\scriptscriptstyle 2}(t)] \ s_{\scriptscriptstyle 01} &= H^{-\!1}(arepsilon)[u_{\scriptscriptstyle 1}(0)u_{\scriptscriptstyle 2}(t) - u_{\scriptscriptstyle 2}(0)u_{\scriptscriptstyle 1}(t)] \ s_{\scriptscriptstyle 10} &= s_{\scriptscriptstyle 00}^{\cdot} = rac{d}{dt}s_{\scriptscriptstyle 00} \ s_{\scriptscriptstyle 11} &= s_{\scriptscriptstyle 01}^{\cdot} = rac{d}{dt}s_{\scriptscriptstyle 01} \ \end{array}$$

and

$$H(\varepsilon) = u_1(0)u_2(0) - u_2(0)u_1(0)$$

How taking the limit as $\varepsilon \to 0$, we find that

$$\begin{array}{ccc} s_{00}(t,\,\varepsilon,\,\mu) & \longrightarrow x_0 u_{10}(t) \\ s_{01}(t,\,\varepsilon,\,\mu) & \longrightarrow 0 \end{array}.$$

Consequently, $u(t, \varepsilon) \to x_0 u_{10}(t)$. From equation 15 in [2] we find that $u_{10}(t)$ is the solution of the equation

$$(2.5) u \cdot + e^{nt} u = 0$$

and this is what we wished to show.

3. Estimates for the Functions $s_{ij}(t, \varepsilon, \mu)$. In this section we would like to find estimates for the functions $s_{ij}(t, \varepsilon, \mu)$ (i, j = 0, 1). We may do so by solving for $u_{ij}(t)$ $(i = 1, 2; j = 0, 1, \dots, m - 1)$ from equation 15 in [2]. Since this would be rather tedious we will take the simpler approach of estimating $u_i(t, \varepsilon, \mu)$ and $u_i(t, \varepsilon, \mu)$ (i = 1, 2). Multiplying (2.1) by u^* and integrating between 0 and t we obtain:

$$rac{arepsilon u^{*2}}{2} + \int_0^t \!\! u^{*2} + rac{u^2}{2} e^{\mu t} - rac{1}{2} \mu \! \int_0^t \!\! u^2 e^{\mu t} = c \; .$$

Consequently

$$u^{\scriptscriptstyle 2} \leq 2 \, | \, c \, | \, + \, \mu \! \int_{\scriptscriptstyle 0}^{t} \! \! u^{\scriptscriptstyle 2} e^{\mu t} dt \; .$$

Now using Bellman's lemma, we obtain

$$u^2 \le 2/c/e^{e^{\mu t}}$$
.

For estimating $u^{\bullet}(t)$, we multiply equation (2.1) by $e^{-\mu t}u^{\bullet}$, integrating between 0 and t and using Bellman's lemma we obtain:

(3.2)
$$u^{*2}(t) \leq 2\varepsilon^{-1}/c/e^{2\mu t}$$
.

In [2] page 323 we proved that for all small $\varepsilon \ge 0$ $H(\varepsilon) \ne 0$, therefore we see that (2.3), (3.1), and (3.2) yield,

$$|s_{00}| \le K(\varepsilon) \exp\left(\frac{e^{\mu t}}{2}\right)$$

 $K(\varepsilon)$ is a bounded function in ε , and

$$|s_{01}| \le \bar{K}(\varepsilon) \exp\left(e^{e^{\mu t}/2}\right)$$

 $\bar{K}(\varepsilon)$ is a bounded function in ε .

To obtain an estimate for s_{ij} (i, j = 1, 2) we write equation (2.1) in amatrix form as:

$$U^{\bullet} = AU$$

when

$$A = egin{pmatrix} 0 & 1 \ - ilde{arepsilon}^{_1} \exp{(\mu t)} & - ilde{arepsilon}^{_1} \end{pmatrix}$$
 .

Hence

$$U=\exp\left[\int\!\!A(s)ds
ight]=egin{pmatrix}s_{\scriptscriptstyle 00} & s_{\scriptscriptstyle 01}\slip_{\scriptscriptstyle 01} & s_{\scriptscriptstyle 11}\end{pmatrix}$$

and from the equation

$$(3.5) \qquad (d/dt) \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\bar{\varepsilon}^1 \exp(\mu t) & -\bar{\varepsilon}^1 \end{pmatrix} \\ = \begin{pmatrix} 0 & 1 \\ -\bar{\varepsilon}^1 \exp(\mu t) & -\bar{\varepsilon}^1 \end{pmatrix} \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix}$$

we obtain

(3.6)
$$s_{10} = -s_{01} \varepsilon^{-1} \exp(\mu t)$$

$$s_{_{11}}=s_{_{00}}-arepsilon^{_{-1}}\!s_{_{01}}$$
 .

4. The problem (1.1) in abstract Hilbert space. We shall now consider the problem (1.1) in any Hilbert space H with norm $||\cdot||$.

Since $\{A(t)\}$ is a semi-group of a nonnegative selfadjoint operator in H, with infinitesimal generator A, there is a resolution of the identity E_{μ} such that A(t) has the spectral representation:

$$A(t)=\int_0^\infty\!\!e^{\mu t}dE_\mu$$
 .

We shall next use the functional calculus of the operator A(t). For fixed $\varepsilon > 0$, $t \ge 0$, we define the operator S_{ij} on H by

$$(4.1) S_{ij}(t,\,\varepsilon) = \int_0^\infty s_{ij}(t,\,\varepsilon,\,\mu) dE_\mu (i,\,j=0,\,1)$$

where the $s_{ij}(t, \varepsilon, \mu)$ are defined by (2.3). If we let D denote the dense domain of the operator $e^{A^2(t)}$ for all t, then our estimates (3.2) through (3.7) imply that D is contained in the domain of $S_{ij}(t, \varepsilon)$ for every i, j = 0, 1.

For x_0 and x_1 in D, we write

$$(4.2) x_{\varepsilon}(t) = S_{00}(t, \varepsilon)x_0 + S_{01}(t, \varepsilon)x_1$$

and we see that $x_{\varepsilon}(t)$ is in the domain of A(t) for every $\varepsilon > 0$. We now state the main theorem.

THEOREM. Let $x_{\varepsilon}(t)$ be defined as in (4.2) when x_0 , x_1 are in D. Then $x_{\varepsilon}(t)$ is the unique solution of the Cauchy problem (1.1) and $x_{\varepsilon}(t)$ converges to the solution of (1.2) as $\varepsilon \to 0$.

To prove this theorem we first prove the following lemmas:

LEMMA 1. For $x \in D$, $(d/dt)S_{ij}(t, \varepsilon)x$ exists and

$$(4.3) (d/dt)S_{ij}(t,\varepsilon)x = \int_0^\infty (d/dt)s_{ij}(t,\varepsilon,\mu)dE_{\mu}x (i,j=0,1).$$

Proof. We shall prove the lemma for i=j=0. Since the proofs for the other cases are similar, they will be omitted. For $x \in D$ and $t \ge 0$ fixed, we have:

$$egin{aligned} & \left\| rac{S_{00}(t+arDelta t,\,arepsilon)-S_{00}(t)}{arDelta t} imes -S_{10}(t,\,arepsilon)x
ight\|^2 \ & = \int_0^\infty & \left[rac{s_{00}(t+arDelta t,\,arepsilon,\,\mu)-s_{00}(t,\,arepsilon,\,\mu)}{arDelta t} - s_{10}(t,\,arepsilon,\,\mu)
ight]^2 d \mid\mid E_{\mu}x\mid\mid^2 \ & = \int_0^\infty & [s_{10}(t',\,arepsilon,\,\mu)-s_{10}(t,\,arepsilon,\,\mu)]^2 d \mid\mid E_{\mu}x\mid\mid^2 \ , \end{aligned}$$

where $t \le t' \le t + \Delta t$, using the theorem of the mean and (2.3).

Now there is a T such that $t + \Delta t \leq T$ for all Δt sufficiently small, so that if we use (3.3) through (3.7) we see that

$$egin{aligned} \mid s_{\scriptscriptstyle 10}(t',arepsilon,\,\mu) - s_{\scriptscriptstyle 10}(t,\,arepsilon,\,\mu) \mid & \leq \mid s_{\scriptscriptstyle 10}(t',\,arepsilon,\,\mu) \mid + \mid s_{\scriptscriptstyle 10}(t,\,arepsilon,\,\mu) \mid \ & \leq arepsilon^{-1} e^{\mu T} K(arepsilon) e^{(\imath/2)\,e^{\mu T}} \leq N(arepsilon,\,T) e^{e^{\mu T}} \end{aligned}$$

where $N(\varepsilon, T)$ is a constant depending on T and ε only. Therefore the function $|s_{10}(t', \varepsilon, \mu) - s_{10}(t, \varepsilon, \mu)|^2$ is summable with respect to the measure $d ||E_{\mu}x||^2$ if Δt is sufficiently small. Furthermore,

$$\lim_{t\to 0} [s_{10}(t',\varepsilon,\mu)-s_{11}(t,\varepsilon,\mu)]^2=0.$$

So that the Lebseque dominated convergence theorem yields:

$$\lim_{M o 0} \int_0^\infty [s_{\scriptscriptstyle 10}(t',\,arepsilon,\,\mu) \,-\, s_{\scriptscriptstyle 10}(t,\,arepsilon,\,\mu)]^2 d \mid\mid E_\mu x\mid\mid^2 \,=\, 0$$
 .

This completes the proof of the lemma.

LEMMA 2. For $x \in D$ and $t \ge 0$, we have

(4.4)
$$\lim_{\varepsilon \to 0} \left\| S_{00}(t, \varepsilon) x - \exp\left(-\int A(s) ds\right) x \right\| = 0$$

$$\lim_{\epsilon \to 0} ||S_{01}(t, \varepsilon)x|| = 0.$$

Proof.

$$igg\|S_{00}(t,\,arepsilon)x - \exp\left(-\int\!A(s)ds
ight) imesigg\|^2 \ = \int_0^\infty \left|\left(s_{00}(t,\,arepsilon,\,\mu) - \exp\left(-\int^t\!e^{\mu s}ds
ight)
ight)
ight|^2\!d\mid\!\mid E_\mu x\mid\!\mid^2.$$

From (3.3) we see that $\left[s_{00}(t,\varepsilon,\mu)-\exp\left(-\int^t e^{\mu s}ds\right)\right]^2$ is summable with respect to the measure $d\mid |E_\mu x||^2$ and, as we have seen in (2.4) and (2.5), the integrand converges pointwise to zero. We apply the Lebesgue dominated convergence theorem to conclude that the integral likewise converges to zero as $\varepsilon \to 0$. This proves (4.4). Relation (4.5) follows from (2.4) and (2.5) likewise.

LEMMA 3. Let B be a bounded operator in H. If $x^{\bullet}(t) + Bx(t) = 0$, $0 \le t \le 0$, and x(0) = 0, then $x(t) \equiv 0$.

The proof of the above lemma is in [3] and therefore will be omitted.

The proof of the theorem. That $x_{\epsilon}(t)$ defined by (4.2) is a solu-

tion of (1.1) follows at once from Lemma 1 by direct verification. The uniqueness of $x_{\epsilon}(t)$ follows from Lemma 3 just as in [1]. Finally, since $\exp\left(-\int_{0}^{t}A(s)ds\right)x_{0}$ is the solution of (1.2) Lemma 2 shows that.

$$\lim_{\epsilon o 0} \left\| x_{\epsilon}(t) - \exp\left(-\int_{-\infty}^{t} A(s) ds\right) x_{0}
ight\| = 0$$
 .

This completes the proof of the theorem.

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