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## **STRUCTURE OF NOETHER LATTICES WITH JOIN-PRINCIPAL MAXIMAL ELEMENTS**

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**In this paper we explore the structure of Noether lattices with join-principal maximal elements.**

Results which completely specify the structure of certain special classes of Noether lattices, and relate them to lattices of ideals of Noetherian rings, have been obtained in [1], [2], [3], [4], [7], and [8]. For example, in [7] we showed that if every maximal element of a Noether lattice  $\mathcal{L}$  is meet-principal, then  $\mathcal{L}$  is distributive and can be represented as the lattice of ideals of a Noetherian ring. Moreover, for distributive Noether lattices, the condition that every maximal element is meet-principal is equivalent to representability. In a more recent paper [8], we began considering the complementary case of a Noether lattice in which every maximal element is join-principal in order to determine the extent of the relationship between the two situations. There we showed that if 0 is prime in  $\mathcal{L}$  (and every maximal element is join-principal), then  $\mathcal{L}$  is distributive and representable. Hence, if 0 is prime, the assumptions that every maximal element is meet-principal and that every maximal element is join-principal are equivalent, and either implies representability.

In this paper, we continue the investigation begun in [8]. Our results extend the class of Noether lattices for which embedding and structure theorems are known, and also introduce a construction process for Noether lattices which leads to new examples.

In §1, we show that in a local Noether lattice  $(\mathcal{L}, M)$  in which  $M$  is join-principal and not a prime of 0, the maximal element  $M$  has a minimal base  $E_1, \dots, E_k$  of independent principal elements (i.e.,  $E_i \wedge (E_1 \vee \dots \vee \widehat{E}_i \vee \dots \vee E_k) = 0$  for  $i = 1, \dots, k$ ). And we use this result to show that if  $M$  is join-principal and not a prime of 0, then  $\mathcal{L}$  is distributive. In §2, we obtain structure and embedding theorems for distributive local Noether lattices with join-principal maximal elements. In §3, we investigate some of the consequences of our results outside of the local case.

We adopt the terminology of [5].

1. Let  $(\mathcal{L}, M)$  be a local Noether lattice and let  $B \in \mathcal{L}$ . The quotient  $B/MB$  is a finite dimensional complemented modular lattice and the number of elements in any minimal set of principal elements with join  $B$  is the dimension of the quotient  $B/MB$  ([4], [6]). Hence

if  $E_1, \dots, E_s$  is any set of principal elements with the property that the elements  $E_i \vee MB$  are independent in  $B/MB$ , then  $E_1, \dots, E_s$  can be extended to a minimal base for  $B$ . We will have occasion to use these observations in what follows.

In this section we show that if  $(\mathcal{L}, M)$  is a local Noether lattice in which  $M$  is join-principal and not a prime of 0, then  $\mathcal{L}$  is distributive.

We begin with a lemma.

**LEMMA 1.1.** *Let  $(\mathcal{L}, M)$  be a local Noether lattice in which  $M$  is join-principal and not a prime of 0. Let  $E_1, \dots, E_k$  be a minimal base for  $M$  and, for each  $i = 1, \dots, k$ , set  $C_i = E_1 \vee \dots \vee \hat{E}_i \vee \dots \vee E_k$ . Then each of the elements  $C_i (i = 1, \dots, k)$  is prime.*

*Proof.* Since  $M$  is principal in  $\mathcal{L}/C_i$  ( $i = 1, \dots, k$ ), each of the elements  $C_i$  is either prime or  $M$ -primary [7]. Assume that  $C_r$  is  $M$ -primary. And let  $n$  be the least positive integer such that  $E_r^{n+1} \leq C_r$ . Then  $E_r^{n+1} \leq MC_r$ . For, if not, there exist principal elements  $F_1, \dots, F_s$  among  $E_1, \dots, \hat{E}_r, \dots, E_k$  such that  $E_r^{n+1}, F_1, \dots, F_s$  is a minimal base for  $C_r$ . But then  $E_r, F_1, \dots, F_s$  is a minimal base for  $M = E_r \vee C_r$ . Since  $C_r$ , by definition, has fewer elements in a minimal base than  $M$ , this is a contradiction. Hence  $E_r^{n+1} \leq MC_r$ , as claimed. Consequently,  $M^{n+1} \leq MC_r$ , and therefore

$$E_r^n \leq M^n \vee (0 : M) = M^{n+1} : M = MC_r : M = C_r \vee (0 : M) = C_r,$$

since  $M$  is join-principal and not a prime of 0. Since  $E_r^n \not\leq C_r$ , this leads to a contradiction. Hence, each of the elements  $C_i$  is prime.

**LEMMA 1.2.** *Let  $(\mathcal{L}, M)$  be a local Noether lattice in which  $M$  is join-principal and not a prime of 0. Then, in the notation of Lemma 1.1,  $C_1 \wedge \dots \wedge C_k = 0$ .*

*Proof.* Let  $E_1, \dots, E_k$  and  $C_1, \dots, C_k$  be as in Lemma 1.1. We first show that for  $1 \leq r < s \leq k$ ,  $E_r E_s = 0$ . Hence, suppose that  $E_r E_s \neq 0$ , and let  $n$  be a positive integer such that  $E_r E_s \leq M^n$  and  $E_r E_s \not\leq M^{n+1}$ . Then  $E_r E_s$  can be used in a minimal base for  $M^n$ . Now, since  $M$  is join-principal and not a prime of 0, it follows from the relation  $M^{n+k+n} = M^{nk}(E_1^n \vee \dots \vee E_k^n)$  that the elements  $E_1^n, \dots, E_k^n$  form a minimal base for  $M^n$ . Hence, for some  $i$ ,  $1 \leq i \leq k$ ,  $M^n = E_r E_s \vee E_1^n \vee \dots \vee \hat{E}_i^n \vee \dots \vee E_k^n$ . But then  $M^n \leq C_i$ , which contradicts Lemma 1.1. It now follows that, for each  $s$  ( $1 \leq s \leq k$ ),  $C_s \wedge E_s = (C_s : E_s)E_s = C_s E_s = 0$ , since  $C_s$  is prime and  $E_s \not\leq C_s$ . Hence by modularity  $C_1 \wedge \dots \wedge C_s = E_{s+1} \vee \dots \vee E_k$  for  $s \leq k$ , so that  $C_1 \wedge \dots \wedge C_k = 0$ .

We are now in a position to establish the main result of the section.

**THEOREM 1.3.** *Let  $(\mathcal{L}, M)$  be a local Noether lattice in which  $M$  is a join-principal and not a prime of  $0$ . Then  $\mathcal{L}$  is distributive.*

*Proof.* Let  $E_1, \dots, E_k$  and  $C_1, \dots, C_k$  be as in Lemma 1.1. A simple inductive argument using modularity proves that

$$(\vee E_i^{j(i)}) \wedge (\vee E_i^{k(i)}) = \vee E_i^{\max(j(i), k(i))}$$

with the convention that  $E_i^\infty$  means  $0$ . Thus it suffices to show that the only principal elements in  $\mathcal{L}$  are  $0, I$  and the powers  $E_i^n$  of the elements  $E_1, \dots, E_k$ . If  $k = 1$ , the result is immediate, so assume  $k \geq 2$ . Let  $E$  be any principal element of  $\mathcal{L}$  different from  $0$  and  $I$ . We assume that the elements  $E_1, \dots, E_k$  are arranged so that  $E \leq C_i$  for  $i > r$  and  $E \not\leq C_i$  for  $i \leq r$ . Set  $C = C_1 \wedge \dots \wedge C_r$  and consider  $\mathcal{L}/C$ . Since  $M$  is principal in each of the local Noether lattices  $\mathcal{L}/C_i$  ( $i = 1, \dots, k$ ), it follows by Lemmas 1.1 and 1.2 that the primes of  $\mathcal{L}/C$  are just  $M$  and  $C_1, \dots, C_r$ . Hence, by the choice of  $E$ , the element  $E \vee C$  is  $M$ -primary in  $\mathcal{L}/C$ , and therefore also in  $\mathcal{L}$ . Let  $n$  be a positive integer such that  $M^{n+1} \leq E \vee C$  and  $M^n \not\leq E \vee C$ , then, by modularity,

$$\begin{aligned} M^{n+1} \vee C &= C \vee ((M^{n+1} \vee C) \wedge E) \\ &= C \vee ((M^{n+1} \vee C) : E)E. \end{aligned}$$

Hence, either  $M^{n+1} \leq C \vee ME$  or  $(M^{n+1} \vee C) : E = I$ . In the first case, however,

$$M^n \leq M^{n+1} : M \leq (C \vee ME) : M = (C : M) \vee E = C \vee E,$$

which contradicts the choice of  $n$ . Hence  $(M^{n+1} \vee C) : E = I$  and  $E \leq M^{n+1} \vee C$ . Then  $E \vee C = M^{n+1} \vee C$ , so by the join-irreducibility of principal elements in a local Noether lattice, it follows that  $E \vee C = E_1^{\varphi(1)} \dots E_k^{\varphi(k)} \vee C$ , for some nonnegative integers  $\varphi(1), \dots, \varphi(k)$ . On the other hand,  $E_i \leq C$  for  $i > r$  and  $E \not\leq C$ , so  $\varphi(i) = 0$  for  $i > r$ . Now, if  $i \neq j$  and  $1 \leq j \leq r$ , then  $E_i \vee C \leq C_j$ . It follows that  $r \leq 1$ , and hence that  $E \leq C_2 \wedge \dots \wedge C_k$ . Then by the proof of Lemma 1.2,  $C_2 \wedge \dots \wedge C_k = E_1$  and  $ME_1^n = E_1^{n+1}$ , for all  $n$ . Hence, there exists a positive integer  $u$  such that  $E \leq E_1^u$  and  $E \not\leq ME_1^u = E_1^{u+1}$ . Since  $E_1$  is principal, it is now immediate that  $E = E_1^u$ .

We note that if  $(\mathcal{L}, M)$  is a local Noether lattice in which  $M^2 = 0$ , then  $M$  is join-principal. Since such a Noether lattice need not be distributive, the statement of Theorem 1.3 need not be valid without the assumption that  $M$  is not a prime of  $0$ . On the other hand, if  $\mathcal{L}$

is an arbitrary Noether lattice in which every maximal element is join-principal, then the number of maximal primes associated with  $0$  is finite. Hence, at most finitely many of the localizations  $\mathcal{L}_M$  ( $M$  maximal) are nondistributive.

2. Let  $(\mathcal{L}_1, M_1)$  and  $(\mathcal{L}_2, M_2)$  be local Noether lattices, and let  $\mathcal{L} = \{(A, B) \in \mathcal{L}_1 \oplus \mathcal{L}_2; A = I \text{ if and only if } B = I\}$ . It is clear that  $\mathcal{L}$  is a sub-multiplicative-lattice of  $\mathcal{L}_1 \oplus \mathcal{L}_2$ . Moreover, if  $E_1$  and  $E_2$  are principal elements of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ , respectively, with  $E_1 \neq I$  and  $E_2 \neq I$ , then the elements  $(E_1, 0)$  and  $(0, E_2)$  are principal in  $\mathcal{L}$ . Hence  $\mathcal{L}$  is a local Noether lattice with maximal element  $(M_1, M_2)$ . We refer to  $\mathcal{L}$  as the *local direct sum of  $\mathcal{L}_1$  and  $\mathcal{L}_2$* . An alternative characterization is given by  $\mathcal{L} = (M_1 | 0 \oplus M_2 | 0) \cup \{(I, I)\}$ .

In this section we continue our investigation of a local Noether lattice  $(\mathcal{L}, M)$  with join-principal maximal element. However, we drop the hypothesis that  $M$  is not a prime of  $0$  and consider, instead, the general distributive case. Our main result is that a distributive local Noether lattice  $(\mathcal{L}, M)$ , in which  $M$  is join-principal, is the local direct sum of local Noether lattices with principal maximal elements. We begin with an extension of Lemma 1.2.

LEMMA 2.1. *Let  $(\mathcal{L}, M)$  be a distributive local Noether lattice in which  $M$  is join-principal. Let  $E_1, \dots, E_k$  be a minimal base for  $M$ . Then  $E_i \wedge E_j = 0$  for all  $i \neq j$ .*

*Proof.* For each  $i = 1, \dots, k$ , set  $C_i = E_1 \vee \dots \vee \widehat{E}_i \vee \dots \vee E_k$ . Then

$$M = M^2 : M = (MC_i \vee E_i^2) : M = C_i \vee (E_i^2 : M)$$

and

$$E_i \vee (E_i^2 : M) = (ME_i \vee E_i^2) : M = ME_i : M = E_i \vee (0 : M),$$

so because

$$(E_i^2 : M) = (E_i^2 : M) \wedge (E_i \vee 0 : M) = 0 : M \vee ((E_i^2 : M) \wedge E_i)$$

by modularity, we have that

$$M = C_i \vee (0 : M) \vee ((E_i^2 : M) \wedge E_i) = C_i \vee (0 : M) \vee (E_i^2 : ME_i)E_i,$$

$i = 1, \dots, k$ . Since principal elements are join-irreducible in a local Noether lattice, since  $\mathcal{L}$  is distributive, and since  $E_i \not\leq C_i$ , it follows that either  $E_i \leq 0 : M$  or  $E_i \leq (E_i^2 : ME_i)E_i$ ,  $i = 1, \dots, k$ .

Assume that  $E_r \leq (E_r^2 : ME_r)E_r$ . Then  $E_r^2 : ME_r = I$ , so  $ME_r = E_r^2$ . Hence  $M = ME_r : E_r = E_r^2 : E_r = E_r \vee (0 : E_r)$ . It follows that  $E_i \leq E_r \vee (0 : E_r)$  for all  $i$ , and that  $E_i \leq 0 : E_r$  for  $i \neq r$  since  $\mathcal{L}$  is distributive and  $E_i$  is join-irreducible. Therefore  $C_i E_i = 0$  ( $i = 1, \dots, k$ ).

Now, assume that  $1 \leq i < j \leq k$  and let  $E$  be a principal element such that  $E \leq E_i \wedge E_j$ . Suppose that  $E \neq 0$  and choose integers  $u$  and  $v$  such that  $E \leq E_i^u \wedge E_j^v$ ,  $E \not\leq E_i^{u+1}$  and  $E \not\leq E_j^{v+1}$ . Then  $E \leq E_i^u$  and  $E \not\leq ME_i^u$ , so  $E = E_i^u$ . Similarly  $E = E_j^v$ , so  $E_i^u = E = E_j^v$ . Then  $u > 1$  and  $v > 1$ , so  $ME_i^{u-1} = ME_j^{v-1}$ . It follows that  $E_i^{u-1} \vee (0:M) = E_j^{v-1} \vee (0:M)$ , so that either  $E_i^{u-1} \leq E_j^{v-1}$  or  $E_i^{u-1} \leq 0:M$ . In either case,  $E_i^u = 0$ . Hence  $E = 0$  and  $E_i \wedge E_j = 0$ .

**THEOREM 2.2.** *Let  $(\mathcal{L}, M)$  be a distributive local Noether lattice. Then  $M$  is join-principal if, and only if,  $\mathcal{L}$  is the (finite) local direct sum of local Noether lattices with principal maximal elements.*

*Proof.* Assume that  $(\mathcal{L}, M)$  is a distributive local Noether lattice in which  $M$  is join-principal. Let  $E_1, \dots, E_k$  be a minimal base for  $M$ . And for each  $i = 1, \dots, k$ , let  $(\mathcal{L}_i, M_i)$  be a local Noether lattice such that  $M_i$  is principal and  $M_i^n = 0$  if, and only if,  $E_i^n = 0$ . Since  $\mathcal{L}$  is distributive, it follows by Lemma 2.1 and [2] that every element  $A \in \mathcal{L}$  has a unique minimal basis consisting of powers of the elements  $E_1, \dots, E_k$ . If we set  $E_i^\infty = 0$  and  $E_i^0 = I$ , then it is clear that the map  $E_1^{n_1} \vee \dots \vee E_k^{n_k} \rightarrow (M_1^{n_1}, \dots, M_k^{n_k})$  is a multiplicative lattice isomorphism of  $\mathcal{L}$  onto the local direct sum of  $\mathcal{L}_1, \dots, \mathcal{L}_k$ .

The converse is clear.

**COROLLARY 2.3.** *Let  $(\mathcal{L}, M)$  be a distributive local Noether lattice in which  $M$  is join-principal. Then  $\mathcal{L}$  is Noether-lattice-embeddable in the lattice of ideals of a homomorphic image of a regular local ring.*

*Proof.* By Corollary 2.2,  $\mathcal{L}$  is the local direct sum of local Noether lattices  $(\mathcal{L}_1, M_1), \dots, (\mathcal{L}_k, M_k)$ , where, for each  $i$ ,  $M_i$  is principal in  $\mathcal{L}_i$ . If  $M_i$  is nilpotent in  $\mathcal{L}_i$ , let  $n_i$  be the least positive integer such that  $M_i^{n_i} = 0$ ; otherwise, let  $n_i = \infty$ . Let  $RL_k$  be the regular local Noether lattice introduced in [1], and let  $X_1, \dots, X_k$  be the minimal base for the maximal element of  $RL_k$ . Let  $A$  be the join of the elements  $X_i X_j$  and  $X_i^{n_i}$  (where  $X_i^\infty = 0$ ). Then  $\mathcal{L}$  is clearly isomorphic to  $RL_k/A$ . Since  $RL_k$  is Noether-lattice-embeddable in the lattice of ideals of a regular local ring, [1], it follows that  $RL_k/A$  and  $\mathcal{L}$  are embeddable in the lattice of ideals of a homomorphic image of a regular local ring.

3. In this section we interpret some of the implications of the results of §§ 1 and 2 outside of the local case.

We begin with a new characterization of the representable distributive Noether lattices.

**THEOREM 3.1.** *Let  $\mathcal{L}$  be a Noether lattice. Then  $\mathcal{L}$  is distributive and representable as the lattice of ideals of a Noetherian ring if, and only if, for each maximal element  $M$  of  $\mathcal{L}$ ,  $M$  is join-principal and  $O_M$  is meet-irreducible.*

*Proof.* If  $\mathcal{L}$  is distributive and representable, then each maximal element  $M$  is principal [7]. Consequently,  $\mathcal{L}_M$  is a quotient of a regular local Noether lattice of altitude 1, and  $O_M$  is meet-irreducible.

Now, assume that  $\mathcal{L}$  is a Noether lattice such that, for every maximal element  $M$ ,  $M$  is join-principal and  $O_M$  is meet-irreducible. Fix  $M$  and consider  $\mathcal{L}_M$ . If  $\{M\}$  is not a prime of 0 in  $\mathcal{L}_M$ , then by Lemma 2.1,  $O_M$  is meet-irreducible if, and only if,  $\{M\}$  is principal. On the other hand, if  $\{M\}$  is a prime of 0 in  $\mathcal{L}_M$ , then  $\{M\}$  is the only prime of 0. In this case, let  $E$  be any principal element such that  $E \leq 0:\{M\}$ . Then  $\{M\}E = 0$ , so  $E$  is a point in  $\mathcal{L}_M$ . Since the meet of any two points is 0 and  $O_M$  is irreducible by assumption, it follows that  $0:\{M\}$  is itself a point and that  $0:\{M\} \leq A$ , for every  $A \neq 0$ . Now, assume that  $\{M\} \neq 0:\{M\}$ , and let  $F$  be a principal element such that  $F \leq \{M\}$ ,  $F \not\leq \{M\}^2$  and  $\{M\}F \neq 0$ . Then  $F$  is  $\{M\}$ -primary, so there is a nonnegative integer  $n$  such that  $\{M\}^n \not\leq F$  and  $\{M\}^{n+1} \leq F$ . Hence  $\{M\}^{n+1} = \{M\}^{n+1} \wedge F = (\{M\}^{n+1}:F)F$ , and therefore either  $\{M\}^{n+1}:F = I$  or  $\{M\}^{n+1} \leq \{M\}F$ . In the first case,  $\{M\}^{n+1} = F$ , so  $\{M\} = F$  by the choice of  $F$ . In the second case,

$$\{M\}^n \leq \{M\}^{n+1}:\{M\} = \{M\}F:\{M\} = F \vee (0:\{M\}) = F,$$

a contradiction. Hence  $\{M\}$  is principal in  $\mathcal{L}_M$ . It now follows by [7] that  $\mathcal{L}$  is distributive and representable.

Recall that a Noether lattice  $\mathcal{L}$  satisfies the *weak union condition* if, given elements  $A, B$  and  $C$  such that  $A \not\leq B$  and  $A \not\leq C$ , it follows that there exists a principal element  $E \leq A$  such that  $E \not\leq B$  and  $E \not\leq C$ . This concept was used in [7] to characterize the distributive Noether lattices which are representable. It is easy to see that if  $\mathcal{L}$  is a Noether lattice which satisfies the weak union condition, then every localization  $\mathcal{L}_M$  has the (weaker) property that, given primes  $P_1, \dots, P_k$  and an element  $A$  such that  $A \not\leq P_i$  ( $i = 1, \dots, k$ ), there exists a principal element  $E \leq A$  such that  $E \not\leq P_i$  ( $i = 1, \dots, k$ ). We say that a Noether lattice with this latter property satisfies the *union condition on primes*.

**THEOREM 3.2.** *Let  $\mathcal{L}$  be a distributive Noether lattice such that, for every maximal element  $M$ ,  $\mathcal{L}_M$  satisfies the union condition on primes. If 0 has no embedded primes and if every maximal element is join-principal, then  $\mathcal{L}$  is Noether-lattice-embeddable in the lattice of ideals of a Noetherian ring.*

*Proof.* Let  $0 = Q_1 \wedge \cdots \wedge Q_k$  be a normal decomposition in which  $Q_i$  is  $P_i$ -primary. And let  $M$  be a maximal element of  $\mathcal{L}$ . If  $M$  is a prime of  $0$ , then  $M$  is a minimal prime. On the other hand, by Lemma 1.1, if  $M$  is not a prime of  $0$ , then  $0$  is prime in  $\mathcal{L}_M$ . Hence, if we assume that  $P_1, \dots, P_s$  are nonmaximal primes and that  $P_{s+1}, \dots, P_k$  are maximal primes, we have that

$$\mathcal{L} \cong \mathcal{L}/P_1 \oplus \cdots \oplus \mathcal{L}/P_s \oplus \mathcal{L}/Q_{s+1} \oplus \cdots \oplus \mathcal{L}/Q_k.$$

Then each of the summands  $\mathcal{L}/P_i$  ( $i = 1, \dots, s$ ) is isomorphic to the lattice of ideals of some Noetherian ring [8], and each of the summands  $\mathcal{L}/Q_i$  ( $i = s + 1, \dots, k$ ) is Noether-lattice-embeddable in the lattice of ideals of a Noetherian ring (Corollary 2.3). The conclusion is now immediate.

By the results of [9], it is easy to see that any Noether lattice of the type described in Theorem 3.2 has the property that every element has a unique normal decomposition. On the other hand, a Noether lattice with this latter property is the direct sum of local Noether lattices with nilpotent maximal elements and one-dimensional Noether lattices in which  $0$  is prime [9]. These observations lead to the following, the proof of which is similar to the proof of Theorem 3.2:

**THEOREM 3.3.** *Let  $\mathcal{L}$  be a Noether lattice in which each maximal element is join-principal. Then the following are equivalent:*

- (i) *Each element has a unique normal decomposition.*
- (ii)  *$\mathcal{L}$  satisfies the union condition on primes and  $0$  has no embedded primes.*
- (iii)  *$\mathcal{L}$  is the (finite) direct sum of Noether lattices with principal maximal elements and local Noether lattices with nilpotent maximal elements.*

*If, in addition,  $\mathcal{L}$  is distributive, then each of the above implies that  $\mathcal{L}$  is Noether-lattice-embeddable in the lattice of ideals of a Noetherian ring.*

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