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## **THE HAUSDORFF MEANS OF DOUBLE FOURIER SERIES AND THE PRINCIPLE OF LOCALIZATION**

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**In the two dimensional case, as in the one dimensional case, the Hausdorff summability method is generated by a Hausdorff weight function. In this paper, we investigate the conditions which must be imposed on this weight function in order that the resulting means of a double Fourier series will display the principle of localization.**

In this article we examine the conditions under which the Hausdorff means of double Fourier series exhibit the principle of localization. As is well known, these means are a generalization of a number of other well known means, including those of Cesàro and Euler. Our results are summarized in Theorems 1 to 4, together with the appropriate corollaries.

Let  $[c, d; a, b]$  denote a rectangle with vertices at  $(a, b)$ ,  $(a, d)$ ,  $(c, b)$  and  $(c, d)$ ,  $a \leq c$ ,  $b \leq d$ . For  $0 < \delta < \pi$ , let  $R(\delta) = [\delta, \delta; -\delta, -\delta]$ ,  $N(\delta) = [\pi, \delta; -\pi, -\delta] \cup [\delta, \pi; -\delta, -\pi]$ ,  $C(\delta) = [\pi, \pi; -\pi, -\pi] \sim N(\delta)$ , and  $E(\delta) = N(\delta) \sim R(\delta)$ . For  $0 < \tau \leq 1/2$ , let  $\mathcal{A}(\tau) = [1 - \tau, 1 - \tau; \tau, \tau]$ , and let  $\theta(\tau) = [1, 1; 0, 0] \sim \mathcal{A}(\tau)$ . Then  $N(\delta)$  is a cross-neighborhood of the origin,  $E(\delta)$  is the deleted cross-neighborhood, and  $\theta(\tau)$  is the  $\tau$ -neighborhood of the boundary of the unit square  $[1, 1; 0, 0]$ .

Let  $f(x, y)$  be a  $2\pi$ -periodic function, Lebesgue integrable in the period square, and let  $\{s_{mn}(x, y)\}$  be the corresponding sequence of partial sums of the Fourier series of  $f(x, y)$ . In the sequel we relate all such calculations to the origin, so that we will be examining the sequence  $\{s_{mn}(0, 0)\}$ , which we denote simply by  $\{s_{mn}\}$ . As is easily shown,

$$\begin{aligned} s_{mn} &= \frac{1}{4\pi^2} \int_{-\pi, -\pi}^{\pi, \pi} f(s, t) \frac{\sin(m + 1/2)s}{\sin s/2} \frac{\sin(n + 1/2)t}{\sin t/2} ds dt \\ (1) \quad &= \frac{1}{4\pi^2} \left\{ \int_{R(\delta)} + \int_{E(\delta)} + \int_{C(\delta)} \right\} \\ &= r_{mn} + e_{mn} + c_{mn}. \end{aligned}$$

Now suppose that a regular linear summability method  $H$  ([6], Vol. 1, p. 74) is applied to the sequence  $\{s_{mn}\}$  and let  $\{h_{mn}\}$  denote the corresponding sequence of transforms under the method  $H$ . Then

$$\begin{aligned} h_{mn} &= H\{s_{mn}\} \\ (2) \quad &= H\{r_{mn}\} + H\{e_{mn}\} + H\{c_{mn}\} \\ &= \alpha_{mn} + \beta_{mn} + \gamma_{mn}. \end{aligned}$$

We will say that the principle of localization holds for the sequence  $\{s_{mn}\}$  if for arbitrary, fixed  $\delta$ ,  $s_{mn} = r_{mn} + o(1)$ ,  $m, n \rightarrow \infty$ , and that it holds restrictedly if  $s_{mn} = r_{mn} + o(1)$ ,  $m, n \rightarrow \infty$ , under certain specified restrictions on the subscripts  $m$  and  $n$ . Similarly, we will say that the principle of localization holds for the sequence  $\{h_{mn}\}$  if  $h_{mn} = \alpha_{mn} + o(1)$ ,  $m, n \rightarrow \infty$ , and that it holds restrictedly if

$$h_{mn} = \alpha_{mn} + o(1), m, n \rightarrow \infty,$$

under certain specified restrictions on the subscripts  $m$  and  $n$ .

It follows that a necessary and sufficient condition for the principle of localization to hold in the first case is that

$$e_{mn} + c_{mn} = o(1), m, n \rightarrow \infty,$$

and in the second case the necessary and sufficient condition is that  $\beta_{mn} + \gamma_{mn} = o(1)$ ,  $m, n \rightarrow \infty$ . Since  $f(x, y)$  is assumed to be Lebesgue integrable, the sequence  $\{c_{mn}\}$  is a null sequence. Since the method  $H$  is assumed to be regular, it follows that the sequence  $\{\gamma_{mn}\}$  is also a null sequence. Thus to prove that the principle of localization holds for an arbitrary, Lebesgue integrable function, it is necessary and sufficient to prove that, for an arbitrary fixed  $\delta$ ,  $\delta > 0$ ,  $e_{mn} = o(1)$  and  $\beta_{mn} = o(1)$ ,  $m, n \rightarrow \infty$ , in the respective cases.

Now let  $H$  be a regular Hausdorff summability method. This method has been investigated in detail by Hallenbach [2]. From the foregoing, it follows that to prove that the principle of localization holds for this method, it is necessary and sufficient to prove that the sequence  $\{\beta_{mn}\}$  is a null sequence, where

$$(3) \quad \beta_{mn} = \sum_{0,0}^{m,n} \binom{m}{k} \binom{n}{l} e_{kl} \int_{0,0}^{1,1} u^k (1-u)^{m-k} v^l (1-v)^{n-l} dg(u, v),$$

where the Hausdorff weight function  $g(u, v)$  is of bounded variation in the unit square  $[1, 1; 0, 0]$ ,  $g(u, 0) = g(u, 0^+) = g(0, v) = g(0^+, v) = 0$ , and  $g(1, 1) = 1$ .

We will say that  $g(u, v)$  satisfies a Lipschitz condition of order one and Lipschitz constant  $M$  on a region  $R \subset [1, 1; 0, 0]$  if  $g(u, v)$  is continuous on  $R$  and if

$$|g(u'', v'') - g(u', v'') - g(u'', v') + g(u', v')| \leq M|(u'' - u')(v'' - v')|$$

whenever the rectangle  $[u'', v''; u', v']$  contains only points of  $R$ . It is easily seen that in such a case, the absolute value of the measure  $dg(u, v)$  is majorized by  $Mdu dv$  on  $R$ .

**Summary of the main results.** We assume that  $g(u, v)$  is a regular Hausdorff weight function, so that the corresponding Hausdorff

method is regular. From (3), we have

$$(4) \quad \beta_{mn} = \sum_{0,0}^{m,n} \binom{m}{k} \binom{n}{l} e_{kl} \left\{ \int_{A(\tau)} + \int_{A'(\tau)} \right\} u^k (1-u)^{m-k} v^l (1-v)^{n-l} dg(u, v) \\ = \delta_{mn} + \theta_{mn}.$$

**THEOREM 1.** *If a  $2\pi$ -periodic function is Lebesgue integrable in the period square, then  $\delta_{mn} = o(1)$ ,  $m, n \rightarrow \infty$ , provided that for some fixed but otherwise arbitrary  $\varepsilon$ ,  $0 < \varepsilon \leq 1$ , and  $\kappa = \tau(1-\tau)\delta^2/12$ , we have that  $m \leq \varepsilon^{-1}e^{\kappa n}$  and  $n \leq \varepsilon^{-1}e^{\kappa m}$ , a fortiori provided that*

$$\varepsilon m^{-k} \leq m/n \leq \varepsilon^{-1} n^k$$

*for an arbitrary, fixed positive integer  $k$ , as  $m, n \rightarrow \infty$ .*

**COROLLARY 1.** *If  $\{h_{mn}\}$  is a sequence of regular Hausdorff transforms of the sequence  $\{s_{mn}\}$ , corresponding to a  $2\pi$ -periodic function which is Lebesgue integrable on the period square, then the question of whether the sequence  $\{h_{mn}: \varepsilon m^{-k} \leq m/n \leq \varepsilon^{-1} n^k\}$  exhibits the principle of localization depends entirely on the behavior of the Hausdorff weight function in an arbitrarily small  $\tau$ -neighborhood of the boundary of the unit square  $[1, 1; 0, 0]$ .*

**COROLLARY 2.** *If, in Corollary 1, the sequence  $\{h_{mn}\}$  corresponds to a Hausdorff weight function  $g(u, v)$  such that for some  $\tau > 0$ , the measure  $dg(u, v)$  is identically zero in a  $\tau$ -neighborhood of the boundary of the unit square, then the sequence  $\{h_{mn}: \varepsilon m^{-k} \leq m/n \leq \varepsilon^{-1} n^k\}$  exhibits the principle of localization. In particular, if  $0 < \alpha < 1$  and  $0 < \beta < 1$ , and if  $\{\varepsilon_{mn}\}$  is the sequence of the Euler  $(\varepsilon; \alpha, \beta)$  transforms of the sequence  $\{s_{mn}\}$ , corresponding to a  $2\pi$ -periodic function Lebesgue integrable on the period square, then the sequence*

$$\{\varepsilon_{mn}: \varepsilon m^{-k} \leq m/n \leq \varepsilon^{-1} n^k\}$$

*exhibits the principle of localization.*

Corollary 1 is an obvious consequence of Theorem 1. This is also the case with the first part of Corollary 2. The second part of Corollary 2 then follows by observing that the Euler  $(\varepsilon; \alpha, \beta)$  means are Hausdorff means corresponding to a function  $g(u, v)$  such that the measure  $dg(u, v)$  is identically zero except at the point  $(\alpha, \beta)$ , where  $dg(u, v) = 1$ .

**THEOREM 2.** *Let  $\sigma$  be any subset of  $\theta(\tau)$  such that on every simply connected subset of  $\sigma$ , the Hausdorff weight function  $g(u, v)$  satisfies a Lipschitz condition of order one with Lipschitz constant  $M$ . Let*

$$(5) \quad \sigma_{mn} = \sum_{0,0}^{m,n} \binom{m}{k} \binom{n}{l} e_{kl} \int_{\sigma} u^k (1-u)^{m-k} v^l (1-v)^{n-l} dg(u, v).$$

Then we have

(a) if  $f(x, y) \in L_p(N(\delta))$ ,  $1 < p < \infty$ , then  $\sigma_{mn} = o(1)$ ,  $m, n \rightarrow \infty$ ,  $\varepsilon m^{-\lambda} \leq m/n \leq \varepsilon^{-1} n^{\lambda}$ , where  $\varepsilon$  and  $\lambda$  are fixed but arbitrary,  $0 < \varepsilon \leq 1$ , and  $0 < \lambda < p - 1$ .

(b) if  $f(x, y) \in B(N(\delta))$ , then  $\sigma_{mn} = o(1)$ ,  $m, n \rightarrow \infty$ ,  $m \leq \varepsilon^{-1} e^{n^{\lambda}}$ ,  $n \leq \varepsilon^{-1} e^{m^{\lambda}}$ , where now  $\varepsilon$  and  $\lambda$  are fixed,  $0 < \varepsilon \leq 1$  and  $0 < \lambda < 1$ .

In particular, the contribution to the  $m$ nth term of the sequence  $\{\beta_{mn}\}$ , due to integration with respect to  $g(u, v)$  over any subset of the unit square on which  $g(u, v)$  satisfies a Lipschitz condition of order one with Lipschitz constant  $M$ , is  $o(1)$ ,  $m, n \rightarrow \infty$ ,  $\varepsilon m^{-\lambda} \leq m/n \leq \varepsilon^{-1} n^{\lambda}$ ,  $0 < \lambda < p - 1$ , if  $f(x, y) \in L_p(N(\delta))$ ; and this contribution is

$$o(1), m, n \rightarrow \infty, m \leq \varepsilon^{-1} e^{n^{\lambda}}, n \leq \varepsilon^{-1} e^{m^{\lambda}}, 0 < \lambda < 1,$$

if  $f(x, y) \in B(N(\delta))$ .

COROLLARY. Let  $\{c_{mn}\}$  denote the sequence of the Cesáro  $(C; \alpha, \beta)$  means of  $\{s_{mn}\}$ , corresponding to a  $2\pi$ -periodic function  $f(x, y)$ . If  $1 \leq \alpha, \beta$ , and if  $f(x, y) \in B(N(\delta))$ , then for arbitrary fixed  $\varepsilon, \lambda$ ,  $0 < \varepsilon \leq 1$ ,  $0 < \lambda < 1$ , the sequence  $\{c_{mn}; m \leq \varepsilon^{-1} e^{n^{\lambda}}, n \leq \varepsilon^{-1} e^{m^{\lambda}}\}$  exhibits the principle of localization. If  $1 \leq \alpha, \beta$ , and if  $f(x, y) \in L_p(N(\delta))$ ,  $1 < p < \infty$ , then for arbitrary, fixed  $\varepsilon, \lambda$ ,  $0 < \varepsilon \leq 1$ ,  $0 < \lambda < p - 1$ , the sequence

$$\{c_{mn}; \varepsilon m^{-\lambda} \leq m/n \leq \varepsilon^{-1} n^{\lambda}\}$$

exhibits the principle of localization.

The corollary follows from the theorem by taking  $\sigma = [1, 1; 0, 0]$  and observing that the  $(C; \alpha, \beta)$  means are Hausdorff means corresponding to the function  $g(u, v) = \{1 - (1 - u)^{\alpha}\} \{1 - (1 - v)^{\beta}\}$ . Then  $|dg(u, v)| = |\alpha\beta(1 - u)^{\alpha-1}(1 - v)^{\beta-1}|dudv \leq Mdudv$  for some constant  $M \geq \alpha\beta$  if  $1 \leq \alpha, \beta$ .

Since it is already known ([6], Vol. II, p. 304) that for bounded  $f(x, y)$ , the sequence  $\{c_{mn}\}$  exhibits the principle of localization without the restrictions imposed on the subscripts  $m, n$  in the above corollary, it would seem that our estimates in Theorem 2 can be improved or a new proof devised to give a better result. Attempts to achieve this have been unsuccessful so far.

THEOREM 3. Suppose that the Hausdorff weight function  $g(u, v)$  is discontinuous along finitely many lines

$$\omega = \{u = u_i, i = 1, 2, \dots, k; v = v_j, j = 1, 2, \dots, l\},$$

such that for some  $\tau > 0$ ,  $\tau \leq u_i \leq 1 - \tau$ ,  $\tau \leq v_j \leq 1 - \tau$  for all  $i, j$ . Let  $g_i(v) = g(u_i^+, v) - g(u_i^-, v)$  and  $g_j(u) = g(u, v_j^+) - g(u, v_j^-)$ , and suppose that the functions  $\{g_i(v), g_j(u)\}$  all satisfy a Lipschitz condition of order one and Lipschitz constant  $M$ . Let

$$(6) \quad \omega_{mn} = \sum_{0,0}^{m,n} \binom{m}{k} \binom{n}{l} e_{kl} \int_{\omega} u^k (1-u)^{m-k} v^l (1-v)^{n-l} dg(u, v)$$

be the contribution to  $\beta_{mn}$  due to integration with respect to  $g(u, v)$  along these lines. Then we have

(a) if  $f(x, y) \in L[\pi, \pi; -\pi, -\pi]$ , then  $\omega_{mn} = o(1)$ ,  $m, n \rightarrow \infty$ , provided  $\varepsilon \leq m/n \leq \varepsilon^{-1}$ , where  $0 < \varepsilon \leq 1$  is fixed but otherwise arbitrary.

(b) if  $f(x, y) \in L_p(N(\delta))$ , then

$$\omega_{mn} = o(1), m, n \rightarrow \infty, \varepsilon m^{-\lambda} \leq m/n \leq \varepsilon^{-1} n^{\lambda},$$

where  $0 < \varepsilon \leq 1$ , and  $0 < \lambda < p - 1$  are fixed but otherwise arbitrary.

(c) if  $f(x, y) \in B(N(\delta))$ , then  $\omega_{mn} = o(1)$ ,  $m, n \rightarrow \infty$ ,  $m \leq \varepsilon^{-1} e^{n^{\lambda}}$ ,  $n \leq \varepsilon^{-1} e^{m^{\lambda}}$ , where  $\varepsilon$  is as before, and  $\lambda$  is again fixed,  $0 < \lambda < 1$ .

From Theorems 1, 2 and 3, a number of general statements pertaining to the principle of localization for the sequence  $\{h_{mn}\}$  of regular Hausdorff transforms of the sequence  $\{s_{mn}\}$  can be made immediately. We give one example. From Theorems 1 and 3, we get

**THEOREM 4.** Suppose that  $f(x, y) \in L[\pi, \pi; -\pi, -\pi]$ , that  $f(x, y)$  is periodic of period  $2\pi$ , and that  $\{s_{mn}\}$  is the sequence of partial sums of the Fourier series of  $f(x, y)$ . If the Hausdorff weight function  $g(u, v)$  is such that for some  $\tau > 0$  the measure  $dg(u, v)$  is identically zero in the  $\tau$ -neighborhood of the boundary of the unit square, except along finitely many lines  $u = u_i$ ,  $v = v_j$ ,  $\tau \leq u_i \leq 1 - \tau$ ,  $\tau \leq v_j \leq 1 - \tau$  for all  $i, j$ , along which the difference functions  $g_i(v)$ ,  $g_j(u)$ , as defined in Theorem 3, all satisfy a Lipschitz condition of order one and Lipschitz constant  $M$ , then the corresponding sequence of Hausdorff transforms  $\{h_{mn}\}$  exhibits the principle of localization restrictedly, with the restriction that for arbitrary fixed  $\varepsilon$ ,  $0 < \varepsilon \leq 1$ , we have

$$\varepsilon \leq m/n \leq \varepsilon^{-1}.$$

**Preliminary Lemmas.** We collect here a few lemmas to facilitate the proofs of our results.

**LEMMA 1.** Let

$$\begin{aligned} \rho_1 \sin \alpha &= u \sin s, \quad \rho_1 \cos \alpha = 1 - u + u \cos s \\ \rho_2 \sin \beta &= v \sin t, \quad \rho_2 \cos \beta = 1 - v + v \cos t. \end{aligned}$$

Then

$$(7) \quad \sum_{0,0}^{m,n} \binom{m}{k} \binom{n}{l} \frac{\sin(k+1/2)s}{\sin s/2} \frac{\sin(l+1/2)t}{\sin t/2} u^k (1-u)^{m-k} v^l (1-v)^{n-l} \\ = \rho_1^m \rho_2^n \phi(m; \alpha, s) \phi(n; \beta, t)$$

where

$$\phi(m; \alpha, s) = \sin m\alpha \cot s/2 + \cos m\alpha \\ \phi(n; \beta, t) = \sin n\beta \cot t/2 + \cos n\beta .$$

*Proof.* The proof is essentially due to Szász [3, p. 443].

$$(8) \quad \sum_0^m \binom{m}{k} \sin(k+1/2)s u^k (1-u)^{m-k} \\ = \operatorname{Im} \{ (1-u + ue^{is})^m e^{is/2} \} \\ = \operatorname{Im} \{ \{ \rho_1 (\cos \alpha + i \sin \alpha) \}^m (\cos s/2 + i \sin s/2) \} \\ = \rho_1^m \operatorname{Im} \{ (\cos m\alpha + i \sin m\alpha) (\cos s/2 + i \sin s/2) \} \\ = \rho_1^m (\sin m\alpha \cos s/2 + \cos m\alpha \sin s/2) .$$

Dividing through by  $\sin s/2$  and noting that (7) is a product of two sums of the type (8) completes the proof.

LEMMA 2. For small values of  $t$ , say  $|t| \leq \delta$ , and  $0 \leq v \leq 1$ ,

$$(9) \quad \sin n\beta = \sin ntv + 2 \cos n(tv + \tfrac{1}{2}rv(1-v)t^3) \sin nrv(1-v)t^3/2 ,$$

where  $|r| = |r(t, v)| \leq \delta'$ .

*Proof.* This result is due to Szász [3, p. 449], taking into account that  $\sin(-x) = -\sin x$ .

LEMMA 3.  $|\phi(m; \alpha, s)| \leq \cot s/2 + 1$ ,  $0 < s \leq \pi$ , so that for  $0 < \delta \leq s \leq \pi$ ,  $|\phi(m; \alpha, s)| \leq \gamma < \infty$  uniformly in  $m$ ,  $\alpha$  and  $s$ .

*Proof.* The lemma is obvious.

LEMMA 4.

$$\phi(n; \beta, t) = 2 \frac{\sin n\beta}{t} + \psi'(n; t, v)$$

where  $\psi'(n; t, v)$  is bounded absolutely and uniformly,  $0 \leq v \leq 1$ ,  $0 < |t| \leq \pi$ ,  $n = 1, 2, \dots$ .

*Proof.* From Lemma 1, we have

$$\begin{aligned}\phi(n; \beta, t) &= 2 \frac{\sin n\beta}{t} + \sin n\beta \{\cot t/2 - 2/t\} + \cos n\beta \\ &= 2 \frac{\sin n\beta}{t} + \psi'(n; t, v) .\end{aligned}$$

LEMMA 5. For small values of  $t$ , say  $|t| \leq \delta$ , and  $0 \leq v \leq 1$ ,

$$\frac{\sin n\beta}{t} = \frac{\sin nt v}{t} + \psi''(n; t, v) \{nv(1-v)t^2/12\} ,$$

where  $|\psi''(n; t, v)| \leq 12\delta'$ .

*Proof.* The lemma follows immediately from Lemma 2 on division by  $t$  since  $|\cos x| \leq 1$  and  $|\sin x| \leq |x|$ . Thus the second term on the right in (9) is bounded absolutely by the absolute value of  $nrv(1-v)t^3$ . Thus

$$\sin n\beta = \sin nt v + \psi''(n; t, v) \{nv(1-v)t^3/12\} ,$$

where  $|\psi''(n; t, v)| \leq |12r(t, v)| \leq 12\delta'$ .

LEMMA 6. For  $|s| \leq \pi$  and  $0 \leq u \leq 1$ ,

$$\rho_1^m \leq e^{-mu(1-u)s^2/12} .$$

*Proof.* From the definition in Lemma 1, we have

$$\begin{aligned}\rho_1^2 &= 1 - 2u(1-u)(1 - \cos s) \\ &= 1 - 2u(1-u)\{(s^2/2! - s^4/4!) + (s^6/6! - s^8/8!) + \dots\} \\ &\leq 1 - 2u(1-u)s^2(1/2! - s^2/4!) \\ &\leq 1 - u/6(1-u)s^2 \\ &\leq e^{-u/6(1-u)s^2}\end{aligned}$$

since under the restrictions on  $s$  and  $u$ ,  $u(1-u)$ , and each of the paired terms in  $s$ , is nonnegative, and  $1/2! - s^2/4! > 1/12$ .

REMARK. It is clear that under similar restrictions on  $t$  and  $v$ , we have

$$\rho_2^n \leq e^{-nv(1-v)t^2/12} .$$

LEMMA 7. In the interval  $0 \leq v \leq 1$  and  $|t| \leq \delta$ ,

$$\rho_2^n \phi(n; \beta, t) = 2\rho_2^n \frac{\sin nt v}{t} + \psi(n; t, v)$$

where  $\psi(n; t, v)$  is bounded absolutely and uniformly in  $n, t$  and  $v$ ,



say  $|\psi(n; t, v)| \leq \delta''$ .

*Proof.* By Lemmas 4 and 5,

$$(10) \quad \phi(n; \beta, t) = 2 \frac{\sin nt v}{t} + 2\psi''(n; t, v)nv(1-v)t^2/12 + \psi'(n; t, v).$$

Then

$$\begin{aligned} \rho_2^n \phi(n; \beta, t) &= 2\rho_2^n \frac{\sin nt v}{t} + 2\psi''(n; t, v)\rho_2^n(n/12)v(1-v)t^2 \\ &\quad + \rho_2^n \psi'(n; t, v) \\ &= 2\rho_2^n \frac{\sin nt v}{t} + \psi(n; t, v), \end{aligned}$$

where  $\psi(n; t, v)$  is bounded absolutely and uniformly, since  $\psi'(n; t, v)$  and  $\psi''(n; t, v)$  are so bounded, and

$$\rho_2^n(n/12)v(1-v)t^2 \leq (n/12)v(1-v)t^2 e^{-nv(1-v)t^2/12} \leq e^{-1}$$

for all values of  $n, v$  and  $t$ , since  $ze^{-z} \leq e^{-1}$  for  $z \geq 0$ .

LEMMA 8. For  $n$  and  $k$  large enough,  $0 \leq v \leq 1$  and  $|t| \leq \delta$ ,

$$|\rho_2^n \phi(n; \beta, t)| \leq 4n$$

and

$$|\phi(n; \beta, t)| \leq 4kn.$$

*Proof.* The first part follows immediately from Lemma 7 since  $\rho_2^n \leq 1$ ,  $\psi(n; t, v)$  is bounded, and  $|\sin nt v/t| \leq nv \leq n$ . The second part follows from (10) since  $\psi'(n; t, v)$  and  $\psi''(n; t, v)$  are bounded, and  $nv(1-v)t^2/12 < n$  for  $0 \leq t \leq \pi$ .

LEMMA 9. If  $K$  and  $\kappa$  are any fixed, positive numbers, however, large or small, then

$$K\{me^{-\kappa n} + ne^{-\kappa m}\} = o(1), \quad m, n \rightarrow \infty,$$

provided that for some fixed  $\lambda$ ,  $0 < \lambda < \kappa$ , and fixed  $\varepsilon$ ,  $0 < \varepsilon \leq 1$ , we have  $m \leq \varepsilon^{-1}e^{\lambda n}$  and  $n \leq \varepsilon^{-1}e^{\lambda m}$  as  $m, n \rightarrow \infty$ , a fortiori, provided that  $\varepsilon \leq m/n \leq \varepsilon^{-1}$  as  $m, n \rightarrow \infty$ .

*Proof.* We prove only that under these conditions,  $Kme^{-\kappa n} = o(1)$ ,  $m, n \rightarrow \infty$ , the proof of the other part being the same. But then

$$\begin{aligned} Kme^{-\kappa n} &\leq K\varepsilon^{-1}e^{-(\kappa-\lambda)n} \\ &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

LEMMA 10. For  $0 < \delta \leq s \leq \pi$ ,

$$\int_0^1 \rho_1^m du < cm^{-1},$$

where  $c = 48/\delta^2$ .

*Proof.* By Lemma 6,

$$\begin{aligned} \int_0^1 \rho_1^m du &\leq \int_0^1 e^{-m u (1-u) s^2/12} du \\ &\leq \int_0^1 e^{-m u (1-u) \delta^2/12} du. \end{aligned}$$

Now since the integrand on the right is symmetric in  $u$  and  $1-u$ , and  $u(1-u) \geq u/2$ ,  $0 \leq u \leq 1/2$ , we get

$$\begin{aligned} \int_0^1 \rho_1^m du &\leq 2 \int_0^{1/2} e^{-m u (1-u) \delta^2/12} du \\ &\leq 2 \int_0^{1/2} e^{-m u \delta^2/24} du \\ &< 48/m\delta^2 = cm^{-1}. \end{aligned}$$

LEMMA 11. For  $0 \leq v \leq 1$  and  $0 < \delta \leq \pi$ ,

$$\int_0^\delta \left| \frac{\sin ntv}{t} \right| dt < 2 \log n, \quad n \geq 10.$$

*Proof.* If  $v = 0$ , the lemma is obvious. If  $v \neq 0$ , set  $ntv = t'$ . Then

$$\begin{aligned} \int_0^\delta \left| \frac{\sin ntv}{t} \right| dt &= \int_0^{n\delta v} \left| \frac{\sin t}{t} \right| dt \\ &\leq \int_0^{\pi n} \left| \frac{\sin t}{t} \right| dt \\ &< \int_0^1 \frac{\sin t}{t} dt + \int_1^{\pi n} \frac{dt}{t} \\ &< \{1 + \log \pi + \log n\} \\ &< 2 \log n, \quad n \geq 10. \end{aligned}$$

LEMMA 12. For  $0 \leq v \leq 1$ ,  $0 < \delta \leq \pi$  and  $1/p + 1/q = 1$ ,  $1 < p < \infty$ , we have

$$\left\{ \int_0^\delta \left| \frac{\sin ntv}{t} \right|^q dt \right\}^{1/q} < 2n^{1/p} \log n, \quad n \geq 10.$$

*Proof.* Again, if  $v = 0$ , the result is obvious. Otherwise set

$ntv = t'$ . Then

$$\begin{aligned} \left\{ \int_0^\delta \left| \frac{\sin ntv}{t} \right|^q dt \right\}^{1/q} &= \left\{ \int_0^{nv\delta} \left| nv \frac{\sin t}{t} \right|^q \frac{dt}{nv} \right\}^{1/q} \\ &\leq \left\{ (nv)^{q-1} \int_0^{\pi n} \left| \frac{\sin t}{t} \right|^q dt \right\}^{1/q} \\ &\leq n^{1/p} \left\{ \int_0^{\pi n} \left| \frac{\sin t}{t} \right| dt \right\}^{1/q} \\ &< n^{1/p} \{1 + \log \pi + \log n\}^{1/q} \\ &< 2n^{1/p} \log n, \quad n \geq 10. \end{aligned}$$

LEMMA 13. Given an arbitrary fixed number  $K < \infty$ , and fixed  $\varepsilon$  and  $\lambda$ ,  $0 < \varepsilon \leq 1$ ,  $0 < \lambda < p - 1$ , where  $1 < p < \infty$ , then

$$K \left\{ \frac{m^{1/p} \log m}{n} + \frac{n^{1/p} \log n}{m} \right\} = o(1), \quad m, n \rightarrow \infty,$$

provided that  $n/m \leq \varepsilon^{-1} m^\lambda$  and  $m/n \leq \varepsilon^{-1} n^\lambda$  as  $m, n \rightarrow \infty$ .

*Proof.* Again we satisfy ourselves by proving that under these restrictions,  $K(m^{1/p} \log m)/n = o(1)$ ,  $m, n \rightarrow \infty$ , the proof of the other part being the same. But then  $m < \varepsilon^{-1} n^{1+\lambda}$ , and so, taking  $\varepsilon = 1$  for convenience,

$$\begin{aligned} K \frac{m^{1/p} \log m}{n} &< K \frac{n^{1/p+\lambda/p} \log n^{1+\lambda}}{n} \\ &= K(1 + \lambda) \frac{\log n}{n^{1-1/p-\lambda/p}} \\ &= o(1), \quad n \rightarrow \infty, \end{aligned}$$

provided that  $\lambda < p - 1$ .

**Proof of the main results.** By the earlier remarks, to establish the conditions under which the sequence  $\{h_{mn}\}$  will exhibit the principle of localization, it is necessary and sufficient to establish the conditions under which the sequence  $\{\beta_{mn}\}$  is a null sequence. By (1) and (3),

$$\begin{aligned} \beta_{mn} &= \frac{1}{4\pi^2} \sum_{0,0}^{m,n} \binom{m}{k} \binom{n}{l} \int_{E(\delta)} f(s, t) \frac{\sin(k + 1/2)s}{\sin s/2} \frac{\sin(l + 1/2)t}{\sin t/2} ds dt \\ &\quad \times \int_{0,0}^{1,1} u^k (1-u)^{m-k} v^l (1-v)^{n-l} dg(u, v). \end{aligned}$$

Interchanging the order of summation and integration, which is permissible in this case since the sum is finite, and applying Lemma 1, we get

$$\begin{aligned}
\beta_{mn} &= \frac{1}{4\pi^2} \int_{E(\delta)} f(s, t) ds dt \int_{0,0}^{1,1} \rho_1^m \rho_2^n \phi(m; \alpha, s) \phi(n; \beta, t) dg(u, v) \\
&= \frac{1}{4\pi^2} \left\{ \int_{\delta, -\delta}^{\pi, \delta} + \int_{-\pi, -\delta}^{-\delta, \delta} + \int_{-\delta, \delta}^{\delta, \pi} + \int_{-\delta, -\pi}^{\delta, -\delta} \right\} f(s, t) ds dt \\
(11) \quad &\times \int_{0,0}^{1,1} \rho_1^m \rho_2^n \phi(m; \alpha, s) \phi(n; \beta, t) dg(u, v) \\
&= \lambda_{mn}^1 + \lambda_{mn}^2 + \lambda_{mn}^3 + \lambda_{mn}^4.
\end{aligned}$$

We assume that the mass points of  $dg(u, v)$  are all bounded uniformly from the boundary of the unit square, and that  $\tau$  is small enough so that  $\theta(\tau)$  contains no mass points of  $dg(u, v)$ . We also assume that  $g(u, v)$  satisfies a Lipschitz condition of order one and Lipschitz constant  $M$  in every simply connected region of  $\theta(\tau)$  over which  $g(u, v)$  is continuous, and we let  $\sigma$  denote the union of all such regions. Finally, we assume that in  $\theta(\tau)$ ,  $g(u, v)$  has finitely many lines of discontinuity,  $u = u_i$ ,  $i = 1, 2, \dots, k$ ,  $v = v_j$ ,  $j = 1, 2, \dots, l$ , such that  $\tau \leq u_i$ ,  $v_j \leq 1 - \tau$  for all  $i, j$ , and that the difference functions  $\{g_i(v), g_j(u)\}$ , where  $g_i(v) = g(u_i^+, v) - g(u_i^-, v)$  and

$$g_j(u) = g(u, v_j^+) - g(u, v_j^-),$$

all satisfy a Lipschitz condition of order one and Lipschitz constant  $M$  on  $\theta(\tau)$ . We denote the set of these lines by  $\omega$ .

Thus,  $\lambda_{mn}^i$ ,  $i = 1, 2, 3, 4$ , has three components, namely, due to integration over  $\Delta(\tau)$ , then due to integration over  $\sigma$ , and finally due to integration over  $\omega$ . We denote these components by  $\delta_{mn}^i$ ,  $\sigma_{mn}^i$  and  $\omega_{mn}^i$ , so that

$$(12) \quad \lambda_{mn}^i = \delta_{mn}^i + \sigma_{mn}^i + \omega_{mn}^i, \quad i = 1, 2, 3, 4,$$

and by (4), (5) and (6), we have

$$\begin{aligned}
(13) \quad \delta_{mn} &= \delta_{mn}^1 + \delta_{mn}^2 + \delta_{mn}^3 + \delta_{mn}^4 \\
\sigma_{mn} &= \sigma_{mn}^1 + \sigma_{mn}^2 + \sigma_{mn}^3 + \sigma_{mn}^4 \\
\omega_{mn} &= \omega_{mn}^1 + \omega_{mn}^2 + \omega_{mn}^3 + \omega_{mn}^4
\end{aligned}$$

*Proof of Theorem 1.* From (11) and (12), we get

$$\begin{aligned}
\left| \delta_{mn}^1 \right| &= \frac{1}{4\pi^2} \left| \int_{\delta, -\delta}^{\pi, \delta} f(s, t) ds dt \int_{\Delta(\tau)} \rho_1^m \rho_2^n \phi(m; \alpha, s) \phi(n; \beta, t) dg(u, v) \right| \\
&\leq \frac{1}{4\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| ds dt \int_{\Delta(\tau)} \rho_1^m \rho_2^n |\phi(m; \alpha, s)| |\phi(n; \beta, t)| dg(u, v) \\
&\leq \frac{kn\gamma}{\pi^2} e^{-m\tau(1-\tau)\delta^2/12} V(g) \int_{\delta, -\delta}^{\pi, \delta} e^{-n\tau(1-\tau)t^2/12} |f(s, t)| ds dt \\
&= K_1 n e^{-\kappa m} I_1(n),
\end{aligned}$$

where we have applied Lemmas 3, 6 and 8, and set  $\kappa = \tau(1 - \tau)\delta^2/12$ ,  $K_1 = k\gamma V(g)/\pi^2$ , where  $V(g)$  denotes the total variation of  $g(u, v)$  in the unit square, and

$$I_1(n) = \int_{\delta, -\delta}^{\pi, \delta} e^{-n\tau(1-\tau)t^2/12} |f(s, t)| ds dt ,$$

and used the observation that in  $\Delta(\tau)$ ,  $\rho_2^n$  is bounded by  $e^{-n\tau(1-\tau)t^2/12}$ . Now  $e^{-n\tau(1-\tau)t^2/12} |f(s, t)| \downarrow 0$ ,  $n \rightarrow \infty$ , except on a set of measure zero, so that  $I_1(n) \downarrow 0$ ,  $n \rightarrow \infty$ .

That  $|\delta_{mn}^2| \leq K_2 n e^{-\kappa m} I_2(n)$ , where  $K_2$  and  $I_2(n)$  have the corresponding relation to the rectangle  $[-\delta, \delta; -\pi, -\delta]$ , is proved in the same manner. To show that  $|\delta_{mn}^3| \leq K_3 m e^{-\kappa n} I_3(m)$  and  $|\delta_{mn}^4| \leq K_4 m e^{-\kappa n} I_4(m)$ , we interchange the roles of  $s$  and  $t$ , and thus of  $m$  and  $n$ . Here,  $\kappa$  has the same meaning throughout, and the constants  $K_i$ ,  $i = 1, 2, 3, 4$ , are all finite. Thus

$$(14) \quad |\delta_{mn}| \leq |\delta_{mn}^1| + |\delta_{mn}^2| + |\delta_{mn}^3| + |\delta_{mn}^4| \\ \leq K \{ n e^{-\kappa m} (I_1(n) + I_2(n)) + m e^{-\kappa n} (I_3(m) + I_4(m)) \}$$

where  $K = \max \{K_1, K_2, K_3, K_4\}$ . By the same argument as used to prove that  $I_1(n) \downarrow 0$ ,  $n \rightarrow \infty$ , we have that  $I_2(n) \downarrow 0$ ,  $n \rightarrow \infty$ , and  $(I_3(m) + I_4(m)) \downarrow 0$ ,  $m \rightarrow \infty$ . Applying this to the right hand side of (14), we get  $\delta_{mn} = o(1)$ ,  $m, n \rightarrow \infty$ , provided that  $m \leq \varepsilon^{-1} e^{\kappa n}$  and  $n \leq \varepsilon^{-1} e^{\kappa m}$  as  $m, n \rightarrow \infty$ . This completes the proof of the theorem.

*Proof of Theorem 2.* Taking  $\tau = 1/2$ , we may assume that the Lebesgue measure of  $\sigma$  is unity, that is, the measure of  $[1, 1; 0, 0] \sim \sigma$  is zero. As before, we have

$$|\sigma_{mn}^1| \leq \frac{1}{4\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| ds dt \int_{\sigma} \rho_1^m \rho_2^n |\phi(m; \alpha, s)| |\phi(n; \beta, t)| |dg(u, v)| ,$$

and since on  $\sigma$  the measure  $|dg(u, v)|$  is majorized by  $M du dv$ , we have by Lemmas 3 and 7,

$$(15) \quad |\sigma_{mn}^1| \leq \frac{M\gamma}{2\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| ds dt \int_{0,0}^{1,1} \rho_1^m \frac{\sin ntv}{t} |dudv| \\ + \frac{M\gamma}{4\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| ds dt \int_{0,0}^{1,1} \rho_1^m |\psi(n; t, v)| |dudv| .$$

Now

$$\int_{0,0}^{1,1} \rho_1^m |\psi(n; t, v)| |dudv| = \int_0^1 \rho_1^m du \int_0^1 |\psi(n; t, v)| dv \\ \leq \delta'' c m^{-1} ,$$

uniformly in  $t, m$  and  $n$ ,  $0 < \delta \leq s \leq \pi$ , by Lemmas 7 and 10. Thus the last term on the right in (15) is bounded by  $(M\gamma Ic\delta'')/(4\pi^2 m) = o(1)$ ,  $m \rightarrow \infty$ , where we have set

$$I = \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| ds dt.$$

Next, applying Lemma 10 and changing the order of integration in the first term on the right in (15), we get

$$(16) \quad \begin{aligned} |\sigma_{mn}^i| &\leq \frac{M\gamma c}{2\pi^2 m} \int_0^1 \left\{ \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| \left| \frac{\sin nt v}{t} \right| ds dt \right\} dv \\ &\quad + o(1), \quad m \rightarrow \infty. \end{aligned}$$

Now suppose that  $f(s, t) \in B(N(\delta))$ , that is,  $|f(s, t)| \leq B < \infty$  almost everywhere in  $N(\delta)$ . Then

$$\begin{aligned} |\sigma_{mn}^i| &\leq \frac{BM\gamma c\pi}{2\pi^2 m} \int_0^1 \left\{ \int_{-\delta}^{\delta} \left| \frac{\sin nt v}{t} \right| dt \right\} dv + o(1) \\ &= \frac{BM\gamma c}{\pi m} \int_0^1 \left\{ \int_0^{\delta} \left| \frac{\sin nt v}{t} \right| dt \right\} dv + o(1) \\ &< 2 \frac{BM\gamma c}{\pi m} \log n + o(1) \\ &= K \frac{\log n}{m} + o(1), \quad m \rightarrow \infty, \quad n \geq 10 \end{aligned}$$

by Lemma 11. Thus  $\sigma_{mn}^i = o(1)$ ,  $m, n \rightarrow \infty$ , provided that for some fixed but otherwise arbitrary  $\varepsilon$  and  $\lambda$ ,  $0 < \varepsilon \leq 1, 0 < \lambda < 1$ , we have  $n \leq \varepsilon^{-1} e^{m^\lambda}$  as  $m, n \rightarrow \infty$ .

That  $\sigma_{mn}^2 = o(1)$ ,  $m, n \rightarrow \infty$ ;  $n \leq \varepsilon^{-1} e^{m^\lambda}$ , is proved in a similar manner. To prove that  $\sigma_{mn}^3 + \sigma_{mn}^4 = o(1)$ ,  $m, n \rightarrow \infty$ ,  $m \leq \varepsilon^{-1} e^{n^\lambda}$ , we again reverse the roles of  $s$  and  $t$ , and so of  $m$  and  $n$ . This completes the proof of Theorem 2 for the case where  $f(s, t) \in B(N(\delta))$ .

If  $f(s, t) \in L_p(N(\delta))$ ,  $1 < p < \infty$ , then in (16),

$$(17) \quad \begin{aligned} &\int_{\delta, -\delta}^{\pi, \delta} |\hat{f}(s, t)| \left| \frac{\sin nt v}{t} \right| ds dt \\ &\leq \left\{ \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)|^p ds dt \right\}^{1/p} \left\{ \int_{\delta, -\delta}^{\pi, \delta} \left| \frac{\sin nt v}{t} \right|^q ds dt \right\}^{1/q} \\ &< 2In^{1/p} \log n \\ &= K' n^{1/p} \log n, \quad n \geq 10, \end{aligned}$$

where we equated the first integral to  $I$ , and applied Lemma 12 to the second integral. Then by (16),

$$\begin{aligned}
 |\sigma_{mn}^1| &\leq \frac{K'M\gamma c}{2\pi^2} \cdot \frac{n^{1/p} \log n}{m} + o(1) \\
 &\leq K_1 \frac{n^{1/p} \log n}{m}, \quad m \rightarrow \infty, \quad n \geq 10,
 \end{aligned}$$

the  $o(1)$  term being absorbed in the first term.

Proceeding as in the first part of the proof with respect to  $\sigma_{mn}^i$ ,  $i = 2, 3, 4$ , and combining the results, we get

$$\begin{aligned}
 |\sigma_{mn}| &\leq K \left\{ \frac{n^{1/p} \log n}{m} + \frac{m^{1/p} \log m}{n} \right\}, \quad m, n \geq 10 \\
 &= o(1), \quad m, n \rightarrow \infty
 \end{aligned}$$

by Lemma 13, provided that  $m/n \leq \varepsilon^{-1}n^\lambda$  and  $n/m \leq \varepsilon^{-1}m^\lambda$ , or equivalently provided that  $\varepsilon m^{-\lambda} \leq m/n \leq \varepsilon^{-1}n^\lambda$ ,  $0 < \varepsilon \leq 1$ ,  $0 < \lambda < p - 1$ . This completes the proof of Theorem 2.

*Proof of Theorem 3.* Let  $\omega_{mn}^{1,i}$  be the contribution to  $\omega_{mn}^1$  due to integration along the line  $u = u_i$ , and let  $\omega_{mn}^{1,j}$  be the contribution to  $\omega_{mn}^1$  due to integration along the line  $v = v_j$ . Then

$$\omega_{mn}^{1,i} = \frac{1}{4\pi^2} \int_{\delta, -\delta}^{\pi, \delta} f(s, t) ds dt \int_{u_i^-, 0}^{u_i^+, 1} \rho_1^m \rho_2^n \phi(m; \alpha, s) \phi(n; \beta, t) dg(u, v).$$

By Lemma 6,  $\rho_1^m \leq e^{-\kappa m}$ , where again  $\kappa = \tau(1 - \tau)\delta^2/12$ . Then by Lemmas 3 and 8,

$$\begin{aligned}
 |\omega_{mn}^{1,i}| &\leq \frac{Mn\gamma e^{-\kappa m}}{\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| ds dt \\
 &= \frac{M\gamma I_1}{\pi^2} n e^{-\kappa m} \\
 &= K'_1 n e^{-\kappa m}
 \end{aligned}$$

where  $I_1$  and  $K'_1$  have the obvious meanings,  $K'_1$  being an absolute constant independent of  $i$ . Since there are at most  $k < \infty$  such lines, the total contribution of these  $k$  lines to  $\omega_{mn}^1$  does not exceed

$$(18) \quad kK'_1 n e^{-\kappa m} = K' n e^{-\kappa m}.$$

Next, we have

$$(19) \quad \omega_{mn}^{1,j} = \frac{1}{4\pi^2} \int_{\delta, -\delta}^{\pi, \delta} f(s, t) ds dt \int_{0, v_j^-}^{1, v_j^+} \rho_1^m \rho_2^n \phi(m; \alpha, s) \phi(n; \beta, t) dg(u, v),$$

and so, by Lemmas 3, 6 and 8,

$$\begin{aligned}
|\omega_{mn}^{1,j}| &\leq \frac{k n \gamma M}{\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| e^{-n\tau(1-\tau)t^2/12} ds dt \int_0^1 \rho_1^m du \\
&< \frac{k \gamma M c}{\pi^2} \frac{n}{m} I_1(n) \\
&= K_1'' \frac{n}{m} I_1(n)
\end{aligned}$$

where  $k$  now is the constant of Lemma 8, and  $I_1(n)$  is the same as in the proof of Theorem 1. Here again  $K_1''$  is an absolute constant independent of  $j$ , and  $I_1(n) \downarrow 0$ ,  $n \rightarrow \infty$ . Since there are at most  $l < \infty$  such lines, the total contribution of these to  $\omega_{mn}^1$  is bounded absolutely by

$$(20) \quad l K_1'' \frac{n}{m} I_1(n) = K'' \frac{n}{m} I_1(n).$$

Thus by (18) and (20),

$$\begin{aligned}
|\omega_{mn}^1| &\leq K' n e^{-\kappa m} + K'' \frac{n}{m} I_1(n) \\
&\leq K_1(n e^{-\kappa m} + \frac{n}{m} I_1(n)).
\end{aligned}$$

By similar reasoning, we get

$$\begin{aligned}
|\omega_{mn}^2| &\leq K_2(n e^{-\kappa m} + \frac{n}{m} I_2(n)) \\
|\omega_{mn}^3| &\leq K_3(m e^{-\kappa n} + \frac{m}{n} I_3(m)) \\
|\omega_{mn}^4| &\leq K_4(m e^{-\kappa n} + \frac{m}{n} I_4(m)).
\end{aligned}$$

Taking  $K = 2 \cdot \max \{K_1, K_2, K_3, K_4\}$  and combining the results, we have

$$\begin{aligned}
|\omega_{mn}| &\leq K \{m e^{-\kappa n} + n e^{-\kappa m}\} \\
&\quad + K \left\{ \frac{n}{m} (I_1(n) + I_2(n)) + \frac{m}{n} (I_3(m) + I_4(m)) \right\} \\
&= o(1), \quad m, n \rightarrow 0, \quad \varepsilon \leq m/n \leq \varepsilon^{-1},
\end{aligned}$$

where the first part goes to zero by Lemma 9, and the second part because of the convergence of the integrals  $I_1(n)$ ,  $I_2(n)$ ,  $I_3(m)$  and  $I_4(m)$  to zero. This completes the proof for  $f(s, t) \in L[\pi, \pi; -\pi, -\pi]$ .

If  $f(s, t) \in L_p(N(\delta))$ ,  $1 < p < \infty$ , then



$$\begin{aligned}
|\omega_{mn}^{1,i}| &\leq \frac{M\gamma e^{-\kappa m}}{4\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| ds dt \int_0^1 |\rho_2^n \phi(n; \beta, t)| dv \\
&\leq \frac{M\gamma e^{-\kappa m}}{2\pi^2} \int_0^1 \left\{ \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| \left| \frac{\sin nt v}{t} \right| ds dt \right\} dv \\
&\quad + \frac{M\gamma e^{-\kappa m}}{4\pi^2} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| ds dt \int_0^1 |\psi(n; t, v)| dv
\end{aligned}$$

by Lemma 7. By (17), the integral in brackets in the first term on the right is bounded by  $K_1' n^{1/p} \log n$ . Thus the first term on the right is bounded  $M\gamma K_1' e^{-\kappa m} n^{1/p} \log n / 2\pi^2$ . Since  $\psi(n; t, v)$  is bounded absolutely and uniformly, the second term is  $o(1)$  as  $m \rightarrow \infty$ . Since there are at most  $k$  such lines, their total contribution to  $\omega_{mn}^1$  is bounded by

$$(21) \quad \frac{kM\gamma K_1'}{2\pi^2} e^{-\kappa m} n^{1/p} \log n + k \cdot o(1) = K' e^{-\kappa m} n^{1/p} \log n, \quad m \rightarrow \infty.$$

On the other hand, by (19),

$$\begin{aligned}
|\omega_{mn}^{1,j}| &\leq \frac{\gamma M c}{2\pi^2 m} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| \left| \frac{\sin nt v_j}{t} \right| ds dt \\
&\quad + \frac{\gamma M c}{4\pi^2 m} \int_{\delta, -\delta}^{\pi, \delta} |f(s, t)| |\psi(n; t, v_j)| ds dt \\
&= \frac{\gamma M c K_1'}{2\pi^2} \frac{n^{1/p} \log n}{m} + o(1), \quad m \rightarrow \infty \\
&= K_1'' \frac{n^{1/p} \log n}{m} + o(1), \quad m \rightarrow \infty.
\end{aligned}$$

The contribution of finitely many, say  $l$ , such lines is then bounded by

$$(22) \quad l K_1'' \frac{n^{1/p} \log n}{m} + l \cdot o(1) = K'' \frac{n^{1/p} \log n}{m}, \quad m \rightarrow \infty,$$

the  $o(1)$  term being absorbed in the first term. Combining (21) and (22), and setting  $K_1 = \max\{K', K''\}$ , we have

$$\begin{aligned}
|\omega_{mn}^1| &\leq K_1 n^{1/p} \log n \left\{ e^{-\kappa m} + \frac{1}{m} \right\}, \quad m \rightarrow \infty \\
&= o(1), \quad m, n \rightarrow \infty, \quad n/m \leq \varepsilon^{-1} m^\lambda, \quad 0 < \lambda < p-1
\end{aligned}$$

by Lemma 13, since for  $m$  sufficiently large,  $e^{-\kappa m} < m^{-1}$ .

Proceeding as before with regard to  $\omega_{mn}^i$ ,  $i = 2, 3, 4$ , and combining the results, we get

$$\omega_{mn} = o(1), \quad m, n \rightarrow \infty, \quad \varepsilon m^{-\lambda} \leq m/n \leq \varepsilon^{-1} n^\lambda,$$

where  $0 < \lambda < p - 1$  is arbitrary but fixed.

If  $f(s, t) \in B(N(\delta))$ , then a similar calculation yields the result

$$\omega_{mn} = o(1), \quad m, n \rightarrow \infty, \quad m \leq \varepsilon^{-1}e^{n^\lambda}, \quad n \leq \varepsilon^{-1}e^{m^\lambda},$$

where  $\varepsilon$  and  $\lambda$  are arbitrary but fixed,  $0 < \varepsilon \leq 1$ , and now  $0 < \lambda < 1$ . Since the calculations are rather obvious, we avoid the details. This completes the proof of Theorem 3.

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Gregory Frank Bachelis and Haskell Paul Rosenthal, <i>On unconditionally converging series and biorthogonal systems in a Banach space</i> .....	1
Richard William Beals, <i>On spectral theory and scattering for elliptic operators with singular potentials</i> .....	7
J. Lennart (John) Berggren, <i>Solvable and supersolvable groups in which every element is conjugate to its inverse</i> .....	21
Lindsay Nathan Childs, <i>On covering spaces and Galois extensions</i> .....	29
William Jay Davis, David William Dean and Ivan Singer, <i>Multipliers and unconditional convergence of biorthogonal expansions</i> .....	35
Leroy John Derr, <i>Triangular matrices with the isoclinal property</i> .....	41
Paul Erdős, Robert James McEliece and Herbert Taylor, <i>Ramsey bounds for graph products</i> .....	45
Edward Graham Evans, Jr., <i>On epimorphisms to finitely generated modules</i> .....	47
Hector O. Fattorini, <i>The abstract Goursat problem</i> .....	51
Robert Dutton Fray and David Paul Roselle, <i>Weighted lattice paths</i> .....	85
Thomas L. Goulding and Augusto H. Ortiz, <i>Structure of semiprime <math>(p, q)</math> radicals</i> .....	97
E. W. Johnson and J. P. Lediaev, <i>Structure of Noether lattices with join-principal maximal elements</i> .....	101
David Samuel Kinderlehrer, <i>The regularity of minimal surfaces defined over slit domains</i> .....	109
Alistair H. Lachlan, <i>The transcendental rank of a theory</i> .....	119
Frank David Lesley, <i>Differentiability of minimal surfaces at the boundary</i> ....	123
Wolfgang Liebert, <i>Characterization of the endomorphism rings of divisible torsion modules and reduced complete torsion-free modules over complete discrete valuation rings</i> .....	141
Lawrence Carlton Moore, <i>Strictly increasing Riesz norms</i> .....	171
Raymond Moos Redheffer, <i>An inequality for the Hilbert transform</i> .....	181
James Ted Rogers Jr., <i>Mapping solenoids onto strongly self-entwined, circle-like continua</i> .....	213
Sherman K. Stein, <i>B-sets and planar maps</i> .....	217
Darrell R. Turnidge, <i>Torsion theories and rings of quotients of Morita equivalent rings</i> .....	225
Fred Ustina, <i>The Hausdorff means of double Fourier series and the principle of localization</i> .....	235
Stanley Joseph Wertheimer, <i>Quasi-compactness and decompositions for arbitrary relations</i> .....	253
Howard Henry Wicke and John Mays Worrell Jr., <i>On the open continuous images of paracompact Čech complete spaces</i> .....	265