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**ON EXTENSIONS OF HOMEOMORPHISMS TO  
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## ON EXTENSIONS OF HOMEOMORPHISMS TO HOMEOMORPHISMS

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Let  $h: P \rightarrow Q$  be a homeomorphism between two compact subsets of the topological spaces  $X$  and  $Y$  respectively.

Conditions on the decompositions of  $X \setminus P$  and  $Y \setminus Q$  are found such that there exists a homeomorphism  $H$  of  $X$  onto  $Y$  which is an extension of  $h$ .

It is shown that if  $P$  and  $Q$  are compact subsets of the one dimensional space  $R_\omega$  consisting of all rational points of the Hilbert space  $l_2$  then any homeomorphism between  $P$  and  $Q$  can be extended to a homeomorphism of  $R_\omega$  onto itself. Thus an example of a one dimensional space having a very high degree of homogeneity is obtained.

A generalization of a theorem of B. Knaster and M. Reichbach (Reichaw) is also given.

Let  $h: P \rightarrow Q$  be a homeomorphism between two compact subsets  $P \subset X$ ,  $Q \subset Y$  of the topological spaces  $X$  and  $Y$ . The problem of finding conditions under which there exists a homeomorphism  $H: X \rightarrow Y$  which is an extension of  $h$  has been considered by a number of authors (see [7], [9]). It was shown ([7]) that one can extend homeomorphisms given between two compact subsets of the Cantor set to a self homeomorphisms of the Cantor set under certain conditions. There are other examples where one can extend homeomorphisms in totally disconnected spaces.

In this paper theorems on extensions of homeomorphisms between subsets of two topological spaces to a homeomorphism of the whole spaces are proved. Some results concerning the degree of homogeneity of spaces are obtained. The theorems obtained here apply mostly to totally disconnected spaces.

In §1 a generalizations of a theorem of B. Knaster and M. Reichbach (see [7]) from metric separable spaces to regular spaces is given. It is applied to extend homeomorphisms in a non separable lacunar subset of some Banach space.

In §2 a theorem an extension of homeomorphisms in metric spaces is proved. It is applied to the subspace  $R_\omega$  of the Hilbert space  $l_2$  consisting of all points  $x = \{x_n\}_1^\infty$  such that  $x_n$  is rational for each  $n$ . As was shown by P. Erdős (see [4] or [5] p. 13)  $R_\omega$  has dimension 1. We show that *every* homeomorphism between two compact subsets of the space  $R_\omega$  can be extended to a self homeomorphism of  $R_\omega$ . Thus an example of a *finite dimensional*, but *not zero dimensional* space having a very high degree of homogeneity is obtained. This result is

related to a problem posed by B. Knaster in [12]. At the end some problems concerning extensions of homeomorphisms in the Knaster-Kuratowski biconnected set ([8], or [5]) are posed.

NOTATION. In the sequel we use the logical connectives  $\vee$  (or)  $\wedge$  (and)  $\Rightarrow$  (implies).  $N$  denotes the set of natural numbers  $Z$  the set of integers and  $R$  the set of real numbers.  $\text{card}(A)$  or  $\overline{A}$  denotes the cardinality of  $A$ , nbd. stands for "neighborhood" and  $S(p, \varepsilon)$  denotes the ball of radius  $\varepsilon$  and centre  $p$  in a metric space. Finally all homeomorphisms, are "onto".

1. In this section two theorems on extensions of homeomorphisms to homeomorphisms are proved. The first theorem generalizes Theorem 1 of [7] from separable metric spaces to regular spaces. The second theorem follows from the first one and is applied to extend homeomorphisms in lacunar<sup>1</sup> subspace of some Banach space.

DEFINITION 1.1. A directed set<sup>2</sup>  $A$  will be called sequentially directed if  $A = \bigcup_{i=1}^{\infty} A_i$  where  $A_i$  are disjoint and the ordering in  $A$  is defined by: Two elements of the same  $A_i$  are incomparable and if  $\alpha' \in A_i$ ,  $\alpha'' \in A_j$  and  $i < j$  then  $\alpha' < \alpha''$ .

We note that in a sequentially directed set every non cofinal subset has an upper bound.

DEFINITION 1.2. Let  $A = \{\alpha | \alpha \in A\}$ ,  $B = \{\beta | \beta \in B\}$  be directed sets. A map  $f: A \rightarrow B$  will be called cofinality preserving if:

(i)  $\alpha_1 \neq \alpha_2 \Rightarrow f(\alpha_1) \neq f(\alpha_2)$

(ii) for every cofinal subset  $C \subset A$ ,  $f(C)$  is a cofinal subset of  $B$ .

(iii) for every cofinal subset  $D$ ,  $D \subset f(A)$ ,  $f^{-1}(D)$  is a cofinal subset of  $A$ .

A map  $f$  satisfying conditions (i) and (ii) will be called semi cofinality preserving.

LEMMA 1.1. Let  $A$  and  $B$  be directed sets. Let  $f$  and  $g$  be cofinality preserving maps  $f: A \rightarrow B$ ,  $g: B \rightarrow A$ . Then there exists a bijection  $k: A \rightarrow B$  such that  $k$  is also cofinality preserving and for every  $\alpha \in A$  either  $k(\alpha) = f(\alpha)$  or  $k(\alpha) = g^{-1}(\alpha)$ .

*Proof.* The proof is similar to the proof of the Cantor Bernstein theorem. The following lemma is trivial.

<sup>1</sup> A lacunar space is a space in which every compact set is nowhere dense. (See [10]).

<sup>2</sup> A directed set  $A$  is a partially ordered set such that for every  $\alpha', \alpha'' \in A$  there exists  $\alpha''' \in A$  with  $\alpha''' > \alpha' \wedge \alpha''' > \alpha''$  ([6]).

**LEMMA 1.2.** *Let  $A$  and  $B$  be two directed sets in which every non cofinal subset has an upper bound. If  $f: A \rightarrow B$  is monotone and semi cofinality preserving then  $f$  is cofinality preserving.*

**DEFINITION 1.3.** Let  $P \subset X$  be a compact subset of  $X$ . A decomposition of  $X \setminus P$  is a family  $\{X_\alpha | \alpha \in A\}$  such that  $X \setminus P = \bigcup_{\alpha \in A} X_\alpha$ , where  $X_\alpha$  are open, closed and disjoint subsets of  $X$  and  $A$  is a directed set of indices.

A decomposition  $\{X_\alpha | \alpha \in A\}$  of  $X \setminus P$  is called regular if the following conditions hold:

- (1) for every  $p \in P$  and every nbd.  $U_p$  of  $p$  there exists an  $\alpha_0$  and a nbd.  $\tilde{U}$  of  $p$ ,  $\tilde{U} \subset U_p$ , such that for  $\alpha > \alpha_0$   $X_\alpha \cap \tilde{U} \neq \emptyset \Rightarrow X_\alpha \subset U_p$ .
- (2) for every  $p \in P$ , every nbd.  $U_p$  of  $p$  and every  $\alpha_0$  the set  $U_p \setminus \{X_\alpha | \alpha \succ \alpha_0\}$  is a nbd. of  $p$ .
- (3) for every cofinal subset  $C$  of  $A$  there exists a point  $p \in P$  and a cofinal subset  $C_p \subset C$  such that for every nbd.  $U_p$  of  $p$  there exists  $\alpha_0$  such that  $U_p \cap X_\alpha \neq \emptyset$  for  $\alpha \in C_p \wedge \alpha > \alpha_0$ . ( $\alpha_0$  depends on  $U_p$ ).

**DEFINITION 1.4.** Let  $\{X_\alpha | \alpha \in A\}$  and  $\{Y_\beta | \beta \in B\}$  be decompositions of  $X \setminus P$  and  $Y \setminus Q$  respectively. Let  $h: P \rightarrow Q$  be a homeomorphism. We say (similarly to [7]) that  $\{X_\alpha\}$  and  $\{Y_\beta\}$  approach  $P$  and  $Q$  according to  $h$  if the following properties hold:

- (4) There exists a cofinality preserving map  $f: A \rightarrow B$  such that  $X_\alpha$  is homeomorphic with  $Y_{f(\alpha)}$ .
- (4a) There exists a cofinality preserving map  $g: B \rightarrow A$  such that  $Y_\beta$  is homeomorphic with  $X_{g(\beta)}$ .
- (5) for every pair of points  $(p, q)$  with  $p \in P, q = h(p) \in Q$  and for every nbd.  $V$  of  $q$  there exists a nbd.  $U$  of  $p$  and  $\alpha_0$  such that for  $\alpha > \alpha_0$

$$X_\alpha \cap U \neq \emptyset \Rightarrow Y_{f(\alpha)} \cap V \neq \emptyset$$

- (5a) for every pair of points  $(q, p)$  with  $q \in Q, p = h^{-1}(q) \in P$  and for every nbd.  $U$  of  $p$  there exists a nbd.  $V$  of  $q$  and  $\beta_0$  such that for  $\beta > \beta_0$ .

$$Y_\beta \cap V \neq \emptyset \Rightarrow X_{g(\beta)} \cap U \neq \emptyset$$

**THEOREM 1.1.** *Let  $X$  and  $Y$  be regular spaces and let  $h: P \rightarrow Q$  be a homeomorphism between compact subsets  $P \subset X, Q \subset Y$ . Let  $\{X_\alpha | \alpha \in A\}$  and  $\{Y_\beta | \beta \in B\}$  be regular decompositions of  $X \setminus P$  and  $Y \setminus Q$  respectively and let  $\{X_\alpha\}$  and  $\{Y_\beta\}$  approach  $P$  and  $Q$  according to  $h$ . Then there exists an extension of  $h$  to a homeomorphism  $H: X \rightarrow Y$ .*

*Proof.* Let  $\theta_\alpha: X_\alpha \rightarrow Y_{f(\alpha)}$   $\psi_\beta: Y_\beta \rightarrow X_{g(\beta)}$  be the homeomorphisms given by (4) and (4a). Let  $k$  be the cofinality preserving map of  $A$

onto  $B$  given by Lemma 1. Denote:

$$A_f = \{\alpha \mid \alpha \in A \wedge k(\alpha) = f(\alpha)\}$$

and

$$A_g = \{\alpha \mid \alpha \in A \wedge k(\alpha) = g^{-1}(\alpha)\} .$$

Define  $H$  by:

$$H(x) = \begin{cases} h(x) & x \in P \\ \theta_\alpha(x) & x \in X_\alpha \wedge \alpha \in A_f \\ \psi_\alpha^{-1}(x) & x \in X_\alpha \wedge \alpha \in A_g \setminus A_f . \end{cases}$$

Clearly  $H$  is a one-to-one mapping of  $X$  onto  $Y$ . By the symmetry of our assumptions it suffices to prove that  $H$  is continuous. Continuity of  $H$  is obvious at every point  $x \in X \setminus P$ . We shall show that for every point  $q \in Q$  and an arbitrary nbd.  $V$  of  $q$  there exists a nbd.  $U$  of  $p = h^{-1}(q)$  such that  $H(U) \subset V$ .

Denote by  $\hat{H}$  the map  $H$  restricted to  $X \setminus P$ . It suffices to show that there exists a nbd.  $\hat{U}$  of  $p$  such that  $\hat{H}(\hat{U}) \subset V$ .

Let (1a) (2a) and (3a) denote properties obtained from (1), (2) and (3) by replacing  $X, p, P, U, \alpha, C$  by  $Y, q, Q, V, \beta, D$  respectively. Let  $\tilde{V} \subset V$  be the nbd. of  $q$  given by (1a). By (5) there exists a nbd.  $\hat{U}_1$  of  $p$  contained in  $U$  such that

$$(6) \quad \alpha > \alpha_0 \wedge X_\alpha \cap \hat{U}_1 \neq \emptyset \Rightarrow \theta_\alpha(X_\alpha) \cap \tilde{V} \neq \emptyset .$$

By (2) there exists a nbd.  $U_1$  of  $p$  such that

$$(7) \quad X_\alpha \cap U_1 \neq \emptyset \Rightarrow \theta_\alpha(X_\alpha) \subset V .$$

If there are no sets  $X_\alpha$  contained in  $U_1$  for which  $H$  is defined by  $\psi_\alpha^{-1}$  then obviously  $H$  is continuous at  $p$ . Thus it remains to consider the case that there exist sets  $X_\alpha$  satisfying:

$$(8) \quad X_\alpha \subset U_1$$

$$(9) \quad \text{for } X_\alpha, H \text{ is defined by } \psi_\alpha^{-1}$$

$$(10) \quad H(X_\alpha) \not\subset V .$$

We denote these sets by  $\hat{X}_\alpha$  and the set of their indices by  $A' = A'(U_1)$ . We prove first the following proposition (\*).

(\*) The nbd.  $U_1$  can be chosen so that  $A'(U_1)$  is not cofinal.

Indeed, suppose that for some  $U_1, A'$  is cofinal. By definition of  $\tilde{V}$  there exists for this  $U_1$  a cofinal subset of indices  $\alpha'$  such that

$$(11) \quad \hat{X}_{\alpha'} \subset U_1 \text{ and } \psi_{\alpha'}^{-1}(\hat{X}_{\alpha'}) \cap \tilde{V} = \emptyset$$

hence there exists a cofinal subset  $A'' = A''(U_1)$  of indices  $\alpha''$  and a

point  $q_1$  such that (3a) is satisfied. Clearly  $q_1 \in \tilde{V}$ . By (3) there exists a cofinal subset  $A''' = A'''(U_1)$  of indices  $\alpha'''(A''' \subset A'')$  and a point  $p_1$  such that every nbd.  $U$  of  $p_1$  intersects all sets  $\hat{X}_{\alpha'''}$  with  $\alpha''' > \alpha_0'''$  ( $\alpha_0'''$  depends on  $U$ ). By regularity of  $X$  and by  $\hat{X}_{\alpha'} \subset U_1$

$$(12) \quad p_1 \in \bar{U}_1 \text{ (the closure of } U_1 \text{) .}$$

By (5) and (5a) we have  $h(p_1) = q_1$ .

Indeed assume  $h^{-1}(q_1) = p_2 \neq p_1$ . Let  $U_1$  and  $U_2$  be disjoint nbds of  $p_1$  and  $p_2$  and let  $\tilde{U}_1 \subset U_1, \tilde{U}_2 \subset U_2$  be the nbds of  $p_1$  and  $p_2$  given by (1). There exists an index  $\alpha_0'''$  such that for  $\alpha''' > \alpha_0'''$  we have  $\hat{X}_{\alpha'''} \cap \tilde{U}_1 \neq \emptyset$ .

According to (5a) there exists for  $\tilde{U}_2$  a nbd.  $V$  of  $q_1$  and a  $\beta_0$  such that for  $\beta > \beta_0: Y_\beta \cap V \neq \emptyset \Rightarrow \hat{X}_{\alpha'''} \cap \tilde{U}_2 \neq \emptyset$  but this is impossible since  $\hat{X}_{\alpha'''} \subset U_1$  for  $\alpha > \alpha_0'''$ .

Suppose now to the contrary that (\*) does not hold. Then for every  $U_1$  there exists a point  $p_{u_1}$  and a point  $q_{u_1}$  such that

$$p_{u_1} \in \bar{U}_1, q_{u_1} \in \tilde{V} \text{ and } h(p_{u_1}) = q_{u_1} .$$

But then the generalized sequence  $\{p_{u_1}\}$  converges to  $p$  which contradicts the continuity of  $h$  at  $p$ . Thus (\*) holds. Consider now the set  $\hat{U} = U_1 \cup \{\hat{X}_\alpha | \alpha \in A'\}$ . By (\*)  $A'$  is not a cofinal subset of  $A$ . Thus by (2)  $\hat{U}$  is a nbd. of  $p$  and  $\hat{H}(\hat{U}) \subset V$ . Theorem 1 is proved.

From now on  $X$  and  $Y$  will denote metric spaces and the decompositions of  $X \setminus P$  and  $Y \setminus Q$  will be assumed to have sequentially directed sets of indices  $A$  and  $B, A = \bigcup_{i=1}^\infty A_i, B = \bigcup_{i=1}^\infty B_i$ .

**THEOREM 1.2.** *Let  $h: P \rightarrow Q$  be a homeomorphism. The following conditions are sufficient for the existence of a homeomorphism  $H: X \rightarrow Y$  which is an extension of  $h$ :*

(13) *for every  $i, \bar{A}_i = \bar{B}_i = M$  where  $M$  is some fixed infinite cardinal.*

(14) *for every  $\alpha \in A_i$*

$$\delta(X_\alpha) < \frac{1}{2^i}$$

(14a) *for every  $\beta \in B_i$*

$$\delta(Y_\beta) < \frac{1}{2^i}$$

(15) *for every  $\alpha \in A_i$*

$$d(i) < \rho(X_\alpha, P) < \frac{1}{2^{i-1}}$$

(15a) *for every  $\beta \in B_i$*

$$d(i) < \rho(Y_\beta, Q) < \frac{1}{2^{i-1}}$$

where  $d(i) > 0$ .

(16) for every  $\alpha, \beta$  there exists a homeomorphism

$$h_{\alpha\beta}: X_\alpha \rightarrow Y_\beta .$$

(17) for every  $p \in P$  and  $\varepsilon > 0$  there exists an  $i_0$  such that

$$\text{card} \{ \alpha \mid \alpha \in A_{i_0} \wedge X_\alpha \cap S(P, \varepsilon) \neq \emptyset \} = M$$

(17a) for every  $q \in Q$  and  $\varepsilon > 0$  there exists a  $j_0$  such that

$$\text{card} \{ \beta \mid \beta \in B_{j_0} \wedge Y_\beta \cap S(q, \varepsilon) \neq \emptyset \} = M .$$

*Proof.* It suffices to show that all assumptions of Theorem 1 are satisfied. Clearly  $\{X_\alpha \mid \alpha \in A\}$  and  $\{Y_\beta \mid \beta \in B\}$  are regular decompositions of  $X \setminus P$  and  $Y \setminus Q$ . To show that  $\{X_\alpha\}$  and  $\{Y_\beta\}$  approach  $P$  and  $Q$  according to  $h$  it suffices to construct (by the symmetry of our assumptions) a monotone semi-cofinality preserving map  $f: A \rightarrow B$  such that (4) and (5) hold. Let us well order  $A_i$  and  $B_i$  into type  $\omega(M)$  where  $\omega(M)$  denotes the first ordinal of cardinality  $M$ .

Let  $j: N \rightarrow N$  satisfy:

$$(18) \quad \frac{1}{2^{j(i)-1}} < \frac{d(i)}{2} \quad \text{for all } i \in N$$

$$(19) \quad j(1) < j(2) \cdots < j(i-1) < j(i) \cdots$$

$$(20) \quad j(i) > j_0 \quad \text{where } j_0 \text{ satisfies (17a) with } \varepsilon = \frac{d(i)}{2}$$

For every  $X_\alpha$  (where  $\alpha \in A_i$ ) there exists a point  $p_\alpha \in P$  such that  $\rho(X_\alpha, p_\alpha) = \rho(X_\alpha, P)$ . Take the point  $q_\alpha = h(p_\alpha) \in Q$ . By (17a) and (20) there exists an injection  $f_i: A_i \rightarrow B_{j(i)}$  such that,

$$Y_{\beta(f_i(\alpha))} \cap S\left(q_\alpha, \frac{d(i)}{2}\right) \neq \emptyset .$$

The union  $f = \bigcup_{i=1}^\infty f_i$  is a semi cofinality preserving monotone map  $f: A \rightarrow B$  satisfying:

$$(21) \quad \begin{cases} \alpha \in A_i \Rightarrow f(\alpha) \in B_{j(i)} \\ \rho(Y_{f(\alpha)}, h(p_\alpha)) < \rho(X_\alpha, p_\alpha) . \end{cases}$$

Thus (4) holds. By (18), (19), (21) also (5) holds.

Let  $\mathcal{N}_\Sigma$  denote the Banach space of all bounded functions from an infinite set  $\Sigma$  to the reals,

$$\mathcal{M}_\Sigma = \{f \mid f: \Sigma \rightarrow \mathbb{R}, \sup_{\sigma \in \Sigma} |f(\sigma)| < \infty\}$$

with the norm  $\|f\| = \sup_{\sigma \in \Sigma} |f(\sigma)|$ .

Let  $W$  denote the subset of  $\mathcal{M}_\Sigma$  consisting of rational valued functions. Clearly  $W$  is a lacunar space. In the following theorem an application of Theorem 2 to extension of homeomorphisms in the space  $W$  is given

**THEOREM 1.3.** *Each homeomorphism  $h: P \rightarrow Q$  between compact subsets of  $W$  has an extension to a self homeomorphism of  $W$ .*

*Proof.* We shall write  $W \setminus P(W \setminus Q)$  as union

$$\cup \{X_\alpha \mid \alpha \in A\} \cup \{Y_\beta \mid \beta \in B\}$$

where  $X_\alpha(Y_\beta)$  are ‘‘cubes’’ such that all assumptions of Theorem 2 will be satisfied.

For every function  $\alpha \in Z^\Sigma$  from  $\Sigma$  to the set of integers  $Z$  denote by  $X_\alpha^1$  the cube:

$$(22) \quad X_\alpha^1 = \{f \mid f \in W \wedge \sqrt{2} + \alpha(\sigma) < f(\sigma) < \sqrt{2} + \alpha(\sigma) + 1 \text{ for all } \sigma\}.$$

All such cubes are homeomorphic, mutually disjoint, closed and open subsets of  $W$ . Let  $\mathcal{F}_1 = \{X_\alpha^1 \mid \alpha \in Z^\Sigma\}$ . Clearly  $\overline{\mathcal{F}_1} = 2^{\overline{\Sigma}}$ . Define

$$(23) \quad A_1 = \{\alpha \mid \alpha \in Z^\Sigma \wedge \rho(X_\alpha^1, P) > 1\}.$$

Since  $P$  is bounded (as a compact set) we have also:  $\overline{A_1} = 2^{\overline{\Sigma}}$ . For every function  $\alpha \in Z^\Sigma$  denote by  $X_\alpha^2$  the cube:

$$(24) \quad X_\alpha^2 = \left\{ f \mid f \in W \wedge \sqrt{2} + \frac{\alpha(\sigma)}{6} < f(\sigma) < \sqrt{2} + \frac{\alpha(\sigma) + 1}{6} \text{ for all } \sigma \right\}.$$

Let  $\mathcal{F}_2 = \{X_\alpha^2 \mid \alpha \in Z^\Sigma \wedge X_\alpha^2 \subset W \setminus \bigcup_{\alpha \in A_1} X_\alpha^1\}$  and define

$$(25) \quad A_2 = \{\alpha \mid \alpha \in Z^\Sigma \wedge \rho(X_\alpha^2, P) > \frac{1}{6} \wedge X_\alpha^2 \in \mathcal{F}_2\}.$$

There exists for every  $p \in P$  at least one cube  $X_{\alpha_0}^1$  in  $\mathcal{F}_1$  such that  $X_{\alpha_0}^1 \cap P = \emptyset$  and  $X_{\alpha_0}^1 \cap S(p, 1) \neq \emptyset$ , (hence  $\alpha_0 \in A_1$ ). The set  $\{X_\alpha^2 \mid X_\alpha^2 \subset X_{\alpha_0}^1 \wedge \alpha \in A_2\}$  has cardinality  $2^{\overline{\Sigma}}$  and therefore also  $\overline{A_2} = 2^{\overline{\Sigma}}$ . Thus (17) holds for  $\varepsilon = 1$  and  $i_0 = 2$ .

By induction we define sets  $A_i$  for  $i = 3, 4 \dots$  and sets of cubes  $\{X_\alpha \mid \alpha \in A_i\}$  satisfying:

$$(26) \quad A_i = 2^{\overline{\Sigma}} \quad (i = 1, 2 \dots)$$



$$(27) \quad \delta(X_\alpha) < 1/6^{i-1} \quad \text{for } \alpha \in A_i$$

$$(28) \quad \frac{1}{6^{i-1}} < \rho(X_\alpha, P) < \frac{2}{6^{i-2}} \quad \text{for } \alpha \in A_i .$$

Obviously (26) (27) (28) imply (13) (14) (15). Taking  $i_0 = i + 1$  for  $\varepsilon > 1/6^{i-1}$  ( $i = 1, 2, \dots$ ) one obtains that (17) holds with  $M = 2^{\overline{i}}$ . Also  $W \setminus P = \cup \{X_\alpha | \alpha \in A\}$  where  $A = \bigcup_{i=1}^\infty A_i$ .

Similarly  $W \setminus Q$  can be decomposed into sets  $Y_\beta, \beta \in B = \bigcup_{i=1}^\infty B_i$ . Finally assumption (16) of Theorem 2 is satisfied since all the cubes  $X_\alpha$  and  $Y_\beta$  are homeomorphic.

2. In this section a theorem on extension of homeomorphisms to homeomorphisms in metric spaces is proved. It is applied to extend homeomorphisms in the one-dimensional space  $R_\omega$  of all points with rational coordinates in the Hilbert space  $l_2$ . We show that each homeomorphism between two compact subsets of  $R_\omega$  can be extended to a self homeomorphism of  $R_\omega$ . Thus an example of a *finite* dimensional but not zero dimensional space having a very high degree of homogeneity is obtained.

DEFINITION 2.1. Let  $\{X_\alpha | \alpha \in A\}$  be a decomposition of  $X \setminus P$ . For every  $\alpha \in A$  let  $p_\alpha \in P$  be any point such that  $\rho(X_\alpha, P) = \rho(X_\alpha, p_\alpha)$ . The sets  $X_\alpha$  will be called thin with respect to  $P$  if the following conditions hold:

$$(29) \quad \begin{aligned} \frac{1}{2^i} < \rho(X_\alpha, P) < \frac{1}{2^{i-1}} \quad & \text{for } \alpha \in A_i \quad (\text{for } i = 2, 3 \dots) \\ \frac{1}{2} < \rho(X_\alpha, P) \quad & \text{for } \alpha \in A \end{aligned}$$

(30) for every  $p \in P$  and  $\varepsilon > 0$  there exists an  $i_0 = i_0(p, \varepsilon)$  and  $\delta = \delta(p, \varepsilon) > 0$  such that

$$i > i_0 \wedge \alpha \in A_i \wedge S(p, \delta) \cap X_\alpha \neq \emptyset \Rightarrow p_\alpha \in S(p, \varepsilon) .$$

If moreover

(31) for every  $i = 1, 2, \dots \overline{A}_i = M$  where  $M$  is an infinite cardinal and for every  $p \in P$  and every  $d > 1/2^i$  there exist  $M$  indices  $\alpha'$  satisfying  $\alpha' \in A_i, \rho(p_\alpha, p) < 1/4^i$  and  $\rho(X_{\alpha'}, P) < Kd$  where  $K > 1$  is a fixed number then the sets  $X_\alpha$  will be called  $M$  dense with respect to  $P$ .

LEMMA 2.1. Let  $\{X_\alpha | \alpha \in A\}$  and  $\{Y_\beta | \beta \in B\}$  be decompositions of  $X \setminus P$  and  $Y \setminus Q$ . Let  $h: P \rightarrow Q$  be a homeomorphism. The following assumptions suffice for the existence of an extension of  $h$  to a homeomorphism  $H: X \rightarrow Y$ .

(32) The sets  $X_\alpha$  are thin with respect to  $P$ .

(32a) The sets  $Y_\beta$  are thin with respect to  $Q$ .

(33) There exists an injection  $\phi: A \rightarrow B$  such that

$$\alpha \in A_i \Rightarrow \phi(\alpha) \in B_i$$

(33a) There exists an injection  $\psi: B \rightarrow A$  such that

$$\beta \in B_i \Rightarrow \psi(\beta) \in A_i$$

(34) There exist homeomorphisms  $f_\alpha: X_\alpha \rightarrow Y_{\phi(\alpha)}$  satisfying

$$\frac{1}{K}\rho(x, p_\alpha) < \rho(f_\alpha(x), h(p_\alpha)) < K\rho(x, p_\alpha)$$

for every  $x \in X$  where  $K > 1$  is a fixed number

(34a) There exist homeomorphisms  $g_\beta: Y_\beta \rightarrow X_{\psi(\beta)}$  satisfying

$$\frac{1}{K}\rho(y, q_\beta) < \rho(g_\beta(y), h^{-1}(q_\beta)) < K\rho(y, q_\beta)$$

for every  $y \in Y_\beta$  where  $K > 1$  is a fixed number, and  $q_\beta \in Q$  is any point of  $Q$  for which  $\rho(Y_\beta, Q) = \rho(Y_\beta, q_\beta)$ .

(35) for every cofinal (in  $A$ ) sequence  $\{\alpha_s\}$  of indices  $\rho(h(p_{\alpha_s}), q_{\phi(\alpha_s)}) \rightarrow 0$  for  $s \rightarrow \infty$

(35a) for every cofinal (in  $B$ ) sequence  $\{\beta_s\}$  of indices  $\rho(h^{-1}(q_{\beta_s}), p_{\psi(\beta_s)}) \rightarrow 0$  for  $s \rightarrow \infty$ .

*Proof.* By (33) and (33a) there exists a one-to-one mapping  $\theta$  of  $A$  onto  $B$ . Denoting  $A_\phi = \{\alpha \mid \alpha \in A, \theta(\alpha) = \phi(\alpha)\}$  and  $A_\psi = \{\alpha \mid \alpha \in A, \theta(\alpha) = \psi^{-1}(\alpha)\}$  one can assume (by Lemma 1.1) that  $\theta$  is defined so that  $A = A_\phi \cup A_\psi$ . We define  $H$  by:

$$H(x) = \begin{cases} h(x) & x \in P \\ f_\alpha(x) & x \in X_\alpha \wedge \alpha \in A_\phi \\ g_\alpha^{-1}(x) & x \in X \wedge \alpha \in A_\psi \setminus A_\phi. \end{cases}$$

As in Theorem 1.1 it suffices to show that  $H$  is continuous at an arbitrary point  $p \in P$ . Let  $V = S(q, \varepsilon)$  be a given nbd. of  $q = h(p)$ . By the continuity of  $h$  we have:

(a) There exists a nbd.  $U_1$  of  $p$  such that

$$x \in U_1 \cap P \Rightarrow H(x) = h(x) \in V.$$

Let  $U_2$  be a nbd. of  $p$  such that  $h(U_2 \cap P) \subset S(q, \varepsilon/8)$ . By (32) and (30) there exists a  $\delta > 0$  such that  $X_\alpha \cap S(p, \delta) \neq \emptyset \Rightarrow p_\alpha \in U_2$ . Let  $\delta_1 = \min(\varepsilon/3K, \delta)$  and let  $U_3 = U_2 \cap S(p, \delta_1)$ . Then

$$x \in X_\alpha \cap U_3 \Rightarrow \left[ \rho(f_\alpha(x), q) < \rho(f_\alpha(x), h(p_\alpha)) + \rho(h(p_\alpha), q) \right. \\ \left. < K\rho(x, p_\alpha) + \frac{\varepsilon}{8} < \varepsilon \right].$$

Hence

(b) There exists a nbd.  $U_3$  of  $p$  such that

$$x \in X_\alpha \cap U_3 \Rightarrow f_\alpha(x) \in V.$$

We now show that

(c) There exists a nbd.  $U_4$  of  $p$  such that

$$x \in X_{\psi(\beta)} \cap U_4 \Rightarrow g_\beta^{-1}(x) \in V.$$

Indeed, otherwise there exists a cofinal sequence of indices  $\{\alpha_s\}$  such that  $x_s \in X_{\alpha_s}$ ,  $x_s \rightarrow p$  and  $\rho(g_{\beta_s}^{-1}(x_s), q) > \varepsilon$  where  $\beta_s = \psi^{-1}(\alpha_s)$ . But then  $p_{\alpha_s} \rightarrow p$  and by (35a) also  $h^{-1}(q_{\beta_s}) \rightarrow p$ . Thus  $q_{\beta_s} \rightarrow q$ . Now by (34a)

$$\rho(y_s q_{\beta_s}) < K\rho(x_s, h^{-1}(q_{\beta_s})) \quad \text{where } y_s = g_{\beta_s}^{-1}(x_s).$$

This is however impossible because  $\rho(y_s, q_{\beta_s}) > \varepsilon/2$  and  $\rho(x_s, h^{-1}(q_{\beta_s})) < \rho(x_s, p) + \rho(p, h^{-1}(q_{\beta_s})) \rightarrow 0$ . It follows by (a) (b) and (c) that  $H(U_1 \cap U_3 \cap U_4) \subset V$ .

**THEOREM 2.1.** *Let  $\{X_\alpha | \alpha \in A\}$  and  $\{Y_\beta | \beta \in B\}$  be decompositions of  $X \setminus P$  and  $Y \setminus Q$  and let  $h: P \rightarrow Q$  be a homeomorphism.*

Denote for every  $i$  and every  $\alpha \in A_i$  by  $x_\alpha \in X_\alpha$  a point satisfying  $\rho(x_\alpha, P) - \rho(X_\alpha, P) < 1/4^i$ .

Similarly denote for every  $i$  and every  $\beta \in B_i$  by  $y_\beta \in Y_\beta$  a point satisfying  $\rho(y_\beta, Q) - \rho(Y_\beta, Q) < 1/4^i$ .

*The following conditions are sufficient for the existence of a homeomorphism  $H: X \rightarrow Y$  which is an extension of  $h$ .*

(36)  $X_\alpha$  are thin and  $\aleph_0$  dense with respect to  $P$ .

(36a)  $Y_\beta$  are thin and  $\aleph_0$  dense with respect to  $Q$ .

(37) for every  $\alpha \in A_i$  and  $\beta \in B_i$  there exists a homeomorphism  $f_{\alpha\beta}: X_\alpha \rightarrow Y_\beta$  such that

$$f_{\alpha\beta}(x_\alpha) = y_\beta$$

and such that for every  $x \in X_\alpha$

$$\rho(f_{\alpha\beta}(x), y_\beta) \begin{cases} < \text{Max} \left\{ \frac{1}{4^i}, 2\rho(x_\alpha, x) \right\} \\ > \rho(x_\alpha, x) - \frac{1}{4^i} \end{cases}$$

(37a) for every  $\beta \in B_i$  and  $\alpha \in A_i$  there exists a homeomorphism  $g_{\beta\alpha}: Y_\beta \rightarrow X_\alpha$  such that

$$g_{\beta\alpha}(y_\beta) = x_\alpha$$

and such that for every  $y \in Y_\beta$

$$\rho(g_{\beta\alpha}(y), x_\alpha) \begin{cases} < \text{Max} \left\{ \frac{1}{4^i}, 2\rho(y_\beta, y) \right\} \\ > \rho(y, y_\beta) - \frac{1}{4^i} \end{cases}$$

(38) for every  $x \in X_\alpha$

$$\rho(x, p_\alpha) > \frac{1}{2}\rho(x, x_\alpha) + \frac{1}{2}\rho(x_\alpha, p_\alpha)$$

(38a) for every  $y \in Y_\beta$

$$\rho(y, q_\beta) > \frac{1}{2}\rho(y, y_\beta) + \frac{1}{2}\rho(y_\beta, q_\beta) .$$

*Proof.* It suffices to define mappings  $\phi: A \rightarrow B$  and  $\psi: B \rightarrow A$  and homeomorphisms  $f_\alpha: X_\alpha \rightarrow Y_{\phi(\alpha)}$  and  $g_\beta: y_\beta \rightarrow X_{\psi(\beta)}$  so that all assumptions of Lemma 2.1 will be satisfied. We note first that by (36) and (36a) assumptions (32) and (32a) hold. Now denote for a fixed  $i$  by  $\{\alpha_n\}_{n=1}^\infty$  the sequence of elements of  $A_i$  and define  $\phi: A_i \rightarrow B_i$  by induction. Suppose that  $\phi$  has already been defined for  $\alpha_1 \dots \alpha_n$ . Define  $\phi(\alpha_{n+1})$  as follows: By (31) there exists a set  $Y_\beta$  satisfying:

$$(39) \quad \begin{cases} \text{(a)} & \beta \in B_i \\ \text{(b)} & \beta \notin \{\phi(\alpha_1) \dots \phi(\alpha_n)\} \\ \text{(c)} & \rho(q_\beta, h(p_\alpha)) < \frac{1}{4^i} \\ \text{(d)} & \rho(Y_\beta, Q) < K\rho(X_\alpha, P) \text{ here } \alpha = \alpha_{n+1} . \end{cases}$$

Put  $\phi(\alpha_{n+1}) = \beta$  and define  $f_\alpha = f_{\alpha, \phi(\alpha)}: X_\alpha \rightarrow Y_{\phi(\alpha)}$  as the homeomorphism given by (37). Thus  $\phi: A_i \rightarrow B_i$  is defined for every  $i$  and so  $\phi: A \rightarrow B$  is defined.

Similarly we define  $\psi: B \rightarrow A$  and  $g_\beta: Y_\beta \rightarrow X_{\psi(\beta)}$  (again using (31) and (37a)).

Obviously (33) and (33a) are satisfied. By (39c) also (35) and (35a) hold. By (29), (37), (38) and (39), (denoting  $\phi(\alpha)$  by  $\beta$ ) we have:

$$\begin{aligned} \rho(f_\alpha(x), h(p_\alpha)) &< \rho(f_{\alpha\beta}(x), f_{\alpha\beta}(x_\alpha)) + \rho(f_{\alpha\beta}(x_\alpha), q_\beta) + \rho(q_\beta, h(p_\alpha)) \\ &< \frac{1}{4^i} + 2\rho(x_\alpha, x) + K\rho(x_\alpha, p_\alpha) + \frac{3}{4^i} \\ &< (K + 6)\rho(x, p_\alpha) \end{aligned}$$

and

$$\begin{aligned} \rho(f_\alpha(x), h(p_\alpha)) &> \rho(f_\alpha(x), q_\beta) - \rho(q_\beta, h(p_\alpha)) \\ &> \frac{1}{2}\rho(f_{\alpha\beta}(x), f_{\alpha\beta}(x_\alpha)) + \frac{1}{2}\rho(y_\beta, q_\beta) - \rho(q_\beta, h(p_\alpha)) \\ &> \frac{1}{(K + 6)}\rho(x, p_\alpha) . \end{aligned}$$

Thus also property (34) of Lemma 2.1 holds with  $K$  replaced by  $K + 6$ .

REMARK. One could define the notion of “thin and  $M$  dense” using sequences of numbers  $\{t_k\}, \{r_k\}$  satisfying  $t_k \rightarrow 0, r_k/t_k \rightarrow 0$  instead of the sequences  $\{2^{-k}\}, \{4^{-k}\}$  used.

*Extension of homeomorphisms in  $R_\omega$ .* We shall show now that in  $R_\omega$  every homeomorphism between two compact subsets can be extended to a self homeomorphism of  $R_\omega$ . Before proving this we introduce some definitions and notations.

The  $n$ -dimensional cube  $C = \{(x_1 \cdots x_n) | a_i \leq x_i \leq a_i + l, i = 1, \dots, n\}$  will be denoted by  $[a_1 a_2 \cdots a_n; l]$ .

Every cube  $[a_1 \cdots a_n; l]$  can be divided into  $2^n$  cubes  $C_{i_1, i_2, \dots, i_n}$  of the form:

$$C_{i_1, i_2, \dots, i_n} = \left\{ (x_1 \cdots x_n) \mid a_j + i_j \frac{l}{2} \leq x_j \leq a_j + \frac{l}{2}(i_j + 1) \right. \\ \left. \text{for every } j = 1 \cdots n \right\}$$

where  $i_j$  equals 0 or 1. Let  $\hat{C}_1, \hat{C}_2 \cdots \hat{C}_{2^n}$ , be the sequence of these cubes ordered lexicographically. By induction we define (as above) cubes  $\hat{C}_{ij}$  which divide the cube  $\hat{C}_i$  into  $2^n$  cubes and more generally  $\hat{C}_{ij \cdots k}$ .

For a given cube  $C$ , let  $Q(C)$  denote the set of cubes:

$$\{\hat{C}_j | j = 2 \cdots 2^n\} \cup \{\hat{C}_{1j} | j = 2 \cdots 2^n\} \cup \{\hat{C}_{11j} | j = 2 \cdots 2^n\} \cup \dots$$

Let  $\langle C \rangle = \langle a_1 \cdots a_n, l \rangle$  denote the cylinder (in  $l_2$ ) over the cube  $C$  i.e.

$$\langle C \rangle = \{ \{x_i\}_{i=1}^\infty | \{x_i\} \in l_2 \text{ and } \forall_{i \leq n} (a_i \leq x_i \leq a_i + l) \}.$$

We call the  $n$ -dimensional cube  $C = [a_1 \cdots a_n, l]$  the base of the cylinder  $\langle C \rangle$ , and define  $Q\langle C \rangle$  as the set of  $\aleph_0$  cylinders in  $l_2$  whose base is one of the cubes in the set  $Q(C)$ .

$\pi_n$  denotes the projection of  $l_2$  on the subspace of all points of the form  $(x_1 \cdots x_n 0 \cdots 0 \cdots)$ .

Finally for a compact subset  $P \subset l_2$  and for a set  $X_\alpha$  disjoint with  $P$  we denote by  $p_\alpha$  any point of  $P$  for which  $\rho(X_\alpha, P) = \rho(X_\alpha, p_\alpha)$  and by  $x_\alpha$  any point of  $X_\alpha$  satisfying  $\rho(x_\alpha, P) - \rho(X_\alpha, P) < \varepsilon_\alpha$  where  $\varepsilon_\alpha$  is given. The following two lemmas are trivial:

LEMMA 2.2. *Let  $S$  be a compact subset of  $R_\omega$  and let  $\varepsilon > 0$ . There exists  $n_0$  such that for every point  $s \in S$  and every  $n > n_0$   $\rho(s, \pi_n(s)) < \varepsilon$ .*

*Proof.* Indeed it suffices to take any finite  $\varepsilon/3$  net  $\{s_1 s_2 \cdots s_k\}$  in  $S$  and choose  $n_0$  such that  $\rho(s_i, \pi_{n_0}(s_i)) < \varepsilon/3$  for every  $i = 1, 2, \dots, k$ .

**LEMMA 2.3.** *The cylinders of the form*

$$\langle r_1 + \sqrt{2}, r_2 + \sqrt{2}, \dots, r_n + \sqrt{2}; l \rangle$$

are for every  $n \in \mathbb{N}$  and every sequence  $l, r_1 \dots r_n$  of rational numbers closed and open subsets of  $R_\omega$ .

**THEOREM 2.2.** *Any homeomorphism  $h: P \rightarrow Q$  between two compact subsets  $P$  and  $Q$  of  $R_\omega$  can be extended to a self homeomorphism of  $R_\omega$ .*

*Proof.* It suffices to decompose the sets  $R_\omega \setminus P$  and  $R_\omega \setminus Q$  so that all assumptions of Theorem 2.1 hold. Let  $\varepsilon_1 > 0$  and let  $n_1$  be a natural number such that Lemma 2.2 holds with  $S = P$ ,  $n_0 = n_1$  and  $\varepsilon = \varepsilon_1$ . Consider the collection  $F_1$  of all cylinders of the form

$$\left\langle \frac{k_1}{4^{n_1}} + \sqrt{2}, \frac{k_2}{4^{n_1}} + \sqrt{2}, \dots, \frac{k_{n_1}}{4^{n_1}} + \sqrt{2}; \frac{1}{4^{n_1}} \right\rangle$$

where  $k_1 \dots k_{n_1}$  are integers.  $F_1$  is a set of mutually disjoint cylinders. Choose from  $F_1$  the set of all cylinders  $\langle C \rangle$  satisfying  $\rho(\langle C \rangle, P) > \frac{1}{2}$  and denote it by  $G_1$ . Take the (countable) set of cylinders

$$\bigcup_{\langle C \rangle \in G_1} Q\langle C \rangle$$

and denote it by  $\{X_\alpha | \alpha \in A_1\}$  where  $A_1$  is countable. (By Theorem 2.1 one has to decompose  $R_\omega \setminus P$  into sets  $X_\alpha$  where  $\alpha \in \bigcup_{i=1}^\infty A_i$  and the sets  $A_i$  have to be disjoint sets of indices. Therefore we do not assume that  $A_1$  is the set of integers).

Let  $\varepsilon_2$  satisfy  $0 < \varepsilon_2 < \varepsilon_1$  and let  $n_2$  be any natural number such that Lemma 2.2 holds with  $S = P$ ,  $n_0 = n_2$  and  $\varepsilon = \varepsilon_2$ .

Decompose the set  $R_\omega \setminus \bigcup \{\langle C \rangle | \langle C \rangle \in G_1\}$  into cylinders of the form:

$$\left\langle \frac{k_1}{4^{n_2}} + \sqrt{2}, \frac{k_2}{4^{n_2}} + \sqrt{2}, \dots, \frac{k_{n_2}}{4^{n_2}} + \sqrt{2}; \frac{1}{4^{n_2}} \right\rangle$$

where  $k_1 \dots k_{n_2}$  are integers, and denote the obtained set of cylinders by  $F_2$ .  $F_2$  is a set of mutually disjoint cylinders. Let

$$G_2 = \left\{ \langle C \rangle | \langle C \rangle \in F_2 \wedge \rho(\langle C \rangle, P) > \frac{1}{2^2} \right\}.$$

Take the (countable) set of cylinders  $\bigcup_{\langle C \rangle \in G_2} Q\langle C \rangle$  and denote it by  $\{X_\alpha | \alpha \in A_2\}$  where  $A_2$  is countable. By induction one can define for a given sequence  $\varepsilon_k \rightarrow 0$  ( $0 < \varepsilon_k < \varepsilon_{k-1}$ ) and a sequence of natural

numbers  $n_k (n_k > n_{k-1})$  countable sets of cylinders  $\{X_\alpha | \alpha \in A_k\}$  for every  $k = 1, 2, \dots$ .

Clearly  $R_\omega \setminus P = \cup \{X_\alpha | \alpha \in A\}$  where  $A = \bigcup_{i=1}^{\infty} A_i$ .

Similarly  $R_\omega \setminus Q = \cup \{Y_\beta | \beta \in B\}$  where  $B = \bigcup_{i=1}^{\infty} B_i$ . Also we can choose the same sequences  $\{\varepsilon_k\}$  and  $\{n_k\}$  for both decompositions. It is easy to show that for sufficiently fast decreasing sequence of numbers  $\varepsilon_k$  (for example  $\varepsilon_k < 1/8^k$  the sets  $X_\alpha(Y_\beta)$  are thin and  $\aleph_0$  dense with respect to  $P(Q)$ ).

Obviously every cylinder  $X_\alpha, \alpha \in A_k$  is homeomorphic to every cylinder  $Y_\beta, \beta \in B_k$ . Also for every pair of points  $x_\alpha \in X_\alpha (\alpha \in A_k) y_\beta \in Y_\beta (\beta \in B_k)$  there exists a homeomorphism  $f_{\alpha\beta}: X_\alpha \rightarrow Y_\beta$  so that  $f_{\alpha\beta}(x_\alpha) = y_\beta$  and such that (37) is satisfied. Finally (38), (38a) follow from simple geometric properties of the Hilbert space  $l_2$ .

Theorem 2.2 is proved.

*We conclude with two problems.* Let  $X$  denote the biconnected set defined by Knaster and Kuratowski ([8] or [5] p. 22) and let  $p \in X$  be the point such that  $X \setminus \{p\}$  is totally disconnected.

**PROBLEM 1.** Can each homeomorphism between two compact subsets of  $X \setminus \{p\}$  be extended to a self homeomorphism of  $X$ ?

In connection with the result obtained in Theorem 2.2 one can ask:

**PROBLEM 2.** Does there exist for  $n = 2, 3, \dots$   $n$ -dimensional space  $X$  where every homeomorphism between two compact subsets can be extended to a self homeomorphism of  $X$ ?

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