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Let B denote the vector space of bounded analytic functions on the open unit disc. The first part of this paper involves the use of three topologies on B which give rise to various continuity classes of operators from B to B, and the study of the relationships among these classes. The second is the examination of a special class of operators called multipliers. An operator T is a multiplier if for some sequence c_n, we have $T(\sum a_n z^n) = \sum a_n c_n z^n$ for any function $\sum a_n z^n$ in B. We characterize all the multipliers from B into B and study their continuity properties.

2. **Definitions.** Let $D = \{z: |z| < 1\}$ be the open unit disc in the complex plane and let $\Gamma = \{z: |z| = 1\}$ be its boundary.

We shall look at the vector space B in each of three different topologies, using (B, τ) to indicate the space B with topology τ. The topology κ is the compact-open topology, uniform convergence on compact subsets of D, and (B, κ) is a metrizable linear topological space. The topology σ is the usual topology of uniform convergence on D, and (B, σ) is a normed space with norm $||f|| = \sup |f(z)|$ for $|z| < 1$.

The strict topology β on the vector space B is the locally convex topology defined by the collection of seminorms $||f||_\phi = ||f\phi||$ for ϕ a continuous function on D which vanishes at infinity. The topology β was introduced in [1] where it was shown that a sequence of functions is β convergent if and only if it is σ bounded and κ convergent. The three topologies [4] are related by $\kappa \subset \beta \subset \sigma$. One advantage of the strict topology is that (B, β), in contrast to (B, σ), has a nice dual [4]. Also [3] the polynomials are strictly dense in B whereas the uniform closure of the polynomials is just C, those functions in B which are uniformly continuous on D.

Let both τ_1 and τ_2 be one of the three topologies κ, β or σ. Then $[\tau_1: \tau_2]$ will denote the class of all continuous linear operators from (B, τ_1) into (B, τ_2). Thus $[\sigma: \sigma]$ is the algebra of all norm bounded linear operators from (B, σ) into (B, σ). The algebra $[\beta: \beta]$ was essentially introduced in [2] where it was shown that $[\beta: \beta]$ is a closed subalgebra of $[\sigma: \sigma]$ in the induced norm topology.

3. **Operator algebras.** We have defined nine continuity classes $[\tau_1: \tau_2]$, but it will be seen that only five of them are distinct. We

determine the inclusion relationships between the distinct classes. In particular, we show that each continuity class is an algebra and is contained in $[\sigma: \sigma]$.

Since (B, β) is not metrizable [3], it is an important fact that a subset of (B, β) is closed if and only if it is sequentially closed. A proof of this result for subsets which are also subspaces appears in [6]. A private communication (1966) from P. Hessler to R. C. Buck contained a proof of this result for subspaces which R. C. Buck observed also holds for arbitrary subsets. We include this unpublished proof here. Another proof is in [9]. We will use the result to conclude that a linear operator T is in $[\beta: \tau]$ where τ is κ, β or σ, if for any sequence $\{f_n\}$ converging strictly to zero, the sequence $\{Tf_n\}$ converges τ to zero.

THEOREM 1 (Hessler). *A subset of (B, β) is closed if and only if it is sequentially closed.*

Proof. Let V be a sequentially closed subset of (B, β). Let F be a function which is not in V. We show that F is not in the strict closure of V.

Since V is sequentially closed in (B, β) and $\beta \subset \sigma$, V is uniformly closed. Hence there exists a $\delta > 0$ such that $\|f - F\| > 2\delta$ for all f in V. Let $\{K_j\}$ be a sequence of expanding compact sets whose union is all of D. Let P_n be the statement that if f is in V and $\|f - F\| \leqq (n+1)\delta$, then there exists some integer $j \leqq n$ such that for x in K_j, the maximum of $|f(x) - F(x)| > j\delta$. We show by contradiction that we can find a subsequence of $\{K_j\}$ such that P_n holds for all n.

Statement P_1 holds vacuously for K_1. Assume that we have chosen sets $K_1 = K(j, 1), K(j, 2), \cdots, K(j, n-1)$ and $P_1, P_2, \cdots, P_{n-1}$ all hold. Suppose that there is no compact set $K(j, n)$ for which P_n holds. Then for any compact set K after $K(j, n-1)$ in the sequence $\{K_j\}$, there exists a function f_K in V such that $\|f_K - F\| \leqq (n+1)\delta$ and for x in K_j the maximum of $|f_K(x) - F(x)| \leqq j\delta$ for all $j \leqq n-1$ and $\|f_K - F\|_K \leqq n\delta$. Doing this for each such compact set K, we obtain a sequence of functions $\{f_K\}$ which are uniformly bounded, since for any K, $\|f_K - F\| \leqq (n+1)\delta$. Then this sequence has a subsequence which converges κ to some function g. Since the subsequence is uniformly bounded it also converges strictly to g. Since V is sequentially closed, g is in V. Denote this subsequence by $\{f_k\}$ and denote the corresponding compact sets by $\{K_k\}$. Since $\{f_k\}$ converges κ to g we have $\|g - F\| \leqq (n+1)\ \delta$. Since for each x in K_k we have $|f_k(x) - F(x)| \leqq k\ \delta$ for each $k \leqq n-1$ it follows that for each x in K_k, $|g(x) - F(x)| \leqq k\delta$ for each $k \leqq n-1$.

Now fix a compact set K occurring after $K(j, n-1)$. For any compact set S containing K with S in $\{K_k\}$ we have $\|f_S - F\|_K \leqq \|f_S - F\|_S \leqq n\delta$. Then as k increases, the sets K_k expand to D and the functions $\{f_k\}$ converge to g. Therefore we obtain $\|g - F\|_K \leqq n\delta$. We also know that for each x in K_k, $|g(x) - F(x)| \leqq k\delta$ for each $k \leqq n-1$ and that $\|g - F\| \leqq (n+1)\delta$. This contradicts P_{n-1}.

To complete the proof let ϕ be a continuous function on D which vanishes at infinity such that $\phi = 1/k$ on ∂K_k, the boundary of K_k. Then the maximum of $|f(x) - F(x)|$ for x in ∂K_k is equal to the maximum of $|f(x) - F(x)|$ for x in K_k which is larger than $k\delta$. Hence the maximum of $|[f(x) - F(x)]\phi(x)|$ for x in ∂K_k is larger than δ. Hence $\|(f - F)\phi\| = \|f - F\|_\phi > \delta$ and F is not in the strict closure of V.

THEOREM 2. *The continuity classes* $[\tau_1: \tau_2]$, *for* $\tau_i = \kappa, \beta$ *or* σ, *are subsets of* $[\sigma: \sigma]$.

Proof. Let T be in $[\tau_1: \tau_2]$ and let the sequence $\{f_n\}$ converge uniformly to f. We apply the closed graph theorem in (B, σ) and assume that $\lim_{n\to\infty} Tf_n = g$, i.e. the sequence $\{Tf_n\}$ converges uniformly to the function g in B. Then since $\tau_1 \subseteqq \sigma$, $\{f_n\}$ converges τ_1 to f. Therefore $\{Tf_n\}$ converges τ_2 to Tf and hence pointwise to Tf. It follows that $Tf = g$ and hence that T is in $[\sigma: \sigma]$.

From $\kappa \subset \beta \subset \sigma$ follow some obvious inclusions among the continuity classes. We indicate which of these inclusions are proper and which continuity classes are identical.

THEOREM 3. *The following identities hold for the continuity classes.*

(i) $[\sigma: \kappa] = [\sigma: \beta] = [\sigma: \sigma]$

(ii) $[\beta: \kappa] = [\beta: \beta]$

(iii) $[\kappa: \sigma] = [\kappa: \beta]$.

Proof. For the first equality we know from the last theorem that $[\sigma: \kappa] \subseteqq [\sigma: \sigma]$. Since $\kappa \subset \sigma$ we have $[\sigma: \sigma] \subseteqq [\sigma: \kappa]$. Therefore $[\sigma: \kappa] = [\sigma: \sigma]$. Also, since $\kappa \subset \beta$ we have $[\sigma: \kappa] \subseteqq [\sigma: \beta] \subseteqq [\sigma: \sigma]$. Since $[\sigma: \kappa] = [\sigma: \sigma]$ we are done.

For the second part let T be in $[\beta: \kappa]$ and let $\{f_n\}$ converge strictly to zero. Then $\{f_n\}$ is κ convergent and uniformly bounded. Since T is in $[\beta: \kappa]$, $\{Tf_n\}$ is κ convergent. Also, since $[\beta: \kappa] \subseteqq [\sigma: \sigma]$, $\{Tf_n\}$ is uniformly bounded. Therefore $\{Tf_n\}$ converges strictly to zero.

Now let T be in $[\kappa: \beta]$ and let $\{f_n\}$ be a sequence converging κ to zero. Then it is known [6, pp 383] that there exists a sequence

$\{c_n\}$ converging monotonically to infinity such that the sequence $\{c_n f_n\}$ converges κ to zero. Thus $\{T(c_n f_n)\}$ converges strictly to zero. Since strictly convergent sequences are uniformly bounded, there exists a constant M such that $\| T(c_n f_n) \| = c_n \| T(f_n) \| \leqq M$. Hence $\{Tf_n\}$ converges uniformly to zero and T is in $[\kappa: \sigma]$. Since $\beta \subset \sigma$, we have $[\kappa: \sigma] \subseteqq [\kappa: \beta]$ and the result follows.

COROLLARY. *The continuity classes* $[\tau_1: \tau_2]$ *are algebras.*

Proof. The last theorem shows that the only possible distinct continuity classes $[\tau_1: \tau_2]$ satisfy $\tau_1 \supseteqq \tau_2$. The corollary follows immediately.

THEOREM 4. *Among the operator algebras* $[\tau_1: \tau_2]$, *the only distinct ones are* $[\kappa: \sigma]$, $[\kappa: \kappa]$, $[\beta: \sigma]$, $[\beta: \beta]$ *and* $[\sigma: \sigma]$, *and all the proper inclusions between them are given by* $[\kappa: \sigma] \subset [\beta: \sigma] \subset [\beta: \beta] \subset [\sigma: \sigma]$ *and* $[\kappa: \sigma] \subset [\kappa: \kappa] \subset [\beta: \beta]$.

Here is a simple diagram of the situation:

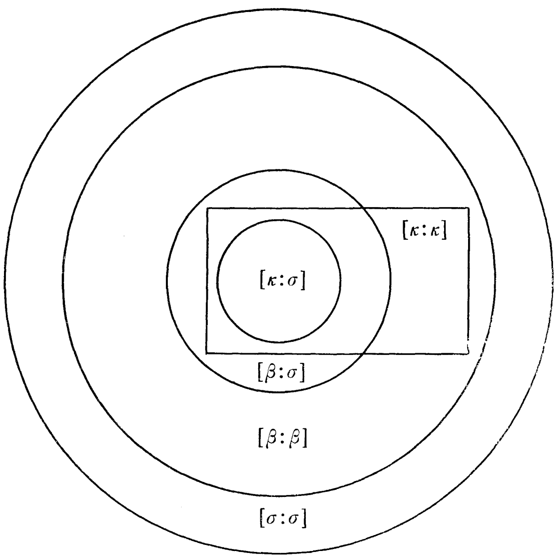

FIGURE 1

Proof. It has been shown that these are the only possible distinct classes. We also know that $[\beta: \sigma] \subseteqq [\beta: \beta]$ and $[\kappa: \sigma] \subseteqq [\kappa: \kappa]$. Consideration of the identity operator shows that these inclusions are proper.

To show that $[\kappa: \kappa]$ is a subset of $[\beta: \beta]$, let T be a linear operator in $[\kappa: \kappa]$ and let $\{f_n\}$ be a sequence converging strictly to zero. Then $\{f_n\}$ converges κ to zero and $\{Tf_n\}$ converges κ to zero. Also $\{f_n\}$ is uniformly bounded and since $[\kappa: \kappa] \subseteqq [\sigma: \sigma]$, $\{Tf_n\}$ is uniformly bounded. Hence $\{Tf_n\}$ converges strictly to zero.

We show that $[\kappa: \kappa]$ is a proper subalgebra of $[\beta: \beta]$ by giving an example of an operator which is in $[\beta: \sigma] \subseteqq [\beta: \beta]$ but not in $[\kappa: \kappa]$. This also shows that $[\beta: \sigma]$ is not contained in $[\kappa: \kappa]$. Let T be the linear operator from B to the constant functions defined by $Tf(z) = \int_0^1 f(x)dx$. Then the sequence $\{f_n\}$, where $f_n(z) = nz^n$, converges κ to zero but $\{Tf_n\}$ converges to 1. Hence T is not in $[\kappa: \kappa]$. Now assume that $\{f_n\}$ converges strictly to zero. Then $\{f_n\}$ converges κ to zero and $\|f_n\| \leqq M$ for some constant M. Let r be a real number satisfying $0 < r < 1$. Let $\|f\|_r = \sup \{|f(z)|: |z| \leqq r\}$. Then

$$|Tf_n(z)| \leqq \left|\int_0^r f_n(x)dx\right| + \left|\int_r^1 f_n(x)dx\right| .$$

Hence $|Tf_n(z)| \leqq r\|f_n\|_r + M(1-r)$. Therefore $\|Tf_n\|$ can be made arbitrarily small by choosing r near 1 and n large.

The bounded linear operator T defined on (C, σ) by $Tf(z) = f(1)$ maps C into the constant functions. By using the Hahn Banach theorem on the associated linear functional given by $Lf = f(1)$, we can extend T to a bounded linear operator on (B, σ). But this operator T is not in $[\beta: \beta]$. Although the sequence $\{z^n\}$ converges strictly to zero, $T(z^n) = 1$ for all n.

Finally we show that $[\kappa: \sigma] \subset [\beta: \sigma]$. Since $\kappa \subset \beta$, we have $[\kappa: \sigma] \subseteqq [\beta: \sigma]$. Let T be the operator defined on B, by $T(\sum a_n z^n) = \sum (a_n/n)z^n$ for any function $\sum a_n z^n$ in B. It follows immediately from Theorem 8 that T is not in $[\kappa: \sigma]$. The corollary to Theorem 7 implies that if T maps B into C, then T will be in $[\beta: \sigma]$. Since $(\sum |a_n|/n)^2 \leqq \sum |a_n|^2 \sum (1/n)^2$, it follows that $\sum a_n(z^n/n)$ is in C.

4. Multipliers and diagonal operators. Assume that T is a linear operator defined on B for which there exists a sequence $\{c_n\}$ such that $T(z^n) = c_n z^n$, for $n = 0, 1 \cdots$. If T is also in $[\kappa: \kappa]$, then for any function $f(z) = \sum a_n z^n$ in B, we have $T(\sum a_n z^n) = \sum a_n c_n z^n$. This holds because the partial sums of $\sum a_n z^n$ converge κ to f. Operators satisfying $T(\sum a_n z^n) = \sum a_n c_n z^n$ for some sequence $\{c_n\}$ are called multipliers. The multipliers from H^p to H^q were first studied by Hardy and Littlewood, (for references and some recent results see [5]). Further results on multipliers can be found in [3] and [7]. Explicitly we have:

DEFINITION. A multiplier on B is a linear operator T such that there exists a sequence $\{c_n\}$ with the property that $T(\sum a_n z^n) = \sum a_n c_n z^n$ for any function $\sum a_n z^n$ in B.

We distinguish another class of closely related operators which leave invariant the one dimensional subspaces generated by z^k. These operators shed light on the relationship between topology and the multipliers.

DEFINITION. A linear operator T defined on B is called diagonal if there exists a sequence $\{c_n\}$ such that $T(z^n) = c_n z^n$, for $n = 0, 1, \cdots$.

Clearly any multiplier is a diagonal operator. At the beginning of this section we showed that any diagonal operator which is also in $[\kappa: \kappa]$ is a multiplier. It is not known whether there exists a diagonal operator mapping B into B which is not a multiplier. The action of any diagonal operator is determined by its action on the polynomials, which are κ dense in B, strictly dense in B, but not uniformly dense in B. Hence it is reasonable to conjecture, although we can not prove it, that the diagonal operators lie in $[\beta: \beta]$. We can show that any diagonal operator in $[\beta: \beta]$ is a multiplier.

THEOREM 5. *Let T be a diagonal operator mapping B into B and assume also that T is in $[\beta: \beta]$. Then T is a multiplier from B into B.*

Proof. Since T is diagonal there is a sequence $\{c_n\}$ such that $T(z^n) = c_n z^n$. We have to show that $T(\sum a_n z^n) = \sum a_n c_n z^n$ for any function $f(z) = \sum a_n z^n$ in B. For any real number r with $0 < r < 1$, let $f_r(z) = f(rz)$. We first prove the theorem for the function f_r. Since T is in $[\sigma: \sigma]$ and the partial sums of the power series for f_r converge uniformly to f_r, it follows that as N approaches infinity, $\sum_{n=0}^{N} a_n r^n T(z^n)$ converges uniformly to Tf_r. Fix x in D and put $T(z^n) = u_n(z)$. Then

$$\left| \sum_{n=N}^{\infty} a_n r^n u_n(x) \right| \leqq \sum_{n=N}^{\infty} |a_n| \, r^n \, |u_n(x)| \leqq ||f|| \, ||T|| \sum_{n=N}^{\infty} r^n ,$$

since $|a_n| \leqq ||f||$. Hence as N approaches infinity, $\sum_{n=0}^{N} a_n r^n \, T(z^n)$ converges pointwise to $\sum_{n=0}^{\infty} a_n r^n \, T(z^n)$. Thus $Tf_r(z) = \sum a_n r^n \, T(z^n)$.

Now if $f(z) = \sum a_n z^n$ is in B, the Cauchy integral formula shows that f_r converges κ to f as r approaches 1 through a sequence of values. Hence for any x in D, we have

$$Tf(x) = \lim_{r \uparrow 1} Tf_r(x) = \lim_{r \uparrow 1} \sum a_n r^n c_n x^n = \sum a_n c_n x^n$$

since the function $\sum a_n c_n x^n w^n$ is analytic in $|w| < 1/|x|$.

5. Multipliers and continuity. In this section we first characterize the multipliers from B into B and then determine those multipliers which lie in the various continuity classes.

The following characterization of the multipliers from B into C, which occurs in [7], suggests how to characterize the multipliers from B into B.

THEOREM PSW [7]. *Let T be a multiplier from B into C, where $T(\sum a_n z^n) = \sum a_n c_n z^n$ for any function $\sum a_n z^n$ in B. Then the sequence $\{c_n\}$ is one side of the sequence of Fourier coefficients of a function in $L^1(\Gamma)$. Conversely, given any such sequence $\{c_n\}$, the operator T defined on B by $T(\sum a_n z^n) = \sum a_n c_n z^n$ is a multiplier from B into C.*

Both Theorem PSW and the next theorem on the multipliers from B into B can be considered as converse forms of Hadamard multiplication theorems. Let $\{c_n\}$ be the sequence associated with a multiplier T. Let $h(z) = \sum c_n z^n$. Then for any function $f(z) = \sum a_n z^n$ in B, we have $Tf(z) = \sum a_n c_n z^n = (h * f)(z)$, the Hadamard product of h and f. We thus solve the problem of determining those functions h such that $h * f$ is in B for every f in B.

A first step in the direction of characterizing the multipliers in various continuity algebras was taken in [3]. Let (C, σ) be C in the topology σ. Given a strictly continuous linear functional L on B, define a linear operator T on B by $T(\sum a_n z^n) = \sum a_n L(z^n) z^n$. Then it was shown that T is a continuous linear operator from (B, β) into (C, σ). Letting $h(z) = \sum L(z^n) z^n$, the result states that $(h * f)(z)$ is in C for any function f in B. Theorem PSW provides a converse.

COROLLARY. *Let $h(z) = \sum c_n z^n$. Assume that $(h * f)(z)$ is in C for every function f in B. Then $c_n = L(z^n)$ for some strictly continuous linear functional L.*

Proof. Since $h * f$ is in C for any function f in B, it follows that [7] the sequence $\{c_n\}$ is one side of the sequence of Fourier coefficients of a function in $L^1(\Gamma)$ and thus that the linear functional L defined on z^n by $L(z^n) = c_n$ can be extended to a strictly continuous functional on all of B.

We know that any diagonal operator in $[\kappa : \kappa]$ is a multiplier from B into B. In fact these are all such multipliers.

THEOREM 6. *Let T be a multiplier from B into B given by $T(\sum a_n z^n) = \sum a_n c_n z^n$. Then there exists an L in the dual of (C, σ)*

such that $c_n = L(z^n)$, *for* $n = 0, 1, \cdots$. *Conversely, given any such* L, *let* T *be defined by* $T(\sum a_n z^n) = \sum a_n L(z^n) z^n$. *Then* T *is in* $[\kappa: \kappa]$. *Furthermore* $||L|| = ||T||$.

Proof. Let L be in the dual of (C, σ) with $L(z^n) = c_n$ and define T as above. Then for fixed x in D, $Tf(x) = \sum a_n L(z^n) x^n = L(\sum a_n x^n z^n)$ because the partial sums of $\sum a_n x^n z^n$ converge uniformly for $|z| < 1$. Let U_x be the operator given by $U_x f(z) = f(xz)$. Then given any r with $0 < r < 1$ and an x in D with $|x| \leqq r$, we have $|Tf(x)| \leqq ||L||$ $||U_x f(z)|| \leqq ||L|| \, ||f||_r$. Hence $||Tf||_r \leqq ||L|| \, ||f||_r$ and $||T|| \leqq ||L||$ and T is in $[\kappa: \kappa]$.

Now let T satisfy the conditions of the theorem and define $h(z) = \sum c_n z^n$. We show that $h(z)$ is analytic in D. Since the function $f(z) = \sum n^{-2} z^n$ is in B, it follows that the function $Tf(z) = \sum c_n n^{-2} z^n$ is in B. Hence $\limsup |c_n|^{1/n} = \limsup |n^{-2} c_n|^{1/n} \leqq 1$.

For any r with $0 < r < 1$, define the linear functional L_r on C by $L_r f = Tf(r)$. Then for $f(z) = \sum a_n z^n$ in B, we have

$$|L_r(f)| = |\sum a_n c_n r^n| \leqq ||f|| \sum |c_n| r^n < \infty$$

because $\sum c_n z^n$ is analytic in D. Hence L_r is in the dual of (C, σ). Now for fixed f in C, $L_r f$ is bounded in norm for all $0 < r < 1$ because Tf is in B. By the uniform boundedness principle, there is an M such that $||L_r|| \leqq M$ for all $0 < r < 1$. By the weak star compactness of the unit ball of the dual of (C, σ) there exists an L in the dual of (C, σ) such that $L_r f$ converges to Lf for every f in C. Letting $f(z) = z^n$ we obtain $L_r(z^n) = r^n c_n$ converging to $L(z^n)$ and to c_n. Hence $c_n = L(z^n)$.

Following the procedure used in the first part of the proof this L now yields an operator T in $[\kappa: \kappa]$. In fact this is equal to the operator which gave L because it agrees with the original operator on the polynomials. From $L_r(f) = Tf(r)$ it follows that $||L_r|| \leqq ||T||$ and hence $||L|| \leqq ||T||$.

COROLLARY. *All the diagonal operators in* $[\beta: \beta]$ *are in* $[\kappa: \kappa]$.

Any uniformly continuous linear functional on C can be extended by the Hahn Banach theorem to a continuous linear functional on the space of all continuous functions on Γ. Corresponding to this functional there is a Radon measure on Γ. Hence the sequences associated with the multipliers from B into B correspond to the Radon measures on Γ. Since $[\beta: \sigma]$ is a subalgebra of $[\kappa: \kappa]$, the multipliers in $[\beta: \sigma]$ will correspond to some subset of the Radon measures.

THEOREM 7. *Let T be a multiplier in $[\beta:\sigma]$ given by $T(\sum a_n z^n) = \sum a_n c_n z^n$. Then $\{c_n\}$ is one side of the sequence of Fourier coefficients of a function in $L^1(\Gamma)$. Conversely given any such sequence $\{c_n\}$, the operator T defined by $T(\sum a_n z^n) = \sum a_n c_n z^n$ is a multiplier in $[\beta:\sigma]$.*

Proof. Let T be a multiplier in $[\beta:\sigma]$. For any function f in B, $\{f_r\}$ converges κ to f and $\|f_r\| \leqq \|f\|$. Hence $\{f_r\}$ converges strictly to f. Therefore if $f(z) = \sum a_n z^n$, $Tf_r(z) = T(\sum a_n r^n z^n) = \sum a_n c_n r^n z^n$ is in C because $\sum a_n c_n z^n$ is analytic in D. Then $\{Tf_r\}$ converges uniformly to Tf which implies that Tf is in C. The result follows from Theorem PSW.

Now let $\{c_n\}$ be one side of the sequence of Fourier coefficients of an L^1 function. Letting $L(z^n) = c_n$, we can ([7] and [10]) extend this to a strictly continuous linear functional on B. As mentioned in the second paragraph after Theorem PSW, the operator T defined by $Tf(z) = \sum a_n L(z^n) z^n$ is a continuous linear operator from (B, β) into (C, σ).

COROLLARY. *The operator T is a multiplier from B into C if and only if T is a multiplier in $[\beta:\sigma]$.*

The last class to consider is $[\kappa:\sigma]$.

THEOREM 8. *Let T be a multiplier in $[\kappa:\sigma]$ given by $T(\sum a_n z^n) = \sum a_n c_n z^n$. Then $\limsup |c_n|^{1/n} < 1$. Conversely any such sequence $\{c_n\}$ defines a multiplier T in $[\kappa:\sigma]$ given by $T(\sum a_n z^n) = \sum a_n c_n z^n$.*

Proof. If T is in $[\kappa:\sigma]$, then there exists an r with $0 < r < 1$ and a constant M such that $\|Tf\| \leqq M\|f\|_r$ for all f in B. Letting $f(z) = z^n$, we obtain $\limsup |c_n|^{1/n} \leqq r$.

Assume now that $\limsup |c_n|^{1/n} = c < 1$. Choose an r such that $c < r < 1$. Then for $f(z) = \sum a_n z^n$ in B, $Tf(z) = \sum a_n c_n z^n$ which is in C. Then $|Tf(z)| \leqq \|f\|_r \sum |c_n| r^{-n} \leqq M\|f\|$ for some constant M and T is in $[\kappa:\sigma]$.

We have shown that the multipliers in $[\kappa:\kappa]$ are the multipliers from B into B and that the multipliers in $[\beta:\sigma]$ are the multipliers from B into C. Let $H(D)$ be the functions analytic in D and let $H(\bar{D})$ be those which are analytic in some open disc containing D.

COROLLARY. *The operator T is a multiplier in $[\kappa:\sigma]$ if and only if it is a multiplier from $H(D)$ into $H(\bar{D})$.*

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