

Pacific Journal of Mathematics

HOW TO RECOGNIZE HOMEOMORPHISMS AND ISOMETRIES

CARLOS R. BORGES

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We study necessary and sufficient conditions for a given function $f; X \rightarrow X$ to be one-to-one, or a homeomorphism or a topological isometry¹, by analyzing the family $\{f^1, f^2, \dots\}$ of its composition iterates.

In § 1, 2 and 3 we develop the key results. (See Theorems 2.1 and 2.3).

In § 4 we obtain a variety of applications, most of which are known, by various different techniques. They all follow easily from our key results:

(a) Conditions which imply that locally nonexpansive maps (in metric spaces or uniform spaces) become local isometries (see Theorem 4.2, Corollary 4.3 and Theorem 4.5, and also [2], [5] and [6]).

(b) Conditions which imply that a given map on a *metrizable* space is a *topological* isometry¹ (see Theorems 4.6 and 4.7, and also [8]).

(c) Some results on compact commutative groups of homeomorphisms (or topological isometries) of X onto X (see Theorems 4.11 and 4.12, and also [4], [10] and [11]).

1. Auxiliary results. For each space X we let id_X denote the identity function on X and, for each continuous function $f: X \rightarrow X$, we let

(a) $f^1 = f, f^n = f \circ f^{n-1}$ for $n > 1$

(b) $P = \bigcap_{n=1}^{\infty} f^n(X)$

(c) $\Gamma(f) = \{f^n | n = 1, 2, \dots\}^-$, the closure being with respect to the co. (i.e. compact-open) topology on the function space X^X .

PROPOSITION 1.1. *If X is locally compact and Hausdorff then X^X (and therefore also $\Gamma(f)$) is a topological semigroup with respect to composition.*

Proof. Immediate from Theorem 2.2 (p. 259) of Dugundji [3].

LEMMA 1.2. *Suppose X is compact Hausdorff and $\{f^1, f^2, \dots\}$ is*

¹ We wish to thank Professors J. Dugundji, W. Comfort and A. D. Wallace for bringing to our attention references [6], [2] and [10] respectively. They strongly motivated us to sharp improvements of our original results.

evenly continuous. Then $\Gamma(f)$ is compact.

Proof. By Theorem 19 (p. 235) of [7], $\Gamma(f)$ is evenly continuous since $\Gamma(f)$ is contained in the pointwise (topology) closure of $\{f^1, f^2, \dots\}$. By Ascoli's Theorem (p. 236 of [7]), $\Gamma(f)$ is compact.

LEMMA 1.3 Suppose X is compact Hausdorff and $\{f^1, f^2, \dots\}$ is evenly continuous. Then $\Gamma(f)$ has an idempotent e (i.e. $e^2 = e$) such that $e(X) = P$ and $e|_P = id_P$.

Proof. By our Lemma 1.2 and Theorem 3.1 of [9], $\Gamma(f)$ has an idempotent e (the unit of the maximal nonempty subgroup $K(f)$ of $\Gamma(f)$).

We show $P \subset e(X)$: Let $y \in P$. Then there exists $x_n \in X$ such that $f^n(x_n) = y$, for each n . Pick a subnet $\{(f^\nu, x_\nu)\}$ of $\{(f^n, x_n)\}$ such that $\lim_\nu f^\nu = e$. Then there exists $w \in X$ and a subnet $\{(f^\lambda, x_\lambda)\}$ of the net $\{(f^\nu, x_\nu)\}$ such that $\lim_\lambda (f^\lambda, x_\lambda) = (e, w)$, because $\Gamma(f) \times X$ is compact. Consequently $e(w) = \lim_\lambda f^\lambda(x_\lambda) = \lim_\lambda y = y$, because of Theorems 4.71 and 2.5 of [1].

We show $e(X) \subset P$: Let $x \in X$ and pick subnet $\{f^\nu\}$ of $\{f^n\}$ such that $\lim_\nu f^\nu = e$. Then $e(x) = \lim_\nu f^\nu(x)$ and, since each $f^n(X)$ is a closed subset of X (because X is compact), $e(x) \in P$.

Clearly $e|_P = id_P$ since $e(e(x)) = e(x)$ for each $x \in X$.

It appears that our main results become more meaningful when stated in terms of a generalization of the concept of "an almost periodic² function" introduced by Whyburn [12], especially because of Lemma 1.5 below.

DEFINITION 1.4. Let $f: X \rightarrow X$ be a function. Then f is said to be *net periodic* if there exists a subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = id_X$ (in X^X).

Clearly net periodicity is equivalent to almost periodicity whenever X^X is metrizable.

LEMMA 1.5. Let $f: X \rightarrow X$ be net periodic. Then f is one-to-one.

Proof. Suppose $f(x) = f(y)$. Then $f^n(x) = f^n(y)$ for all n . Pick subnet $\{f^\nu\}$ of $\{f^n\}$ such that $\lim_\nu f^\nu = id_X$. Then $id_X(x) = \lim_\nu f^\nu(x) = \lim_\nu f^\nu(y) = id_X(y)$ implies that $x = y$. This does the trick.

² For any metric space (X, ρ) a function $f: X \rightarrow X$ is said to be *almost periodic* (see [12]) if for each $\varepsilon > 0$ there exists a positive integer n such that $\rho(f^n(x), x) < \varepsilon$ for all $x \in X$. This is clearly equivalent to saying that there exists a subsequence $\{f^{n_i}\}$ of $\{f^n\}$ which converges uniformly to the identity function of X .

LEMMA 1.6. *Let X be compact Hausdorff and $f: X \rightarrow X$ a continuous function. If f is net periodic then f is an onto homeomorphism.*

Proof. Because of Lemma 1.5 we only need prove that f is onto: Suppose not. Then $X - f(X)$ is an open nonempty subset of X . Let $\{f^\nu\}$ be a subnet of $\{f^n\}$ such that $\lim f^\nu = id_X$. Then for any $a \in X - f(X)$, $\lim_\nu f^\nu(a) = a$ while $f^\nu(X) \subset f(X)$ (i.e. $f^\nu(X) \cap [X - f(X)] = \emptyset$) for all ν . This is a contradiction.

2. Main results.

THEOREM 2.1 *Let X be compact Hausdorff and $f: X \rightarrow X$ a function. If $\{f^1, f^2, \dots\}$ is evenly continuous then $f|P$ is net periodic (and therefore a homeomorphism).*

Proof. By Lemma 1.3, there exists an idempotent $e \in \Gamma(f)$ such that $e|P = id_P$. Consequently there exists a subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = e$ (even if $e = f^n$ for some n) which implies that $\lim_\nu f^\nu|P = e|P = id_P$. Since $f(P) = P$ (because $f^n(X) \supset f^{n+1}(X) \supset P$ for all n) we get that $f^\nu|P = (f|P)^\nu$. Therefore $\lim_\nu (f|P)^\nu = id_P$ and $f|P$ is net periodic.

COROLLARY 2.2. *Let X be compact Hausdorff and $f: X \rightarrow X$ an onto function. If $\{f^1, f^2, \dots\}$ is evenly continuous then f is net periodic (and therefore a homeomorphism).*

Since the converse of Theorem 2.1 is false (for example, let $f: I \rightarrow I$ be defined by $f(t) = t^2$, where I denotes the closed unit interval), the following seems worthy of mention.

THEOREM 2.3. *Let X be any space and $f: X \rightarrow Y$ a continuous function. Then f is one-to-one if and only if $\Gamma(f)$ contains a one-to-one function.*

Proof. The “only if” part is clear. To prove the “if” part let $e: X \rightarrow X$ be any one-to-one function such that $e \in L(f)$. Assume $f(x) = f(y)$. Then $f^n(x) = f^n(y)$ for all n and one easily sees that $e(x) = e(y)$. (This is clear if $e = f^n$ for some n . Otherwise there exists a subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = e$. Therefore $e(x) = \lim_\nu f^\nu(x) = \lim_\nu f^\nu(y) = e(y)$). This shows that f is one-to-one.)

3. Generalizations. With some additional restrictions on f ,

Theorem 2.1 remains valid for a larger class of spaces.

Throughout this section we let E be a *noncompact, locally compact, space*, and $\hat{E} = E \cup \{\infty\}$ be its one-point compactification. Also for each onto map $f: E \rightarrow E$ we consider the three conditions:

- (A) For each sequence $\{x_n\}$ in E , $\lim_n x_n = x \neq \infty$ and $\lim_n f^{i_n}(x) = \infty$ implies that $\lim_n f^{i_n}(x_n) = \infty$,
- (B) For each sequence $\{x_n\}$ in E , $\lim_n x_n = \infty$ implies that $\lim_n f^{i_n}(x_n) = \infty$,
- (C) For each sequence $\{x_n\}$ in E , $\lim_n x_n = \infty$ implies that $\lim_n f(x_n) = \infty$.

In Example 5.1, it is shown that conditions (A) and (C) do not imply (B).

LEMMA 3.1. *Condition (C) is satisfied if f is either (a) uniformly continuous with respect to some compatible metric on \hat{E} (assuming E is separable metrizable), or (b) a homeomorphism.*

Proof. (a) Since \hat{E} is completely metrizable we can continuously extend f to $\hat{f}: \hat{E} \rightarrow \hat{E}$. Since E is not compact, $\hat{f}(\infty) = \infty$. This shows that f satisfies (C).

(b) Suppose $\lim_n x_n = \infty$ but $\lim_n f(x_n) \neq \infty$. Then there exists a subsequence $\{x_\sigma\}$ of $\{x_n\}$ such that $\lim_\sigma x_\sigma = \infty$ $\lim_\sigma f(x_\sigma) = y$ for some $y \in E$. Then $f^{-1}(y) = \lim_\sigma f^{-1} f(x_\sigma) = \lim_\sigma x_\sigma = \infty$, which is impossible.

LEMMA 3.2. *Condition (B) implies (C). (The converse is false—see Example 5.1.)*

Proof. Simply let all $i_n = 1$ in (B).

LEMMA 3.3. *If E is metrizable and $f: E \rightarrow E$ is nonexpansive (with respect to some compatible metric d on E) then f satisfies conditions (A) and (C).*

Proof. (That f satisfies (A) is essentially proved in the proof of Theorem 1.1 of [8]. We present a simpler argument.) Suppose that $\lim_n x_n = x \in E$, $\lim_n f^{i_n}(x) = \infty$ but $\lim_n f^{i_n}(x_n) \neq \infty$. Then there exists compact $C \subset E$, $z \in C$ and subsequence $\{\beta\}$ of the integers such that $f^{i_\beta}(x_\beta) \in C$ and $\lim_\beta f^{i_\beta}(x_\beta) = z$. Since $d(f^{i_\beta}(x_\beta), f^{i_\beta}(x)) \leq d(x_\beta, x)$, we get that $\lim_\beta f^{i_\beta}(x_\beta) = z$, a contradiction. Similarly, one can easily see that f satisfies (C).

THEOREM 3.4. *Let $f: E \rightarrow E$ be an onto function such that*

$\{f^1, f^2, \dots\}$ is evenly continuous and f satisfies conditions (A) and (B). Then f extends continuously to an onto function $\hat{f}: \hat{E} \rightarrow \hat{E}$ such that $\{f^1, f^2, \dots\}$ is evenly continuous. (Therefore, f and \hat{f} are net periodic.)

Proof. Since f satisfies (C), by Lemma 3.2, we can extend f to $\hat{f}: \hat{E} \rightarrow \hat{E}$ continuously, by letting $\hat{f}(\infty) = \infty$. Because of (A) and (B) it is easily seen that $\{\hat{f}^1, \hat{f}^2, \dots\}$ is evenly continuous. Therefore, by Theorem 2.1, \hat{f} is net periodic. This completes the proof.

Example 5.1 shows that if f does not satisfy (B) then f is not necessarily net periodic.

4. Applications.

DEFINITION 4.1. Let (X, ρ) be a metric space. A function $f: X \rightarrow X$ is said to be *locally nonexpansive* if each $p \in X$ has a neighborhood N_p such that $\rho(f(x), f(y)) \leq \rho(x, y)$ for all $x, y \in N_p$. If $N_\infty = X$ then f is said to be *nonexpansive*. If, for some $\varepsilon > 0$, $\rho(x, y) < \varepsilon$ implies that $\rho(f(x), f(y)) \leq \rho(x, y)$, then f is said to be ε -*nonexpansive*. Replacing " \leq " by " $=$ " above, we get the definitions of *local isometry* and of ε -*isometry*.

THEOREM 4.2. Let (X, ρ) be a compact metric space and $f: X \rightarrow X$ a locally nonexpansive onto function. Then f is almost periodic² and therefore a local isometry.

Proof. By compactness of X , there exists $\varepsilon > 0$ such that f is ε -nonexpansive. By the net characterization of even continuity (cf. exercise L on p. 241 of [7]) it is easy to prove that $\{f^1, f^2, \dots\}$ is evenly continuous (for each net (in this case, sequences suffice)

$$\{(f^\nu, x_\nu)\} \subset \{f^1, f^2, \dots\} \times X, \lim_\nu x_\nu = x \quad \text{and} \quad \lim_\nu f^\nu(x) = y$$

implies that

$$\rho(f^\nu(x_\nu), y) \leq \rho(f^\nu(x_\nu), f^\nu(x)) + \rho(f^\nu(x), y) \leq \rho(x_\nu, x) + \rho(f^\nu(x), y)$$

for all ν with $\rho(x_\nu, x) < \varepsilon$, since each f^n is also ε -nonexpansive; therefore $\lim_\nu f^\nu(x_\nu) = y$, which does the trick).

Since f is onto, we get that $P = X$ and therefore that f is almost periodic, by Theorem 2.1 (and footnote 2).

It is now easily seen that f is a local isometry (pick subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = id_X$. Since

$$\rho(x, y) \leq \rho(x, f^\nu(x)) + \rho(f^\nu(x), f^\nu(y)) + \rho(f^\nu(y), y)$$

we get that $\rho(x, y) \leq \lim_\nu \rho(f^\nu(x), f^\nu(y))$ for all $x, y \in X$. If

$$\rho(f(a), f^\nu(b)) < \rho(a, b) < \varepsilon,$$

for some $a, b \in X$, then

$$\rho(f^\nu(a), f^\nu(b)) \leq \rho(f(a), f(b)) < \rho(a, b), \quad \text{for all } \nu,$$

which implies that $\lim_\nu \rho(f^\nu(a), f^\nu(b)) < \rho(a, b)$, a contradiction. This completes the proof.

COROLLARY 4.3 (Satz 1b of [6] and Theorem 1 of [5]). *Let (X, ρ) be a totally bounded metric space and $f: X \rightarrow X$ an ε -nonexpansive map such that $f(X)$ is dense in X . Then f is an ε -isometry and X is dense in $f(X)$.*

Proof. Let \hat{X} be the usual metric completion (in terms of Cauchy sequences—see Theorem 27 on p. 196 of [7]). Since $f(X)$ is dense in X then \hat{X} is also the completion of $f(X)$. Clearly \hat{X} is compact metrizable by a metric $\hat{\rho}$ which extends ρ , and f is extendable to a continuous function (because f is uniformly continuous) $\hat{f}: \hat{X} \rightarrow \hat{X}$ such that \hat{f} is an onto map (because of compactness). It is also easily seen that \hat{f} is ε -nonexpansive (Let $x, y \in \hat{X}$ with $\hat{\rho}(x, y) < \varepsilon$. Pick nets (sequences suffice) $\{x_\nu\}, \{y_\nu\}$ in X with $\lim_\nu x_\nu = x$, $\lim_\nu y_\nu = y$ and $\rho(x_\nu, y_\nu) < \varepsilon$ for all ν (this can be done because of the definition of $\hat{\rho}$). Then $\hat{\rho}(\hat{f}(x), \hat{f}(y)) = \lim_\nu \rho(f(x_\nu), f(y_\nu)) \leq \lim_\nu \rho(x_\nu, y_\nu) = \hat{\rho}(x, y)$, because $\rho(x, y) < \varepsilon$). Theorem 4.2 completes the proof.

DEFINITION 4.4. Let (X, \mathcal{U}) be a uniform space and $\theta = \{\rho_\lambda\}_{\lambda \in \Lambda}$ a gage (see p. 188 of [7]) for \mathcal{U} . A function $f: X \rightarrow X$ is said to be ε -nonexpansive with respect to θ ($\varepsilon < 0$) if $\rho_\lambda(x, y) < \varepsilon$ implies that $\rho_\lambda(f(x), f(y)) \leq \rho_\lambda(x, y)$, for each $\lambda \in \Lambda$. If f is ε -nonexpansive with respect to θ for all $\varepsilon > 0$ then f is said to be nonexpansive with respect to θ . Replacing “<” by “=” above we get the definitions of ε -isometry and of isometry with respect to θ .

THEOREM 4.5. *Let (X, \mathcal{U}) be a uniform compact Hausdorff space and $f: X \rightarrow X$ an onto ε -nonexpansive map with respect to some gage $\theta = \{\rho_\lambda\}_{\lambda \in \Lambda}$ for \mathcal{U} . Then f is net periodic and an ε -isometry with respect to θ .*

Proof. All the statements in the proof of Theorem 4.2 which involve ρ remain valid when ρ is replaced by any $\rho_\lambda \in \theta$. Therefore,

by use of Theorem 19 on p.189 of [7], one immediately gets that $\{f^1, f^2, \dots\}$ is evenly continuous, and f is net periodic and an ε -isometry with respect to θ . This completes the proof.

Brown and Comfort [2] have obtained a similar result for totally bounded Hausdorff uniform spaces.

With respect to topological isometries we get the following results, of which the first shows that the hypothesis that f be a homeomorphism in Corollary 1.2 of [8] is superfluous; the second complements Theorem 1.1 of [8].

THEOREM 4.6. *Let $f: M \rightarrow M$ be a function from the compact metrizable space M onto itself. Then f is a topological isometry if and only if $\{f^1, f^2, \dots\}$ is evenly continuous.*

Proof. Immediate from Corollary 1.2 of [8] and our Theorem 2.1.

THEOREM 4.7. *Let $f: E \rightarrow E$ be a function from the separable locally compact metrizable space E onto itself. Then f is the restriction of a topological isometry $\hat{f}: \hat{E} \rightarrow \hat{E}$ if and only if $\{f^1, f^2, \dots\}$ is evenly continuous and f satisfies conditions (A) and (B).*

Proof. Immediate from Theorems 4.6 and 3.4.

Finally we get some results on transformation groups. But first we need the following three lemmas.

LEMMA 4.8. *Let $f: X \rightarrow X$ be a function from a compact Hausdorff space X onto itself. If $\{f^1, f^2, \dots\}$ is evenly continuous then $\{f^{-1}, f^{-2}, \dots\} \subset \Gamma(f)$ and it is evenly continuous.*

Proof. By Corollary 2.2, f^{-1} is a well-defined function and there exists subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = 1_X$. Therefore

$$\lim_\nu f^{-1} f^\nu = \lim_\nu f^{\nu-1} = f^{-1} 1_X = f^{-1} \in \Gamma(f).$$

Similarly one proves that $\{f^{-1}, f^{-2}, \dots\} \subset \Gamma(f)$. Therefore $\{f^{-1}, f^{-2}, \dots\}$ is evenly continuous, since $\Gamma(f)$ is compact by Lemma 1.2. This completes the proof.

LEMMA 4.9. *Let $f: X \rightarrow X$ be a function from a compact Hausdorff space X onto itself. If $\{f^1, f^2, \dots\}$ is evenly continuous then, $\lim_\nu f^\nu(x) = y$ if and only if $\lim_\nu f^{-\nu}(y) = x$, for any given net $\{f^\nu\}$.*

Proof. (Essentially contained in the proof of Corollary 1.3 of [8].)

Let $\lim_{\nu} f^{\nu}(x) = y$ and assume $\lim_{\nu} f^{-\nu}(y) \neq x$. Then there exists a neighborhood \mathcal{U} of x , some $z \in X - \mathcal{U}$, and a subnet $\{f^{-\alpha}(y)\}$ of $\{f^{-\nu}(y)\}$ with $\lim_{\alpha} f^{-\alpha}(y) = z$. Since $\lim_{\alpha} f^{\alpha}(x) = y$, by the net characterization of even continuity (of $\{f^{-1}, f^{-2}, \dots\}$, by Lemma 4.8) we then get that

$$z = \lim_{\alpha} f^{-\alpha} f^{\alpha}(x) = \lim_{\alpha} x = x,$$

a contradiction. This does the trick.

LEMMA 4.10. *Let $f: X \rightarrow X$ be a function from the compact Hausdorff space X onto itself. If $\{f^1, f^2, \dots\}$ is evenly continuous, then any $g \in \Gamma(f)$ is an onto homeomorphism of X .*

Proof. Pick net $\{f^{\nu}\}$ in the (evenly continuous family) $\{f^1, f^2, \dots\}$ with $\lim_{\nu} f^{\nu} = g$ and assume $g(p) = \lim_{\nu} f^{\nu}(p) = \lim_{\nu} f^{\nu}(q) = g(q)$. Then, by Lemma 4.9, $p = \lim_{\nu} f^{-\nu}(g(p)) = \lim_{\nu} f^{-\nu}(g(q)) = q$, and this shows that g is one-to-one.

For any $p \in X$ and each ν , there exists $p_{\nu} \in X$ with $f^{\nu}(p_{\nu}) = p$. Pick subnet $\{f^{\sigma}\}$ of $\{f^{\nu}\}$ and $w, z \in X$ with $\lim_{\sigma} p_{\sigma} = z$ and $\lim_{\sigma} f^{\sigma}(z) = w$. Then, by even continuity, $w = \lim_{\sigma} f^{\sigma}(p_{\sigma}) = \lim_{\sigma} p = p$.

THEOREM 4.11. *Let $f: X \rightarrow X$ be a function from the compact Hausdorff space X onto itself. If $\{f^1, f^2, \dots\}$ is evenly continuous then $\Gamma(f)$ is a compact commutative group of homeomorphisms of X onto X .*

Proof. Immediate from Lemma 4.10. ($\Gamma(f)$ is commutative because it is the closure of a commutative group.)

This result is similar to Theorem 2 of Wallace [10] and a result of Wallace [11].

THEOREM 4.12. *Let $f: M \rightarrow M$ be a function from the compact metrizable space M onto itself. If $\{f^1, f^2, \dots\}$ is evenly continuous then there exists a compatible metric d for M such that $\Gamma(f)$ is a commutative group of d -isometries acting on M .*

Proof. By Theorem 4.6, f is a d -isometry for some compatible metric d for M . It is easily seen that also each $g \in \Gamma(f)$ is a d -isometry.

5. Counterexamples. The following example illustrates that most of the hypothesis in our main results are not superfluous.

EXAMPLE 5.1. There exists a nonexpansive onto function $f: E^1 \rightarrow E^1$ (E^1 is the real line) such that

(a) f is not net periodic and does not satisfy (B),

(b) of course, f satisfies conditions (A) and (C), and $\{f^1, f^2, \dots\}$ is evenly continuous.

Proof. Let $f: E^1 \rightarrow E^1$ be defined by

$$f(x) = x \quad \text{for each } x < 0$$

$$f(x) = 0 \quad \text{for all } 0 \leq x \leq 1$$

$$f(x) = x - 1 \quad \text{for all } x > 1.$$

Clearly f is onto and nonexpansive. (Note that $f^n(x) = x$ for $x < 0$, $f^n(x) = 0$ for $0 \leq x \leq n$ and $f^n(x) = x - n$ $x > n$. Therefore $\lim_n n = \infty$ but $\lim_n f^n(n) = 0$).

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