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WILLIAM FRANCIS REYNOLDS



# BLOCKS AND $F$ -CLASS ALGEBRAS OF FINITE GROUPS

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For an arbitrary field  $F$  of characteristic  $p \geq 0$ , the usual partitioning of the  $p$ -regular elements of a finite group  $G$  into  $F$ -classes ( $F$ -conjugacy classes) is extended to all of  $G$  in such a way that the  $F$ -classes form a basis of a subalgebra  $Y$  of the class algebra  $Z$  of  $G$  over  $F$ . The primitive idempotents of  $E \otimes_F Y$ , where  $E$  is an algebraic closure of  $F$ , are the same as those of  $Z$ . By means of this fact it is shown that if  $p > 0$  the number of blocks of  $G$  over  $F$  with a given defect group  $D$  is not greater than the number of  $p$ -regular  $F$ -classes  $L$  of  $G$  with defect group  $D$  such that the  $F$ -class sum of  $L$  in  $Z$  is not nilpotent; equality holds if  $O_{p,p',p}(G) = G$  or if  $D$  is Sylow in  $G$ . The results are generalized to arbitrary twisted group algebras of  $G$  over  $F$ .

1. Introduction. The representation theory of a finite group  $G$  over an arbitrary field  $F$  involves certain subsets of  $G$  called  $F$ -conjugacy classes or simply  $F$ -classes [6, p. 164], [9, p. 306]. In this paper we show (Theorem 4) that the  $F$ -class sums in the group algebra  $A$  of  $G$  over  $F$  form a basis of a subalgebra  $Y(A)$  of the center  $Z(A)$  of  $A$ ; we may call  $Y(A)$  the  $F$ -class algebra of  $G$ . (If  $F$  has prime characteristic  $p$ , the definition of the  $p$ -singular  $F$ -classes requires some care.) The crucial property of  $Y(A)$ , from our standpoint, is that its extension  $Y(A)^E$  to an algebra over an algebraic closure  $E$  of  $F$  has precisely the same primitive idempotents as the  $F$ -algebra  $Z(A)$  (Theorem 4); thus the blocks of  $G$  over  $F$  correspond to the primitive idempotents of an algebra over an algebraically closed field. Furthermore we obtain a corresponding result for any twisted group algebra (without any normalization of the factor set) of  $G$  over  $F$  by the methods of [16].

We make use of  $F$ -class algebras in conjunction with methods of Berman and Bovdi (Bódi) [2], [3] to obtain results about the number of blocks of twisted group algebras. In the group-algebra case these results (Theorems 6, 8, and 9) can be summarized as follows.

**THEOREM 1.** *Let  $F$  have prime characteristic  $p$ . For any  $p$ -subgroup  $D$  of  $G$ , the number of blocks of  $G$  over  $F$  with  $D$  as a defect group is less than or equal to the number of  $p$ -regular  $F$ -classes  $L$  of  $G$  with  $D$  as a defect group such that the  $F$ -class sum of  $L$  is not a nilpotent element of  $A$ . Equality holds here if  $O_{p,p',p}(G) = G$*



or if  $D$  is a  $p$ -Sylow subgroup of  $G$ ; in the latter case the nonnilpotence condition can be omitted.

Theorem 1 incorporates generalizations of results of Brauer and Nesbitt [4, Corollaries 1 and 2], [5, (6 D)] as well as of [2] and [3] concerning the case where  $F$  is a splitting field for  $G$ . In [2, Theorem 2] part of the result for  $O_{p, p', p}(G) = G$  is stated for arbitrary  $F$ , but without proof. The  $p$ -Sylow, or "highest defect", result for group algebras over arbitrary  $F$  has been obtained independently by Hubbart [10]; Bovdi's proof of this result is of interest even in the splitting-field case. Treatments of Brauer's results by Rosenberg [17] and Conlon [8] will be referred to frequently. Further references are given below.

In Corollary 2 we generalize a result of Brauer [5, (13A)] on blocks of defect 0. We remark that there is a connection between  $F$ -class algebras and the notion of  $S$ -rings (see [18] for example).

*Added in proof.* L. G. Kovács discovered most of Theorem 1 using vertices and sources, but his proof has appeared only in some unpublished notes written by Andrew Hopkins [9a]. Michler [11a] has independently obtained some interesting related results.

*Terminology.* We have attempted to help a reader interested only in the group-algebra case to skip over the complications caused by twisting. Standard notations, such as  $N_G(H)$ ,  $O_p(G)$ ,  $Z(G)$ , and the vertical line symbol for restrictions of mappings will be used without comment. A  $p'$ -group is one of order not divisible by  $p$ , i.e. such that all its elements are  $p$ -regular; if  $p = 0$ , every finite group is a  $p'$ -group, and a  $p$ -group has order 1. The center and Jacobson radical of an algebra  $X$  are called  $Z(X)$  and  $J(X)$  respectively. We shall follow the notation of [16] except for its categorical machinery.

**2. Representations of a Galois group.** Throughout the paper  $A$  denotes a twisted group algebra of a finite group  $G$  over an arbitrary field  $F$  of characteristic  $p \geq 0$ ; thus  $A$  has a basis  $\{a_g; g \in G\}$  with

$$(2.1) \quad a_g a_{g'} = f(g, g') a_{gg'}, \quad g, g' \in G,$$

for some nonzero  $f(g, g') \in F$ . For any subset  $H$  of  $G$ ,  $A_H$  denotes the subspace of  $A$  with basis  $\{a_h; h \in H\}$ ; if  $H$  is a subgroup,  $A_H$  is a twisted group algebra of  $H$ .  $E$  is a fixed algebraic closure of  $F$ , and  $\mathcal{G}$  is the (untopologized) Galois group of  $E$  over  $F$ . For any  $F$ -space ( $F$ -algebra)  $X$ ,  $X^E = E \otimes_F X$  is the  $E$ -space ( $E$ -algebra) obtained from  $X$  by extension of the ground field. We regard  $X$  as



embedded in  $X^E$  in the usual way; thus  $(A_H)^E = (A^E)_H = A_H^E$ .

We consider two representations of  $\mathcal{G}$  on the  $E$ -space  $A^E$ . First there is the well-known canonical semilinear representation of  $\mathcal{G}$  on  $A^E$ , which we shall call  $P_A$ : for each  $\sigma \in \mathcal{G}$ ,

$$\left[ \sum_{g \in G} w(g) a_g \right] P_A(\sigma) = \sum_{g \in G} w(g)^\sigma a_g, \quad w(g) \in E,$$

where  $w(g)^\sigma$  denotes the image of  $w(g)$  under  $\sigma$ .  $P_A(\sigma)$  is a ring-automorphism of  $A^E$ . (The existence of  $P_A$  does not depend on the fact that  $A$  is a twisted group algebra.)

The second representation of  $\mathcal{G}$  on  $A^E$  is the linear representation  $S_A$  of [16, Theorem 5]. We can describe  $S_A(\sigma)$  by the following restatement of [16, Corollary to Theorem 4].

**THEOREM 2.** *For each  $\sigma \in \mathcal{G}$ , there is a unique  $E$ -linear transformation  $S_A(\sigma)$  of  $A^E$  to  $A^E$  such that:*

(2.2) *For each cyclic subgroup  $\langle g \rangle$  of  $G$ , the restriction of  $S_A(\sigma)$  to  $A_{\langle g \rangle}^E$  is an algebra-automorphism of  $A_{\langle g \rangle}^E$ .*

(2.3) *For each cyclic  $p'$ -subgroup  $\langle g \rangle$  of  $G$ ,  $\psi_j(a S_A(\sigma)) = [\psi_j(a P_A(\sigma))]^{\sigma^{-1}}$  whenever  $a \in A_{\langle g \rangle}^E$  and  $\psi_j$  is an irreducible character of  $A_{\langle g \rangle}^E$ .*

(2.4) *For each cyclic  $p$ -subgroup  $\langle g \rangle$  of  $G$ ,  $S_A(\sigma)$  fixes every element of  $A_{\langle g \rangle}^E$ .*

Here  $\psi_j$  is defined with values in  $E$ . By Theorem 2, the analogue of  $S_A(\sigma)$  for any subgroup  $H$  of  $G$  is

$$(2.5) \quad S_{A_H}(\sigma) = S_A(\sigma)|A_H^E$$

(cf. [16, Theorem 4, (a)]). The group  $\{S_A(\sigma): \sigma \in \mathcal{G}\}$  is finite [16, § 6].

More explicitly: choose any  $n$  divisible by the exponent of  $G$  and write  $n = n_p n_{p'}$  where  $n_p$  is a power of  $p$  and  $n_{p'}$  is not divisible by  $p$ . (If  $p = 0$ ,  $n = n_{p'}$ .) Choose  $m(\sigma)$  so that  $\omega^\sigma = \omega^{m(\sigma)}$  for every  $n_{p'}$ -th root  $\omega$  of 1 in  $E$  and  $m(\sigma) \equiv 1 \pmod{n_p}$ . Then  $\mathcal{G}$  has a permutation representation  $\mathbf{s}_G$  on  $G$  such that

$$(2.6) \quad g \mathbf{s}_G(\sigma) = g^{m(\sigma^{-1})}, \quad g \in G.$$

Then  $a_g S_A(\sigma)$  is a scalar multiple of  $a_{g'}$  in  $A^E$  where  $g' = g \mathbf{s}_G(\sigma)$  ([16, (6.4)] gives a formula for the scalar); thus  $S_A$  acts monomially, with  $\mathbf{s}_G$  as the associated permutation representation (cf. [16, § 3]). In particular if  $A$  is a group algebra, we can take  $a_g = g$ ; then  $g S_A(\sigma) = g \mathbf{s}_G(\sigma)$  [16, (9.2)].

$G$  acts by conjugation both on itself and on  $A^E$  by automorphisms:



$$(2.7) \quad aK_A(x) = a_x^{-1}aa_x, gk_G(x) = x^{-1}gx,$$

for  $a \in A^E$ ,  $g \in G$ ,  $x \in G$  [16, (4.1) and (4.2)];  $K_A$  is a monomial representation of  $G$  with  $k_G$  associated to it. The fixed-point space of  $K_A$  is clearly the center  $Z(A^E) = Z(A)^E$  of  $A^E$ .

In the next proof, and throughout the paper, we shall make tacit use of the basic properties of idempotents of commutative algebras (for example, see [11], especially pp. 54–55). We refer to the primitive idempotents of a commutative algebra as *block idempotents*.

THEOREM 3. *If  $\sigma \in \mathcal{G}$ , then:*

$$(2.8) \quad S_A(\sigma)|Z(A^E) \text{ is an algebra-automorphism of } Z(A^E).$$

$$(2.9) \quad \text{For every block idempotent } d \text{ of } Z(A^E), dS_A(\sigma) = dP_A(\sigma).$$

*Proof.* By [16, (8.1)],  $S_A(\sigma)K_A(x) = K_A(x)S_A(\sigma)$ ; this is obvious in the group-algebra case. Hence  $S_A(\sigma)$  maps  $Z(A^E)$  onto itself. Observe that since  $P_A(\sigma)$  permutes the block idempotents, (2.9) says that  $S_A(\sigma)$  permutes them in the same way. We prove this theorem in three cases of increasing generality.

*Case I.* Suppose that  $A$  is a group algebra. If also  $p = 0$ , the theorem is due to Burnside [7, p. 317, Theorem VII]; our argument generalizes his. To each block idempotent  $d$  of  $Z(A^E)$  there corresponds a “block”  $B[d]$  of  $A^E$  to which are assigned certain irreducible representations  $F_j$  of  $A^E$ , their traces or characters  $\varphi_j$ , and the corresponding principal indecomposable representations  $U_j$ . Then

$$(2.10) \quad d = \sum_g \sum_j \frac{\deg U_j}{|G|} \varphi_j(g^{-1})g$$

where  $g$  runs over the  $p$ -regular elements of  $G$  and  $\varphi_j$  over the irreducible characters of  $B[d]$ : this is Osima’s formula [12, § 2] written in characteristic  $p$ ; for  $p > 0$  we interpret  $(\deg U_j)/|G|$ , which can be written with denominator not divisible by  $p$  [5, (3F)], as an element of the prime subfield of  $F$ . A consideration of characteristic roots shows that  $\varphi_j(g^{m(\sigma)}) = \varphi_j(g)^\sigma$  (cf. [16, Theorem 3]) and (2.9) follows. If  $p = 0$ ,  $Z(A^E)$  is the direct sum of the fields  $dE$ ; since  $S_A(\sigma)|Z(A^E)$  permutes the  $d$ ’s, (2.8) holds. In particular this is true when  $F = \mathbb{Q}$ , in which case any integer relatively prime to the exponent of  $G$  can serve as  $m(\sigma)$ ; an easy reduction modulo  $p$  yields (2.8) for prime characteristic.

*Case II.* Suppose that there is a positive integer  $l$  such that



$f(g, g')^l = 1$  for all  $g, g'$  in (2.1). Then there exists a finite central extension  $G^*$  of  $G$  such that  $A$  is (up to isomorphism) a direct summand of the group algebra  $A^*$  of  $G^*$  over  $F$  [8, pp. 155–156]; then  $A = A^*e^*$  for an idempotent  $e^*$  of  $Z(A^*)$ . Let  $M: \alpha^* \mapsto \alpha^*e^*$  be the projection of  $A^*$  onto  $A$ , and let  $M^E$  be its extension to a projection of  $(A^*)^E$  onto  $A^E$ . For any  $\sigma \in \mathcal{G}$ , set

$$S = S_A(\sigma), S^* = S_{A^*}(\sigma), P = P_A(\sigma), P^* = P_{A^*}(\sigma).$$

By [16, Theorem 4, (a)],  $S^*M^E = M^ES$ . Using Case I we find that  $e^*S^* = e^*P^* = e^*$ , and that for any  $z \in Z(A^E)$ ,

$$zS = (zM^E)S = (zS^*)M^E = (zS^*)e^* = (zS^*)(e^*S^*) = (ze^*)S^* = zS^* ;$$

hence  $S|Z(A^E)$  is a restriction of  $S^*|Z((A^*)^E)$  and (2.8) holds. As for (2.9), if  $d$  is any block idempotent of  $Z(A^E)$ ,  $dS = dS^* = dP^* = dP$ , using Case I and the fact that  $P = P^*|A^E$  by canonicity.

*Case III.* Let  $A$  be arbitrary. By [16, § 9] there exist elements  $c(g)$  of  $E$  such that if we set  $\alpha_g^* = c(g)a_g$ , then  $\{\alpha_g^*; g \in G\}$  is an  $F$ -basis of a twisted group algebra  $A^*$  for  $G$  over  $F$  such that Case II holds for  $A^*$ . We have  $(A^*)^E = A^E$ . For a fixed  $\sigma \in \mathcal{G}$ , set  $S = S_A(\sigma)$ ,  $S^* = S_{A^*}(\sigma)$ ,  $P = P_A(\sigma)$ ,  $P^* = P_{A^*}(\sigma)$ . At once  $P = P^*T$  where  $T$  is the  $E$ -linear transformation of  $A^E$  onto  $A^E$  such that

$$(2.11) \quad \alpha_g T = \frac{c(g)^\sigma}{c(g)} \alpha_g, \quad g \in G.$$

By the proof of [16, (9.3)], the mapping  $g \mapsto c(g)^\sigma/c(g)$  is a 1-cocycle, i.e., a homomorphism of  $G$  into the group of roots of unity of  $E$ ; hence  $T$  is an algebra-automorphism.

We claim that  $S = S^*T$ . In proving this we can replace  $G$  by its cyclic subgroups  $\langle g \rangle$  by (2.5). By (2.2) we can suppose that  $\langle g \rangle$  is either a  $p$ -group or a  $p'$ -group. In the first case  $S$  and  $S^*$  are the identity by (2.4), and so is  $T$  since the homomorphism in (2.11) is trivial. Suppose then that  $G$  is a cyclic  $p'$ -group. Then  $A^E = Z(A^E)$  (see the proof of [16, Theorem 4]) and  $A^E$  is the direct sum of the fields  $dE$  [8, p. 156]. By (2.3)  $\psi_j(dS) = [\psi_j(dP)]^{\sigma^{-1}} = \psi_j(dP)$  for each  $j$  since  $\psi_j(dP)$  is 0 or 1; hence  $dS = dP$  in this case. Similarly  $dS^* = dP^*$ , and  $[d(S^*)^{-1}]S = [d(P^*)^{-1}]P = dT$ ; then  $S = S^*T$  for cyclic  $p'$ -groups and hence for all  $G$ .

Now Case II implies the general case: for since  $S^*|Z(A^E)$  and  $T|Z(A^E)$  are algebra-automorphisms, so is  $S|Z(A^E)$ , while  $dS = (dS^*)T = (dP^*)T = dP$ .

REMARK 1. The argument in Case III shows that (2.3) is equiva-



lent to the condition:

$$(2.12) \quad \text{For each cyclic } p'\text{-subgroup } \langle g \rangle \text{ of } G, \ dS_A(\sigma) = dP_A(\sigma) \\ \text{for every block idempotent } d \text{ of } A_{\langle g \rangle}^E.$$

Hence in Theorem 4 of [16], we can replace condition (b) by our condition (2.9), which is roughly dual to (b). Also condition (c) can be replaced by our stronger condition (2.8).

REMARK 2. Theorem 3 can also be proved using the generalization of (2.10) for twisted group algebras; without proof we state that this formula is

$$(2.13) \quad d = \sum_g \sum_j \frac{\deg U_j}{|G|} \varphi_j(a_g^{-1}) a_g$$

with summations as in (2.9). Since  $d \in Z(A^E)$ , the coefficient of  $a_g$  vanishes unless  $g$  is in a  $K_A$ -regular conjugacy class of  $G$  (see § 3). Passman [13] has shown that only  $p$ -regular  $g$  are needed without deriving (2.13).

3.  $F$ -class algebras. As in [16, § 8], we can combine  $S_A$  and  $K_A$  to form a monomial representation  $D_A$  of the abstract direct product  $\mathcal{G} \times G$  on  $A^E$  by setting

$$(3.1) \quad D_A(\sigma, x) = S_A(\sigma)K_A(x) = K_A(x)S_A(\sigma),$$

$$(3.2) \quad d_G(\sigma, x) = s_G(\sigma)k_G(x) = k_G(x)s_G(\sigma).$$

The following result was suggested by a lemma of Berman [1, Lemma 3.1].

THEOREM 4. *The fixed-point space of  $D_A$  is an  $E$ -subalgebra of  $Z(A^E)$  with identity. Its block idempotents are identical with those of  $Z(A)$ .*

*Proof.* Temporarily denote this space by  $X$ . The first sentence follows from (2.8), for since  $Z(A^E)$  is the fixed-point space of  $K_A$ ,  $X$  is the fixed-point space of the subrepresentation of  $S_A$  on  $Z(A^E)$ . There is a finite normal (not necessarily separable) extension field  $N$  of  $E$  such that every block idempotent  $d$  of  $Z(A^E)$  lies in  $N \otimes_F Z(A)$ .  $P_A$  permutes the  $d$ 's, and by [15, Lemma 2] the block idempotents of  $Z(A)$  are the sums  $\sum d$  over the various orbits. By (2.9) these are also orbits under  $S_A$ ; then the sums  $\sum d$  are the block idempotents of  $X$ .

We shall call the orbits of  $d_G$  the  $F$ -conjugacy classes, or  $F$ -classes, of  $G$ . Since  $gd_G(\sigma, x) = x^{-1}g^{m(\sigma^{-1})}x$  by (2.6), this agrees with the usual



definition [9, p. 306], [1] for the  $p$ -regular elements of  $G$  (cf. the proof of [16, Theorem 6]). The monomial representation  $D_A$  distinguishes certain  $F$ -classes: as in [16, § 3] we say that an  $F$ -class  $L$  is  $D_A$ -regular provided that there exist nonzero  $q(g) \in E$ ,  $g \in L$ , such that  $D_A$  acts as a permutation representation on the elements  $q(g)a_g$  of  $A^E$ . By [16, Lemma 2] if  $g \in L$ , then  $L$  is  $D_A$ -regular if and only if the stabilizer  $\{(\sigma, x) \in \mathcal{G} \times G: a_g D_A(\sigma, x) = a_g\}$  of  $a_g$  under  $D_A$  equals the stabilizer of  $g$  under  $d_G$ . (In the group-algebra case, all  $F$ -classes are  $D_A$ -regular.) By [16, Lemma 1] the dimension of the fixed-point space of  $D_A$  is the number of  $D_A$ -regular  $F$ -classes. In fact an  $E$ -basis is formed by the elements

$$(3.3) \quad y_L = \sum_{g \in L} q(g)a_g$$

as  $L$  ranges over the  $D_A$ -regular  $F$ -classes.

Analogous considerations apply to  $K_A$ : thus we have elements  $z_K$  as  $K$  ranges over the  $K_A$ -regular conjugacy classes of  $G$  which form a well-known basis of the fixed-point space  $Z(A^E)$  as well as of  $Z(A)$  [8, p. 155].

In the group-algebra case we can choose all  $q(g) = 1$  in (3.3) so that the  $y_L$  are the  $F$ -class sums in  $A$ . For general  $A$  it is interesting, although not essential for our later arguments, that we can choose all  $q(g)$  in the ground field  $F$ , so that still  $y_L \in A$ . This statement is equivalent to the following theorem.

**THEOREM 5.** *The fixed-point space  $X$  of  $D_A$  has the form  $Y(A)^E$  for a unique  $F$ -subalgebra  $Y(A)$  of  $Z(A)$ .*

*Proof.* It will suffice to show that the fixed-point space of  $S_A$  has form  $W^E$  for an  $F$ -subspace  $W$  of  $A$ , since this will imply that  $X = W^E \cap Z(A^E) = [W \cap Z(A)]^E$ . By (2.5), (2.2), and (2.5) we can reduce to the case that  $G$  is a cyclic  $p'$ -group. As in Case III of Theorem 3,  $A^E = \bigoplus dE$  and the fixed-point space of  $S_A$  is  $X$ . By Theorem 4 the block idempotents  $e$  of  $X$  are all in  $A$ ; then  $X = \bigoplus eE = (\bigoplus eF)^E$  as required. For general  $G$ ,  $Y(A)$  is unique since  $Y(A) = X \cap A = X \cap Z(A)$ . The statement about the  $y_L$  is true since  $X = \bigoplus_L [Y(A)^E \cap A_L^E] = \bigoplus_L [Y(A) \cap A_L]^E$ .

Henceforth the symbol  $Y(A)$  always denotes this  $F$ -algebra, and the  $y_L$  are chosen in  $A$ , so that they form an  $F$ -basis of it.  $Y(A)$  may be called the  $F$ -class algebra of  $A$ . We could "normalize" the basis  $\{a_g\}$  of  $A$ , changing it so that all  $q(g) = 1$  in (3.3); however we shall not do this in order to avoid conflicting normalizations for subgroups and for conjugacy classes.

We say that an  $F$ -class  $L$  is  $A$ -nonnilpotent provided that (a)  $L$



is  $D_A$ -regular and (b)  $y_L$  is not a nilpotent element of  $Y(A)$ . Here (b) makes sense since  $y_L$  is determined up to a scalar multiple; in terms of radicals it is equivalent to saying that  $y_L \notin J(Y(A))$  or that  $y_L \notin J(Y(A)^E)$ . (It is not always true that  $J(Y(A)^E) = J(Y(A))^E$ : see the example of [15, pp. 12–13].)

REMARK 3. I have not been able to answer the following question even in the group-algebra case: does  $S_A(\sigma)$  map  $J(A)$  into itself?

4. **Counting blocks.** From now on  $p$  will always be prime. For each  $F$ -class  $L$ , call any  $p$ -Sylow subgroup of  $C_G(g)$  for any  $g \in L$  a *defect group* of  $L$ ; this is determined up to conjugacy in  $G$  since  $C_G(g^{m(\sigma)}) = C_G(g)$ . In other words, the defect groups of  $L$  are the same as the defect groups of the conjugacy classes within  $L$ . Each block idempotent  $e$  of  $Z(A)$ , i.e., of  $Y(A)^E$  or of  $Y(A)$ , has form  $e = \sum r[L]y_L$ ,  $r[L] \in F$ , summed over the  $p$ -regular  $D_A$ -regular  $F$ -classes  $L$  (cf. Remark 2). By [17, § 2] and [8, § 3], the largest of the defect groups of those  $L$  for which  $r[L] \neq 0$  form a single conjugacy class of subgroups of  $G$ , called the *defect groups* of  $e$  (in  $A$ ).

The following result is a generalization of the lemma of Brauer that is quoted in its proof.

LEMMA 1. *Let  $D$  be any  $p$ -subgroup of  $G$ , and let  $H = N_G(D)$ . Then there is a bijection of the set of all  $D_A$ -regular  $F$ -classes of  $G$  with defect group  $D$  and the set of all  $D_{AH}$ -regular  $F$ -classes of  $H$  with (unique) defect group  $D$ , given by  $L \mapsto L \cap H$ .*

*Proof.* By a lemma of Brauer [5, (10A)], [17, Lemma 3.4], there is a bijection  $K \mapsto K \cap H$  of all conjugate classes of  $G$  with defect group  $D$  to all conjugate classes of  $H$  with unique defect group  $D$ . For each  $F$ -class  $L$  of  $G$  with defect group  $D$ ,  $L = \bigcup_{\sigma \in \mathcal{S}} K^{[m(\sigma)]}$  where  $K^{[m(\sigma)]} = \{g^{m(\sigma)} : g \in K\}$ , and  $L \cap H = \bigcup (K \cap H)^{[m(\sigma)]}$ ; hence there is a bijection  $L \mapsto L \cap H$  of all  $F$ -classes of  $G$  with defect group  $D$  to all  $F$ -classes of  $H$  with defect group  $D$ . If  $L$  is  $D_A$ -regular and  $h \in L \cap H$ , the stabilizers of  $a_h$  under  $D_A$  and of  $h$  under  $d_G$  are equal; then the stabilizers of  $a_h$  under  $D_{AH}$  and of  $h$  under  $d_H$  are equal, so that  $L \cap H$  is  $D_{AH}$ -regular.

Conversely suppose that  $L \cap H$  is  $D_{AH}$ -regular with defect group  $D$ . The following argument is a refinement of the proof of the Lemma of [14]. Let  $h \in K \cap H \subseteq L \cap H$ , and suppose that  $(\sigma, x) \in \mathcal{S} \times G$  is such that  $hd_G(\sigma, x) = h$ ; we must show that  $a_h D_A(\sigma, x) = a_h$ . Let  $T = \{t \in G : a_h K_A(t) = a_h\}$  be the stabilizer of  $a_h$  under  $K_A$ .  $K \cap H$  is  $K_{AH}$ -regular, i.e.,  $T \cap H = C_H(h)$ . By Brauer's lemma,  $D$  is  $p$ -Sylow in  $C_G(h)$  as well as in  $C_H(h)$ . Since  $C_H(h) \subseteq T \subseteq C_G(h)$ ,  $D$  is  $p$ -Sylow



in  $T$ . Now  $a_h \mathbf{D}_A(\sigma, x) = ca_h$  for some  $c \in E$ ; if  $t \in T$  then

$$\begin{aligned} a_h \mathbf{K}_A(x^{-1}tx) &= c^{-1} a_h \mathbf{D}_A(\sigma, x) \mathbf{K}_A(x^{-1}tx) \\ &= c^{-1} a_h \mathbf{K}_A(t) \mathbf{D}_A(\sigma, x) = c^{-1} a_h \mathbf{D}_A(\sigma, x) = a_h, \end{aligned}$$

so that  $x^{-1}Tx \subseteq T$ ; similarly  $xTx^{-1} \subseteq T$ , so that  $x^{-1}Dx$  is  $p$ -Sylow in  $T$ . Then  $x^{-1}Dx = t^{-1}Dt$  for some  $t \in T$ , and  $xt^{-1} \in N_G(D) = H$ . Now

$$h\mathbf{d}_H(\sigma, xt^{-1}) = h\mathbf{d}_G(\sigma, xt^{-1}) = h\mathbf{d}_G(\sigma, x)\mathbf{k}_G(t)^{-1} = h\mathbf{k}_G(t)^{-1} = h.$$

Since  $L \cap H$  is  $\mathbf{D}_{A_H}$ -regular,  $a_h \mathbf{D}_A(\sigma, xt^{-1}) = a_h$ ; and then  $a_h \mathbf{D}_A(\sigma, x) = a_h \mathbf{K}_A(t) = a_h$  as required.

**LEMMA 2** (cf. [3, Lemma 4]). *Under the assumptions of Lemma 1, the number of  $p$ -regular  $A$ -nonnilpotent  $F$ -classes of  $G$  with defect group  $D$  is not less than the number of  $p$ -regular  $A_H$ -nonnilpotent  $F$ -classes of  $H$  with defect group  $D$ .*

*Proof.* The mapping  $R$  of  $A^E$  into  $A_H^E$  defined by

$$\left[ \sum_{g \in G} w(g)a_g \right] R = \sum_{g \in C} w(g)a_g,$$

where  $C = C_G(D)$ , satisfies  $S_A(\sigma)R = RS_{A_H}(\sigma)$ ; hence the Brauer homomorphism  $R|Z(A^E)$  of  $Z(A^E)$  into  $Z(A_H^E)$  [5, (7B)], [17, Lemma 3.3], [8, § 3] carries  $Y(A)$  into  $Y(A_H)$ . For the basis element  $y_L$  of  $Y(A)$  in (3.3),  $y_L R$  is an analogous element of  $Y(A_H)$  for the  $F$ -class  $L \cap H = L \cap C$ ; if  $y_L$  is nilpotent so is  $y_L R$ . Since  $L$  is  $p$ -regular if and only if  $L \cap H$  is, Lemma 1 implies the result.

The next theorem generalizes [3, Theorem 1], which in turn strengthens [4, Corollary 1] and [12, Corollary 2 to Theorem 9].

**THEOREM 6.** *For any  $p$ -subgroup  $D$  of  $G$ , the number of block idempotents of  $Z(A)$  with defect group  $D$  is not greater than the number of  $p$ -regular  $A$ -nonnilpotent  $F$ -classes of  $G$  with defect group  $D$ .*

*Proof.* By Brauer's first main theorem on blocks, suitably generalized [5, (10B)], [17, Theorem 5.3], [14, Theorem 1] and by Lemma 2, we reduce at once to the case  $G = N_G(D)$ . In this case, let  $V$  be the  $F$ -subspace of  $Z(A)$  with a basis consisting of the elements  $z_K$  (see the paragraph after (3.3)) for the  $\mathbf{K}_A$ -regular conjugacy classes  $K$  of  $G$  with defect group  $D$ . By [17, Lemmas 4.1 and 4.4], [8, p. 166] and [14, p. 281],  $V$  is a (commutative) subalgebra of  $Z(A)$  (possibly without an identity) and the idempotents  $e$  mentioned in the statement are precisely the block idempotents of  $V$ . By Theorem



4 they are the block idempotents of  $U = V \cap Y(A)$ , which is a subalgebra of  $Z(A)$  with a basis consisting of the elements  $y_L$  for the  $D_A$ -regular  $F$ -classes  $L$  of  $G$  with defect group  $D$ . The block idempotents of  $U/J(U)$  are the elements  $e + J(U)$ . Since these are linear combinations of the elements  $y_L + J(U)$  for the  $F$ -classes  $L$  mentioned in the statement, the theorem is proved.

**COROLLARY 1** (cf. [2, Lemma 1]). *The number of block idempotents of  $Z(A)$  is not greater than the number of  $p$ -regular  $A$ -nonnilpotent  $F$ -classes of  $G$ .*

Theorem 6 and its proof, together with the theory of commutative algebras [11], yield the following corollaries, which generalize results of Brauer [5, (13A)] and Bovdi [3, Theorem 3] concerning the case  $D = \{1\}$ .

**COROLLARY 2.** *For any  $p$ -subgroup  $D$  of  $G$ , the number of block idempotents of  $Z(A)$  with defect group  $D$  is the  $E$ -dimension of  $U^E/J(U^E)$ , where  $U$  is defined for  $D$  in  $N_G(D)$ . This equals the  $F$ -dimension of  $U^i$  for sufficiently large  $i$ .*

**COROLLARY 3.** *The following conditions are equivalent, where  $H = N_G(D)$ :*

- (4.1) *There exists a block idempotent of  $Z(A)$  with defect group  $D$ .*
- (4.2) *There exists an  $A_H$ -nonnilpotent  $F$ -class of  $H$  with defect group  $D$ .*
- (4.3) *There exists a  $p$ -regular  $A_H$ -nonnilpotent  $F$ -class of  $H$  with defect group  $D$ .*

Now we obtain some sufficient conditions for equality in Theorem 6. First we consider groups such that  $O_{p,p',p}(G) = G$ .

**THEOREM 7** (cf. [2, Theorems 1 and 2], [3, Theorem 2]). *Assume that  $G$  has normal subgroups  $P$  and  $M$ ,  $P \subseteq M$ , such that  $P$  and  $G/M$  are  $p$ -groups while  $M/P$  is a  $p'$ -group. Then the number of block idempotents of  $Z(A)$  is equal to the number of  $p$ -regular  $A$ -nonnilpotent  $F$ -classes of  $G$ . These coincide with the  $D_A$ -regular  $F$ -classes of  $G$  which are contained in  $O_p(G)$ , and also with the  $p$ -regular  $D_A$ -regular  $F$ -classes of  $G$  with a defect group which contains  $P$ .*

*Proof.* By Burnside's theorem  $Z(P)$  has a normal complement  $Q$  in  $C = C_M(P)$ . Then  $C = Z(P) \times Q$ , and easily  $Q = O_{p'}(M) = O_{p'}(G)$ .

Let  $L$  be any  $p$ -regular  $D_A$ -regular  $F$ -class of  $G$ ; then  $L \subseteq M$ .



We claim that the following conditions on  $L$  are equivalent: (a)  $L \subseteq Q$ ; (b)  $L \subseteq C$ ; (c)  $L$  has a defect group which contains  $P$ ; (d) the  $F$ -classes of  $M$  contained in  $L$  have defect group  $P$ ; (e)  $L$  is  $A$ -nonnilpotent; (f) the conjugacy classes of  $M$  contained in  $L$  are  $A_M$ -nonnilpotent. It is straightforward that (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (d)  $\Leftrightarrow$  (c). Since  $A_Q$  is semisimple [8, p. 156], (a)  $\Rightarrow$  (e). Suppose now that (e) holds; let  $K_1$  be a fixed conjugacy class of  $M$  contained in  $L$ . Then  $L$  is a disjoint union of classes of form  $K = \{gd_G(\sigma, x): g \in K_1\}$  for suitable choices of  $(\sigma, x) \in \mathcal{S} \times G$ . For the element  $y_L = \sum_{g \in L} q(g)a_g$  of (3.3), let  $z_K = \sum_{g \in K} q(g)a_g$ . Then  $y_L = \sum z_K$ , and  $z_K$  is a choice for the basis element of  $Z(A_Q)$  corresponding to  $K$ . Since  $y_L D_A(\sigma, x) = y_L$ ,  $z_{K_1} d_G(\sigma, x) = z_K$ . By (2.8) the elements  $z_K$  are either all nilpotent or all nonnilpotent; since their sum is nonnilpotent, so are they; hence (e)  $\Rightarrow$  (f). Finally (f)  $\Rightarrow$  (b) by the twisted generalization [8, p. 166] of [17, Lemma 4.2].

Let  $e$  be any block idempotent of  $Z(A)$ . Since the expression for  $e$  involves only  $p$ -regular elements,  $e \in Z(A_M)$ . By [15, Lemma 3],  $e \in Z(A_C)$ ; then  $e \in Z(A_Q)$  since (b)  $\Rightarrow$  (a). (Alternatively: by the twisted generalization of [17, Proposition 4.4] which is implicit in [8, § 3], every block idempotent of  $Z(A_M)$  has defect group  $P$ . The proof of Theorem 6 shows that  $e$  is in the algebra  $V$  defined for  $P$  in  $M$ ; then  $e \in Z(A_Q)$  since (d)  $\Rightarrow$  (a).) Therefore the block idempotents of  $Z(A)$  are identical with those of  $Z(A) \cap Z(A_Q)$ , and with those of  $Y(A)^E \cap Y(A_Q)^E$ .  $Z(A_Q^E)$ , being semisimple, is a direct sum of copies of  $E$ ; then so is  $Y(A)^E \cap Y(A_Q)^E$ , and the number of block idempotents of  $Z(A)$  equals the dimension of that algebra, namely the number of  $D_A$ -regular  $F$ -classes of  $G$  which are contained in  $Q$ . Since (a)  $\Leftrightarrow$  (c)  $\Leftrightarrow$  (e), the theorem is proved.

Together with [16, Theorem 6], Theorem 7 implies:

**COROLLARY 4.** *If  $G$  has a normal  $p$ -complement, each block of  $A$  contains exactly one irreducible representation of  $A$  over  $F$ .*

Combining Theorems 6 and 7 we obtain:

**THEOREM 8** (cf. [3, Corollary 3]). *If  $G$  satisfies the hypothesis of Theorem 7, then for every  $p$ -subgroup  $D$  of  $G$  we have equality in Theorem 6.*

We conclude by treating the case of highest defect [5, (6D)], [17, Theorem 6.1], [8, p. 166], [3, Theorem 4], [10]. Our argument, based on [3], differs from that of [17] and [8] in using subalgebras of  $Z(A_H)$  instead of a quotient algebra, and thus avoids counting  $p$ -singular classes.



**THEOREM 9.** *If  $P$  is a  $p$ -Sylow subgroup of  $G$ , the number of block idempotents of  $Z(A)$  with defect group  $P$  is equal to the number of  $p$ -regular  $D_A$ -regular  $F$ -classes of  $G$  with defect group  $P$ . All such  $F$ -classes are  $A$ -nonnilpotent.*

*Proof.* By the first main theorem on blocks, the number of block idempotents in question is equal to the number of block idempotents of  $Z(A_H)$  with defect group  $P$ , where  $H = N_G(P)$ . These are all the block idempotents of  $Z(A_H)$ , as in the proof of Theorem 7; by that theorem, for  $H$ , the number of such block idempotents equals the number of  $p$ -regular  $D_{A_H}$ -regular  $F$ -classes of  $H$  with defect group  $P$ . By the bijection of Lemma 1, this equals the number of the  $F$ -classes of  $G$  mentioned in the first sentence. The  $F$ -classes of  $H$  in question here are all  $A_H$ -nonnilpotent since (c)  $\Rightarrow$  (e) in the proof of Theorem 7; then Lemma 2 implies the second sentence.

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