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SCHLICHT MAPPINGS AND INFINITELY DIVISIBLE KERNELS

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SCHLICHT MAPPINGS AND INFINITELY DIVISIBLE KERNELS

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The purpose of this note is to give a simple condition which is sufficient for a function on a real interval to be the boundary value of a schlicht (univalent) analytic mapping of the upper half plane into itself. This condition leads to a simple transformation which takes (possibly) non-schlicht mappings into schlicht ones. The methods used have applications to probability theory as well; they yield an interesting class of infinitely divisible characteristic functions.

We shall require some facts about infinitely divisible kernels; for a detailed exposition see [5]. If I is a real interval we denote by $L_0(I)$ the set of all continuous complex valued functions which have compact support in I and whose integral over I vanishes. A continuous kernel K(x, y) on $I \times I$ is said to be conditionally positive definite on I if

(1)
$$\iint_{I\times I} K(x, y)\phi(x)\bar{\phi}(y)dxdy \ge 0$$

for all functions $\phi \in L_0(I)$; it is said to be *positive definite on* I if (1) is satisfied for all continuous functions ϕ with compact support in I; it is said to be *infinitely divisible on* I if (for some fixed continuous determination of the argument) the kernel $K^{\alpha}(x, y)$ is positive definite for all $\alpha > 0$.

The connection among these concepts is that a continuous Hermitian kernel K(x, y) with no zeroes is infinitely divisible on I if and only if (for some continuous determination of the argument) the kernel log K(x, y) is conditionally positive definite on I. If K(x, y) > 0for all $x, y \in I$ there is, of course, no difficulty about determining the argument. Finally, the relevance of these notions to function theory is indicated by the following result [6], [4]. If f is a differentiable function we define $K_f(x, y) \equiv [f(x) - f(y)]/(x - y)$ and agree that $K_f(x, x) = f'(x)$.

THEOREM 1. Let f be a continuously differentiable real valued function with positive derivative on a real interval I. The function f possesses an analytic continuation onto the upper half plane which maps the upper half plane into itself if and only if the kernel $K_f(x, y)$ is positive definite on I. This mapping is schlicht if and only if $K_f(x, y)$ is infinitely divisible on I. Although this result completely characterizes the boundary values of schlicht mappings, it is in practice much harder to verify that the kernel $K_f(x, y)$ is infinitely divisible than to test it for positive definiteness. By our remarks above, one must check whether $\log K_f(x, y)$ is conditionally positive definite, but the non-linearity of this expression in f often leads to computational difficulties. In the following, we shall derive a more linear, and hopefully more useful, sufficient condition. Recall that a C^{∞} function ϕ defined on $(0, \infty)$ is completely monotonic if $(-1)^n \phi^{(n)}(x) \ge 0$ for all x > 0 and all $n = 1, 2, 3, \cdots$.

LEMMA 2. Let H(x, y) be a continuous Hermitian kernel on a real interval I such that $\operatorname{Re} \{H(x, y)\} > 0$ and such that -H(x, y) is conditionally positive definite. If ϕ is any completely monotonic function then the kernel $\phi(H(x, y))$ is positive definite on I.

Proof. It is well known that a function ϕ is completely monotonic if and only if there exists a nonnegative measure $d\mu$ such that $\phi(x) = \int_0^{\infty} e^{-xs} d\mu(s)$ for all x > 0 ([8], p. 160); in this event ϕ is analytic in the whole right half plane. But since Re $\{H(x, y)\} > 0$ and exp (-sH(x, y)) is positive definite (even infinitely divisible) for all s > 0, it follows that $\phi(H(x, y)) = \int_0^{\infty} \exp(-sH(x, y)) d\mu(s)$ is convergent and is a positive definite kernel.

An infinitely divisible completely monotonic function ϕ is a function such that ϕ^{α} is completely monotonic for all $\alpha > 0$; if $\phi \neq 0$, a necessary and sufficient condition for this is that the derivative of $-ln\phi$ be completely monotonic ([3], p. 229). Using the lemma and the definition of an infinitely divisible kernel we obtain

COROLLARY 3. Let ϕ be a positive differentiable function on $(0, \infty)$ such that $-\phi'/\phi$ is completely monotonic, and suppose H(x, y) satisfies the hypotheses of Lemma 2. Then $\phi(H(x, y))$ is an infinitely divisible kernel.

Since the function $\phi(x) = 1/x$ satisfies this condition, the following result is immediate.

COROLLARY 4. If H(x, y) satisfies the conditions of Lemma 2, then the kernel 1/H(x, y) is infinitely divisible.

Now suppose that g is a continuously differentiable real valued function on a real interval I, so that $K_g(x, y)$ is a continuous symmetric kernel. If g'(x) > 0 on I then $K_g(x, y)$ is a positive kernel and the inverse function g^{-1} is defined on the interval g(I). Thus, if we assume that $-K_g(x, y)$ is conditionally positive definite on I then we conclude from Corollary 4 that the kernel

$$rac{1}{K_g(x,\,y)} = rac{x-y}{g(x)-g(y)} = rac{g^{-1}(g(x))-g^{-1}(g(y))}{g(x)-g(y)}$$

is infinitely divisible. But this is equivalent to the kernel $K_{g^{-1}}(s, t)$ being infinitely divisible on g(I) and so we may apply Theorem 1 to obtain the conclusion of the following

THEOREM 5. Let g be a continuously differentiable real valued function with positive derivative on a real interval I and suppose that the kernel

$$-K_g(x, y) = -\frac{g(x) - g(y)}{x - y}$$

is conditionally positive definite on I. Then the inverse function g^{-1} has an analytic continuation from g(I) onto the upper half plane which is a schlicht mapping of the upper half plane into itself.

Thus, to ensure that a real function f on a real interval I is the boundary value of a schlicht self-mapping of the upper half plane it is sufficient to check that f'(x) > 0 and that $-K_{f^{-1}}(x, y)$ is conditionally positive definite on $f^{-1}(I)$.

The crucial condition in Theorem 5 is that the kernel $-K_g(x, y)$ be conditionally positive definite, and a great deal is known about functions which satisfy this condition. For example, they are real analytic and are analytically continuable onto the upper half plane, they have a simple integral representation, and they arise as the infinitesimal transformations of the pseudo-semigroup \mathfrak{M}_{∞} of self-mappings of the upper half plane which have real boundary values on I ([6] and [2], pp. 53-54). Furthermore, it is easy to find many non-trivial functions which satisfy this condition. Denote by $\mathfrak{M}_{\infty}(0)$ the class of functions f which are analytic in the upper half plane, map it into itself, are real valued on some open real interval containing zero, and are normalized by the condition f(0) = 0.

LEMMA 6. Let a be a real number, let $b \ge 0$ and let $f \in \mathfrak{M}_{\infty}(0)$. Then the functions $g_0(x) = a$, $g_1(x) = ax$, $g_2(x) = ax^2$, and $g_3(x) = bx^2 f(x)$ are such that $K_{g_1}(x, y)$ is conditionally positive definite on some neighborhood of the origin, i = 0, 1, 2, 3.

Proof. This follows from a direct computation for i = 0, 1, 2

but for i=3 we need to know ([1], p. 63) that $f\in\mathfrak{M}_{\infty}(0)$ if and only if

(2)
$$f(x) = \int_{-\epsilon}^{\epsilon} \frac{x}{1-tx} d\mu(t)$$

for some $\varepsilon > 0$ and some nonnegative bounded measure $d\mu$ on $[-\varepsilon, \varepsilon]$. Thus, since the assertion for i = 3 follows for the special case f(x) = x/(1-tx) by direct computation, it follows for all $f \in \mathfrak{M}_{\infty}(0)$ by linearity.

Using the four types of functions introduced in this lemma we can now use Theorem 5 to construct a wide class of schlicht mappings.

THEOREM 7. Let $f \in \mathfrak{M}_{\infty}(0)$, let $a_1 > 0$, $a_3 \ge 0$, and let a_0 and a_2 be real numbers. Then the function

$$g(x) = a_0 + a_1 x + a_2 x^2 - a_3 x^2 f(x)$$

is such that the inverse function g^{-1} has an analytic continuation from a real neighborhood of a_0 onto the upper half plane which is a schlicht mapping of the upper half plane into itself.

Proof. The kernel $-K_g(x, y)$ is conditionally positive definite by Lemma 6 and g'(x) > 0 in some real neighborhood of zero. The result follows from Theorem 5.

Although this construction provides a wealth of schlicht mappings, it is far from exhaustive: the functions $f(z) = 3[\sqrt[3]{z+1} - 1]$ and $f(z) = \log (z+1)$ are schlicht mappings which are not of this form.

REMARK 1. Linear combinations of the four functions in Lemma 6 are in fact the *only* smooth functions g such that $K_g(x, y)$ is conditionally positive definite. In order to prove this we use the following criterion for a kernel to be conditionally positive definite.

LEMMA 8. Let H(x, y) be a continuous kernel on a real interval I and let $x_0 \in I$. Then H(x, y) is conditionally positive definite on I if and only if the kernel

$$H_{x_0}^*(x, y) \equiv H(x, y) - H(x, x_0) - H(x_0, y) + H(x_0, x_0)$$

is positive definite on I.

Proof. If
$$\phi \in L_0(I)$$
, then
$$\iint_{I \times I} H^*_{x_0}(x, y) \phi(x) \overline{\phi}(y) dx dy = \iint_{I \times I} H(x, y) \phi(x) \overline{\phi}(y) dx dy ,$$

and hence H(x, y) is conditionally positive definite if $H_{x_0}^*(x, y)$ is positive definite. Conversely, suppose H(x, y) is conditionally positive definite and let $\{f_n(x)\}, n = 1, 2, 3, \cdots$ be an approximate identity based at x_0 , i.e., each f_n is a nonnegative continuous function with support in $I \cap [x_0 - n^{-1}, x_0 + n^{-1}]$ and $\int_I f_n(x) dx = 1$ for all n. If ϕ is any continuous function with compact support in I, let $\phi_n(x) \equiv \phi(x) - f_n(x) \int_Y \phi(t) dt$ and observe that $\phi_n \in L_0(I)$ for all large n. Thus,

$$egin{aligned} 0 &\leq \displaystyle \iint_{I imes I} H(x, \, y) \phi_n(x) ar{\phi}_n(y) dx dy \ &= \displaystyle \iint_{I imes I} \Big\{ H(x, \, y) \, - \, \int_I H(x, \, t) f_n(t) dt \, - \, \int_I H(s, \, y) f_n(s) ds \ &+ \, \iint_{I imes I} H(s, \, t) f_n(s) f_n(t) ds dt \Big\} \phi(x) ar{\phi}(y) dx dy \ & o \, \iint_{I imes I} \{ H(x, \, y) \, - \, H(x, \, x_0) \, - \, H(x_0, \, y) \, + \, H(x_0, \, x_0) \} \phi(x) ar{\phi}(y) dx dy \ &= \displaystyle \iint_{I imes I} H_{x_0}^*(x, \, y) \phi(x) ar{\phi}(y) dx dy \end{aligned}$$

as $n \to \infty$. Since ϕ is arbitrary, we conclude that the kernel $H^*_{x_0}(x, y)$ must be positive definite.

LEMMA 9. Let K(x, y) be a continuous kernel on a real interval I. Then xyK(x, y) is positive definite kernel if and only if K(x, y) is a positive definite kernel.

Proof. If zero is not a point of I this is trivial, so suppose $0 \in I$, let $\varepsilon > 0$, and denote by f_{ε} the unique even function such that

$$f_{arepsilon}(x) \equiv egin{cases} 0 & ext{if} \ x \in [0, \ arepsilon] \ arepsilon^{-1}(x-arepsilon) & ext{if} \ x \in [arepsilon, 2arepsilon] \ 1 & ext{if} \ x \ge arepsilon \ . \end{cases}$$

Let $M \equiv \sup_{I \times I} |K(x, y)|$, let ϕ be a continuous function with compact support in *I*, and assume that xyK(x, y) is positive definite on *I*. Then

$$egin{aligned} & \displaystyle \iint_{I imes I} K(x,\,y)\phi(x)ar{\phi}(y)dxdy \ & = \displaystyle \iint_{I imes I} K(x,\,y)\phi(x)ar{\phi}(y)(1\,-\,f_{\epsilon}(y))dxdy \ & + \displaystyle \iint_{I imes I} K(x,\,y)\phi(x)ar{\phi}(y)f_{\epsilon}(y)(1\,-\,f_{\epsilon}(x))dxdy \ & + \displaystyle \iint_{I imes I} K(x,\,y)\phi(x)f_{\epsilon}(x)ar{\phi}(y)f_{\epsilon}(y)dxdy \end{aligned}$$

$$egin{aligned} &\geq - \iint_{I imes I} | \ K(x,\,y) \phi(x) ar{\phi}(y) (1\,-\,f_{arepsilon}(y)) \,|\, dx dy \ &- \iint_{I imes I} | \ K(x,\,y) ar{\phi}(y) f_{arepsilon}(y) \phi(x) (1\,-\,f_{arepsilon}(x)) \,|\, dx dy \ &+ \iint_{I imes I} xy K(x,\,y) x^{-1} \phi(x) f_{arepsilon}(x) y^{-1} ar{\phi}(y) f_{arepsilon}(y) dx dy \ &\geq - 6 \mathcal{M} arepsilon \sup_{I} | \ \phi(x) \,| \int_{I} | \ \phi(x) \,|\, dx \,. \end{aligned}$$

For the last inequality we have used the hypothesis that xyK(x, y) is positive definite and the fact that the function $x^{-1}\phi(x)f_{\varepsilon}(x)$ is a continuous function with compact support in *I*. Since $\varepsilon > 0$ is arbitrary we conclude that

i.e., K(x, y) is positive definite. The converse is trivial.

Now assume that the kernel H(x, y) is of the special form $H(x, y) = K_g(x, y)$, where g is a real valued function which is three times continuously differentiable on an open real interval containing zero. Assume that g(0) = g'(0) = g''(0) = 0. Then $g(x)/x^2$ is continuously differentiable and

$$egin{aligned} H^*_{\scriptscriptstyle \phi}(x,\,y) &= K_{\scriptscriptstyle g}(x,\,y) - K_{\scriptscriptstyle g}(x,\,0) - K_{\scriptscriptstyle g}(0,\,y) + K_{\scriptscriptstyle g}(0,\,0) \ &= xy rac{g(x)}{x^2} - rac{g(y)}{y^2} \ &= xy K_{\scriptscriptstyle h}(x,\,y) \;, \end{aligned}$$

where we set $h(x) \equiv g(x)/x^2$. Thus, Lemma 8 says that $K_g(x, y)$ is conditionally positive definite if and only if $xyK_h(x, y)$ is positive definite, and Lemma 9 says this is equivalent to the kernel $K_h(x, y)$ being positive definite. We conclude that $K_g(x, y)$ is conditionally positive definite if and only if $K_h(x, y)$ is positive definite. But this means that $h(x) = g(x)/x^2 \in \mathfrak{M}_{\infty}(0)$ and hence h has the integral representation (2). The normalization we assumed for g can always be attained by subtracting a suitable quadratic polynomial, since Lemma 6 shows that every such polynomial has a conditionally positive definite difference quotient kernel. We summarize our results as

THEOREM 10. Let g be real valued function on an open real interval I containing zero. The following are equivalent:

(a) The function g is three times continuously differentiable and the kernel $K_g(x, y) = [g(x) - g(y)]/(x - y)$ is conditionally positive definite on I.

(b) The function g has the form

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^2 f(x)$$
,

where a_0 , a_1 , a_2 are real numbers, $a_3 \ge 0$ and $f \in \mathfrak{M}_{\infty}(0)$. (c) The function g has the form

$$g(x) = a_{\scriptscriptstyle 0} + a_{\scriptscriptstyle 1} + a_{\scriptscriptstyle 2} x^2 + \int_{-\varepsilon}^{\varepsilon} rac{x^3 d\mu}{1 - xt}$$

where a_0, a_1, a_2 are real numbers, $\varepsilon \ge 0$ and $d\mu$ is a nonnegative bounded measure.

It should be noted that it is sufficient in (a) to assume only that g is continuously differentiable; the condition on the kernel then implies that g is analytic [6]. This characterization of the functions g such that $K_g(x, y)$ is conditionally positive definite was obtained first by C. FitzGerald [2] using less elementary results on analytic kernels.

REMARK 2. Lemma 2 and its corollaries are also useful in probability theory where one is interested in continuous Hermitian kernels of the form K(x, y) = f(x - y), f(0) = 1. Such a kernel is positive definite if and only if f(x) is the Fourier transform of a (unique) probability measure on the line, i.e., f(x) is a *characteristic* function; this kernel is infinitely divisible if and only if the measure is infinitely divisible. If f(x) is the characteristic function of an infinitely divisible probability measure, then the kernel H(x, y) =lnf(x - y) is conditionally positive definite and has nonpositive real part since $|f(x)| \leq 1$ for all real x. Thus, the kernel $H(x, y) = \lambda - lnf(x - y)$ satisfies the hypotheses of Lemma 2 if $\lambda > 0$ and hence the kernel

$$\frac{\lambda}{\lambda - lnf(x - y)}$$

is infinitely divisible by Corollary 4. But this means that the function $\phi(x) = \lambda/(\lambda - lnf(x))$ is an infinitely divisible characteristic function whenever f is an infinitely divisible characteristic function and $\lambda > 0$. This result was obtained by F. W. Steutel [7] from very different considerations.

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