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Suppose S is a connected, locally connected, complete Moore space having no cut point and in which the Jordan Curve Theorem holds. Thus, suppose S satisfies R. L. Moore's Axioms 0, 1-4. Certain extensions and applications of earlier results of the author are established. In particular, modified forms of the Torhorst theorem and a plane theorem of R. L. Moore are shown to hold in the space S, two theorems concerning the separation of S by compact dendrons are extended to Peano continua and another to certain Menger regular curves. Finally, a general method of constructing certain pathological spaces is given.

In [1], Gref showed that if the space S is metrizable (which does not follow from the axioms indicated), which implies separability [5], [7], and no arc separates S, then many of the known separation theorems of plane topology can be established for the space S. His method of argument, however, is not valid in the present setting and indeed, many of the theorems he establishes are not true for the space here considered.

Extensive use is made of the results and terminology of [6] and [2]. The term "continuous curve" is to mean any connected, locally connected, closed point set, whether compact or not. A point E of a continuous curve M is an endpoint of M if and only if E is an endpoint of every arc in M containing E. If M is compact, this is equivalent to the definition in [8, p. 64].

THEOREM 1 (Modified Torhorst Theorem). *If U is a complementary domain of a Peano continuum M, there exists a Menger regular curve in M which is an irreducible continuum about the boundary of U.*

Proof. Let ω denote a point of U and for each simple closed curve J in M, let D_J denote the interior of J with respect to ω; that is, let D_J denote the complementary domain of J not containing ω. If M contains no simple closed curve, the stated conclusion follows immediately from the fact that M is a Menger regular curve and thus is hereditarily locally connected [8, p. 99]. Let M' denote the set of all points of M not belonging to D_J for any simple closed curve J in M. Clearly, M' is a nonempty closed subset of the compact point set M. Suppose D is a proper domain with respect to M

containing M'. For each point X of $M - D$, there is a simple closed curve J in M such that D_J contains X. Since $M - D$ is compact and D_J is open, there exists a finite collection K of simple closed curves in M such that every point of $M - D$ belongs to D_J for some simple closed curve J in K. It follows with the aid of Theorem 17, Chapter 3 of [6], that there exists a finite disjoint collection L of simple domains covering $M - D$ such that each of them is D_J for some simple closed curve J in M. If D_J belongs to L, it follows with the aid of Theorem 50, Chapter 1 of [6] that $M - D_J$ is connected, and a straightforward argument shows that $M - D_J$ is a continuous curve. It follows that $M - L^*$ is connected. $M - L^*$ is thus a connected, closed point set containing M' and lying in D. It follows that M' is connected. A modification of Jones' argument for Theorem 1 of [3] shows M' is locally connected.

Suppose O and P are points of M' which are not separated from each other in M' by any point. By Theorem 59, Chapter 2 of [6], there is a simple closed curve J in M containing O and P. Let E and F denote points of J such that $E + F$ separates O from P in J. If $E + F$ does not separate O from P in M', then M' contains a θ-curve and this leads to a contradiction. Hence, M' is a regular curve. Let N denote a subset of M' which is an irreducible continuum about the boundary of U. N is a regular curve and each two points of N are separated from each other in N by some pair of points.

THEOREM 2 (Modified Moore Theorem). *If a Peano continuum M separates the point A from the point B, then M contains an arc or a simple closed curve which separates A from B.*

Proof. By Theorem 1, there is a regular curve N lying in M which is an irreducible continuum about the boundary of the component U of $S - M$ containing A. Let V denote the component of $S - N$ containing B and N' a subcontinuum of N irreducible about the boundary of V. If the regular curve N' contains no simple closed curve, it follows from Theorem 5 of [2] that N' contains an arc which separates A from B. If N' contains a simple closed curve J, it follows from Theorem 5 of [3] that J separates A from B.

It might at first appear that a similar argument would establish the stronger result that if U is a complementary domain of a Peano continuum M and A is a point of $S - M - U$, then the outer boundary of U with respect to A is an arc or a simple closed curve. However, this is not true. The set V of the argument above need not be a component of $S - \bar{U}$.

The next two theorems are sharper results than Theorems 9 and

10 of [2].

THEOREM 3. *If U is a connected domain and M is a Peano continuum which is an irreducible continuum about the boundary of U and does not intersect U, then every endpoint of M and every simple closed curve in M is a subset of the boundary of U.*

Proof. It follows from Theorem 1 that M is a regular curve. Suppose E is an endpoint of M not on the boundary of U. There is a region R containing E but no boundary point of U. There is a point A which separates E from $M - R \cdot M$ in M. Let H and K denote mutually separated point sets whose sum is $M - A$ such that H contains $M - R \cdot M$. $H + A$ is a proper subcontinuum of M containing the boundary of U. This is a contradiction. Suppose J is a simple closed curve in M which is not a subset of β, the boundary of U, and D is the component of $S - J$ containing U. There is an arc AXB on J which contains no point of β. If there exists a free segment T of M lying in J, then $M - T$ is a proper subcontinuum of M containing β. Thus, the branch points of M are dense in AXB. Let Y_1, Y_2, Y_3, $\cdots$ denote branch points of M in AXB and Y_1Z_1, Y_2Z_2, Y_3Z_3, $\cdots$ arcs in M such that for each n, Z_n is a boundary point of U and $s(Y_nZ_n)$ is a subset of D. Suppose for some two integers i and j, Y_iZ_i and Y_jZ_j have a point in common. Then $Y_iZ_i + Y_jZ_j$ contains an arc Y_iOY_j lying except for its endpoints in D. A contradiction results in much the same manner as in the argument for Theorem 2. Thus, if i and j are distinct integers, Y_iZ_i and Y_jZ_j are mutually exclusively arcs having one endpoint in β and one endpoint in AXB. This contradicts Theorem 79, Chapter 2 of [6].

THEOREM 4. *No endpoint of S is a boundary point of three complementary domains of a Peano continuum.*

Proof. Suppose the endpoint E of S is a boundary point of the three complementary domains U_1, U_2 and U_3 of the *Peano* continuum M. It follows from Theorem 2 above and Theorems 10, 12 and 13 of [2], that there is an arc A_1E in M which is irreducible with respect to being an arc which separates U_1 from U_2 and such that (1) $S - A_1 E$ has only two components, one, V_1, containing U_1 and the other, V_2, containing U_2, (2) A_1E is the boundary of V_1 and of V_2. One of the sets V_1 and V_2 contains U_3. Suppose that one is V_2. $M \cdot (V_2 + EA_1)$ contains the boundaries of U_2 and U_3 and hence contains an arc EA_2 which separates U_2 from U_3 and which is a subset of the boundary of each of its complementary domains. Let H_2 and H_3 denote the components of $S - EA_2$ containing U_2 and U_3, respectively. One of the sets H_2 and H_3 contains V_1. Suppose that one is H_3. Then $K_2 = H_2$ is a component of $S - (EA_1 + EA_2)$ and has boundary EA_2. $K_1 = V_1$ is a component of $S - (EA_1 + EA_2)$ with boundary EA_1. Let K_3 denote the component of $S - (EA_1 + EA_2)$ containing U_3. There is an integer i, either 1 or 2, such that EA_i contains a sequence X_1, X_2, X_3, $\cdots$ of points converging to E and such that for each n, X_n is accessible from K_3. The set of all points of EA_i accessible from K_i is dense in EA_i. So there is a sequence Y_1, Y_2, Y_3 $\cdots$ of points of EA_i converging to E and accessible from K_i. A slight modification of Jones' argument for Proposition 4 of [4] shows that a contradiction results.

THEOREM 5. *If M is a Peano continuum and U is a component of $S - M$ such that $Bd(\bar{U})$ contains no endpoint of S, then $Bd(U)$ is a Peano continuum.*

Proof. There exists a regular curve N in M which is irreducible about the boundary of $\bar{U}$. It suffices to show that N is a subset of the boundary of U. Suppose X is a point of N. If X is a boundary point of some component V of $S - (N + \bar{U})$, then, since the boundary of V is a simple closed curve J by Theorems 2 and 3 above and Theorem 5 of [2], X belongs to J and J lies in the boundary of U. Suppose X is not a boundary point of any component of $S - (N + \bar{U})$. By an argument similar to that for Theorem 8 of [2] using Theorem 25 of [4], it follows that no region is a subset of a compact regular curve. Thus, there exist components $D_1, D_2, D_3, \cdots$ of $S - (N + \bar{U})$ such that X is a limit point of the sum of their boundaries $J_1, J_2, J_3, \cdots$, respectively. For each n, J_n is a simple closed curve and hence is a subset of the boundary of U. Therefore, X is a boundary point of U.

COROLLARY. *Under the hypothesis of Theorem 5, if E is a component of $S - M$ other than U, then the outer boundary of U with respect to E is a simple closed curve.*

Thus, Theorem 43, Chapter 4 of [6] remains true if Axiom 5 is replaced by the requirement that M contain no endpoint of S.

THEOREM 6. *If U and V are complementary domains of the continuous curve M, E and F are points of M which are boundary points of both U and V and some arc from E to F in M separates U from V, then every arc from E to F in M separates U from V.*

Proof. Suppose A and B are two arcs from E to F in M such that A separates U from V but B does not. It follows from Theorems 10, 12 and 13 of [2] that $S - A$ has only two components and A is the boundary of each of them. Let U' and V' denote these two components containing U and V, respectively. There is an arc CD not intersecting B such that C is in U and D is in V. Let G denote the collection of all components of $A - A \cdot B$ which intersect CD. G is finite since it is a disjoint collection of domains with respect to A covering the compact closed set $A \cdot (CD)$. It follows from a slight modification of the argument for Theorem 110, Chapter 1 of [6] that there do not exist two elements g_1 and g_2 of G and infinitely many components t_1, t_2, t_3, $\cdots$ of $CD - (CD) \cdot A$ such that for each n, t_n has one endpoint in g_1 and one in g_2. Hence, there exist two finite sequences $s(L_1X_1R_1)$, $s(L_2X_2R_2)$, $\cdots$, $s(L_nX_nR_n)$ and Z_1, $Z_2 \cdots$, Z_n such that (1) for each $i \leqq n$, $s(L_iX_iR_i)$ is an element of G containing the point Z_i, (2) the interval X_iZ_i of CD does not interset any element of G distinct from $s(L_iX_iR_i)$, (3) $CX_1 - X_1$ is a subset of U' and for each $i < n$, $s(Z_iX_{i+1})$ and $Z_nD - Z_n$ do not intersect A, (4) $L_iX_iR_i$ is not $L_{i+1}X_{i+1}S_{i+1}$. (Other repetitions are possible.) For each $i \leqslant n$, let $s(L_iY_iR_i)$ denote the component of $B - A \cdot B$ with the endpoints indicated, let J_i denote the simple closed curve $L_iY_iR_iX_iL_i$ and I_i the component of $S - J_i$ not containing E. I_i intersects neither U not V and does not contain F.

I_1 is a subset of either U' or V'. Suppose I_1 is a subset of U'. There is an arc RHT lying except for R and T in V' such that (1) the order RX_1T is true on $s(L_1X_1R_1)$ and (2) the segment RT of A contains $(CD) \cdot (R_1X_1L_1)$. Let I' denote the complementary domain of the simple closed curve $J' = L_1X_1R_1HL_1$ not containing E. I' is a subset of V'. Let $J'' = RHT + L_1Y_1R_1 + (L_1R_1 - RT)$ and I'' its complementary domain not containing E. $I'' = I_1 + I' + s(RT)$. Thus, I'' contains X_1. CX_1 does not intersect $L_1Y_1R_1$ since CD does not intersect B. CX_1 does not intersect $L_1X_1R_1 - RT$ since RT contains $(CD) \cdot (L_1X_1R_1)$ and CX_1 does not intersect $s(RHT)$ since $s(RHT)$ lies in V' and $CX_1 - X_1$ lies in U'. Since CX_1 does not intersect J'', it is a subset of I''. It follows that C does not belong to U. This is a contradiction. Therefore, I_1 is a subset of V'.

There exists an arc POQ lying except for its endpoints in U' such that $s(PQ)$ of $s(L_1X_1R_1)$ contains $CD \cdot L_1X_1R_1$. Let I' denote the complementary domain of $POQX_1P$ not containing E. I' is a subset of U' and $I_1 + I' + s(PX_1Q)$ is the interior I'' with respect to E of $J'' = POQ + L_1Y_1R_1 + (L_1X_1R_1 - PQ)$. The arc Z_1X_2 contains the point Z_1 of I'' and the point X_2 of $S - \bar{I}''$. Hence, $s(Z_1X_2)$ contains a point of J''. If C' is such a point, then C' is on $s(POQ)$ and thus is in U'.

It follows that $s(Z_1X_2)$ is a subset of U'. An induction argument using a modification of the argument above shows that CD is a subset of $U' + EF + I_1 + I_2 + \cdots + I_n$, so that D is not a point of V. This is a contradiction.

THEOREM 7. *If the Peano continuum M separates S and contains one of the complementary domains of each simple closed curve it contains, then every component of $S - M$ has an endpoint of S on its boundary.*

Proof. Suppose U and V are components of $S - M$ and N is a regular curve which is an irreducible continuum about the boundary of U. By Theorem 2, N contains an arc or simple closed curve which separates U from V. Obviously, no simple closed curve in N has this property. It follows from Theorem 5 of [2] that N contains an endpoint of S, which by Theorem 3 is a boundary point of U.

The next result strengthens Theorem 21 of [2]. If the stipulation that M be a compact regular curve is replaced by the stipulation that M be a Peano continuum, the resulting conjecture is false.

THEOREM 8. *If the compact regular curve M does not have infinitely many complementary domains but separates S, then there exist two components of $S - M$ with connected boundaries such that if U is one of them, the boundary of $\bar{U}$ is a simple closed curve or a subset of an arc.*

Proof. The argument procedes by induction on n, the number of components of $S - M$. The case $n = 2$ follows easily with the aid of Theorems 12 and 13 of [2] and Theorem 2 above. Suppose the theorem is true for all integers n greater than 1 and less than k but is not true for $n = k$. Let the statement that the complementary domain U of the closed point set N has property P relative to N mean the boundary of U is connected and the boundary of $\bar{U}$ is a simple closed curve or a subset of an arc. Let M denote a compact regular curve such that $S - M$ has only k components but does not have 2 components having property P relative to M. If some component of $S - M$ has property P relative to M, let U denote it and if no such component exists, let U denote any component of $S - M$.

Suppose there is an endpoint E of S on the boundary of $\bar{U}$. There is a component V of $S - M - U$ whose boundary contains E and there is a subcontinuum N of M not containing E but containing the boundary of every component of $S - M$ other than U and V. Since no compact regular curve contains a region, $S - N$ has only

$k-1$ components and has no more than one component having property P relative to N. This is a contradiction.

Suppose there is no endpoint of S on the boundary of $\bar{U}$. It follows with the aid of Theorems 1 and 2 that there exists a subcontinuum N of M which is irreducible with respect to being a continuum which separates each two components $S-M$ from each other and which is the sum of a finite number of arcs and does not contain infinitely many simple closed curves. It follows from Theorem 5 of [2] and Theorems 1 and 3 above that N contains a simple closed curve J which lies in the boundary of U', the component of $S-N$ containing U. J contains a free segment T of N which does not intersect the boundaries of three components of $S-N$. T does intersect the boundaries of two components of $S-N$, since otherwise, N is not irreducible with respect to being a continuum which separates each two components of $S-M$ from each other. Let $N'=N-T$. N' is a compact regular curve such that $S-N'$ has only $k-1$ components but has no more than one component having property P relative to N'. This is a contradiction.

THEOREM 9. *If k and n are nonnegative integers and M is a Peano continuum which contains only n simple closed curves and only k endpoints of S, then $S-M$ has at most $n+k+1$ components.*

Proof. Suppose J is the only simply closed curve in M. Let D and I denote the components of $S-J$. The boundary of every component of $S-M$ contains either J or an endpoint of S. It follows with the aid of Theorem 4 above and Theorem 4 of [3] that $S-M$ does not have infinitely many components. There is a subset M' of M which is irreducible with respect to being a continuum which separates each two components of $S-M$ from each other. Neither $Cl(C\cdot M')$ nor $Cl(I\cdot M')$ contains J. Thus, each of these closed sets is a subset of a dendron in M'. Let $k(D)$ and $k(I)$ denote the number of endpoints of S in $D\cdot M'$ and $I\cdot M'$, respectively. It follows from a modification of Theorem 14 of [2] similar to Theorem 6 of [2] that $D-D\cdot M'$ and $I-I\cdot M'$ have no more than $k(D)+1$ and $k(I)+1$ components, respectively. Thus, $S-M'$ has at most $k(D)+k(I)+2=k+2$ components. Hence, the theorem is true for $n=1$.

Suppose the theorem is true for $n<p$ and M is a Peano continuum satisfying the hypothesis of the theorem for $n=p$. Define M' as above and let J denote some simple closed curve in M'. (If M' contains no simple closed curve, Theorem 14 of [2] gives the desired result.) It follows that there exist two components U and V of $S-M'$ separated from each other by J and a free segment T of M' in

J which is a subset $Bd(U) \cdot Bd(V)$ and intersects the boundary of no other component of $S - M'$. $M' - T$ is a Peano continuum containing no more than $p - 1$ simple closed curves, only k endpoints of S and has no more than $p + k$ complementary domains. Thus, $S - M$ has no more than $p + k + 1$ components.

The next two theorems are very useful in constructing examples of spaces satisfying Axioms 0, 1-4 and having endpoints. The proof of the first theorem is similar to arguments given in [2] and will be omitted.

THEOREM 10. *If the ray R contains no endpoints of S and does not separate S, then $S - R$, with the subspace topology, satisfies Axioms* 0, 1-4.

THEOREM 11. *Suppose S is the set of all points of a space satisfying Axioms* 0, 1-4, *J is a simple closed curve with complementary domains D and I, AEB is an arc on J, P is a point of $J - AEB$ accessible from I, EF is an arc lying except for E in I and A_1B_1, $A_2B_2 \cdots$ is a sequence of mutually exclusive arcs in I converging to AEB, containing no endpoint of S and such that for each n, E_n is the only point of A_nB_n on EF and the order $FE_nE_{n+1}E$ on EF holds. Suppose $M = (J - P - E) + D + \sum_n(A_nB_n - E_n)$. Suppose finally, that $S - M$, with the subspace topology, is locally connected at P, E and E_n for each n. Then the subspace $S - M$ satisfies Axioms* 0, 1-4 *and E is an endpoint of $S - M$.*

Proof. Without loss of generality, it may be assumed that F is not an endpoint of S. Since M is the sum of countably many closed sets, $S - M$ is an inner limiting set. By Theorem 168, Chapter 1 of [6], the subspace $S - M$ satisfies Axioms 0 and 1. Clearly, $S - M$ satisfies Axiom 2.

Suppose $S - \bar{M}$ is not connected. Let U' and V' denote components of $S - \bar{M}$. U' and V' contain components U and V, respectively, of $S - (\bar{M} + EF)$. Each component of $S - (\bar{M} + EF)$ intersects $A_nB_n - E_n$ for infinitely many n. There exist positive integers $k < i < j < m$ and points X_i on $A_iB_i - E_i$, X_j on $A_jB_j - E_j$, Y_j, Y_k on $A_kB_k - E_k$ and Y_m on $A_mB_m - E_m$ such that X_i and X_j are accessible from U and Y_k and Y_m are accessible from V. Let X_iOX_j and Y_kQY_m denote arcs lying in U and V, respectively. Let J_U denote the simple closed curve formed by X_iOX_j, the intervals or points X_iE_i of A_iB_i and X_jE_j of A_jB_j and the interval E_iE_j of EF, and let I_U denote the interior of J_U with respect to D. Let J_V denote the simple closed curve formed by Y_kQY_m, Y_kE_k, Y_mE_m and E_kE_m, and let I_V denote the interior of J_V with respect to D. $Cl(I_U \cdot M) + EF$ is a compact dendron containing no endpoint of S and whose intersection with J_U is connected. By Theorem 6 of [2], $I_U - I_U \cdot \bar{M}$ is connected. It follows that $I_U - \bar{M}$ is a subset of U and hence does not intersect V. Similarly, $I_V - \bar{M} \cdot I_V$ is a subset of V and does not intersect U. Let J' denote the simple closed curve $E_jE_mY_mQY_kE_kE_iX_iOX_jE_j$ and I' its interior with respect to D. $I' = I_U + I_V + s(E_iE_j)$. It follows that $I' - \bar{M} \cdot I'$ is a connected subset of $S - \bar{M}$ intersecting both U' and V'. This is a contradiction. Thus, $S - \bar{M}$ is connected.

Since $S - \bar{M}$ is connected, P is not a cut point of $S - M$. Suppose some point X is a cut point of $S - M$. There exists an arc XE' lying except for E' in $S - M - EF$ such that E' is on EF but is not E_n for any n. $M + (XE' - E')$ has all the properties of M used in the above argument to show $S - \bar{M}$ is connected. Thus, $S - Cl(M + XE')$ is connected. Hence, $S - M$ has no cut point.

Every arc in $S - M$ containing E contains E_n for sufficiently large n. For suppose EXQ is an arc in $S - M$. Let ZQ' denote an arc lying except for Z and Q' in $S - M$ such that Z is on $J - AEB$ and ZQ' is irreducible from J to EXQ. Let J_A denote the simple closed curve formed by the arc ZQ', the interval $Q'E$ of EXQ and the arc ZAE on J and let I_A denote the interior of J_A with respect to D. Let WO' denote an arc lying except for W and O' in $S - \bar{M}$ such that W is on $J - ZEB$ and WO' is irreducible from J to J_A. Let J_B denote the simple closed curve formed by the arc WO', the arc $O'E$ of J_A not containing A and the arc EBW on J and let I_B denote the interior of J_B with respect to D. I_A and I_B are mutually exclusive. It follows that $I_A + D + s(ZAE)$ is a domain containing $AE - E$ and $I_B + D + s(WBE)$ is a domain containing $BE - E$. Since the sequence A_1B_1, A_2B_2, $\cdots$ converges to AEB and does not intersect D, it follows that for sufficiently large n, A_nB_n intersects both I_A and I_B. Hence, for sufficiently large n, A_nB_n intersects EXQ. Since EXQ is a subset of $S - M$, it follows that E_n is eventually on EXQ. It follows E is an endpoint of $S - M$.

It remains to be shown that Axiom 4 is satisfied. Suppose T is a simple closed curve in $S - M$ not containing P. There is a positive integer N such that T does not intersect EE_N. Let D_T and I_T denote the interior and exterior, respectively, of T with respect to E. There is an arc C on T such that $H = C$ plus all components of $Cl(I_T \cdot M)$ intersecting T is a compact dendron. It follows from Theorem 6 of [2] that $I_T - H \cdot I_T$ is connected. I_T, as a subspace of S, satisfies Axioms 0, 1-4 by Theorem 23, Chapter 3 of [6]. In this subspace, each component of $Cl(I_T \cdot M)$ intersecting T is a ray containing no endpoint of the subspace and, from the argument above, does not separate the subspace. Since there are not infinitely many of these rays, $I_T - I_T \cdot H$ satisfies Axioms 0, 1-4, by Theorem 10. The first part of the proof of this theorem now applies to the subspace $I_T - I_T \cdot H$, with M replaced by $M \cdot I_T - H \cdot I_T$, to show that $I_T - I_T \cdot M$ is connected. Clearly, T is the boundary with respect to $S - M$ of I_T. $Cl(M \cdot D_T)$ is the sum of a finite number of arcs. It follows that there is a compact dendron M' containing $Cl(M \cdot D_T)$, lying in $\bar{D}_T$, intersecting T in a connected set and containing no endpoint of S. By Theorem 6 of [2], $D_T - M' \cdot D_T$ is connected. T is the boundary with respect to $S - M$ of D_T. Thus, $S - M - T$ is the sum of two mutually exclusive connected domains each having boundary T.

Suppose T is a simple closed curve in $S - M$ containing P. Define I_T and D_T as before. As above, $D_T - M \cdot D_T$ is connected and T is the boundary with respect to $S - M$ of D_T and I_T. Suppose $I_T - \bar{M} \cdot I_T$ is not connected and U' and V' are two of its components. For each n, $H_n = J + T + (E_n F + A_1 A_1 + A_2 B_2 + \cdots + A_n B_n)$. $\bar{I}_T$ is a compact regular curve containing no endpoint of S and only two simple closed curves. Hence, it has at most three complementary domains by Theorem 9. D, $I_T - \bar{D} - H_n$ and $D_T - H_n$ are three mutually separated sets whose sum is $S - H_n$. Thus, these three sets are connected. In particular, $I_T - \bar{D} - H_n$ is connected. It follows that the boundaries of U' and V' intersect $A_n B_n - E_n$ for infinitely many n. A modified form of the argument in the second paragraph of this proof shows that a contradiction results. This completes the proof.

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