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## MOORE SPACES AND $w$ $\Delta$ -SPACES

RICHARD EARL HODEL

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R. E. HODEL

*This paper is dedicated to Professor J. H. Roberts  
on the occasion of his sixty-fifth birthday.*

**This paper is a study of conditions under which a  $w\mathcal{A}$ -space is a Moore space. In §2 we introduce the notion of a  $G_\delta^*$ -diagonal and show that every  $w\mathcal{A}$ -space with a  $G_\delta^*$ -diagonal is developable. In §3 we prove that every regular  $\theta$ -refinable  $w\mathcal{A}$ -space with a point-countable separating open cover is a Moore space. In §4 we introduce the class of  $\alpha$ -spaces and show that a regular  $w\mathcal{A}$ -space is a Moore space if and only if it is an  $\alpha$ -space. Finally, in §5 we study a new class of spaces which generalizes both semi-stratifiable and  $w\mathcal{A}$ -spaces.**

1. Preliminaries. We begin with some definitions and known results which will be used throughout this paper. Unless otherwise stated no separation axioms are assumed; however regular spaces are always  $T_1$  and paracompact spaces are always Hausdorff. The set of natural numbers will be denoted by  $N$ .

Let  $X$  be a set,  $\mathcal{G}$  a cover of  $X$ ,  $x$  an element of  $X$ . The *star of  $x$  with respect to  $\mathcal{G}$* , denoted  $\text{st}(x, \mathcal{G})$ , is the union of all elements of  $\mathcal{G}$  containing  $x$ . The *order of  $x$  with respect to  $\mathcal{G}$* , denoted  $\text{ord}(x, \mathcal{G})$ , is the number of elements of  $\mathcal{G}$  containing  $x$ .

A space  $X$  is *developable* if there is a sequence  $\mathcal{G}_1, \mathcal{G}_2, \dots$  of open covers of  $X$  such that, for each  $x$  in  $X$ ,  $\{\text{st}(x, \mathcal{G}_n) : n = 1, 2, \dots\}$  is a fundamental system of neighborhoods of  $x$ . Such a sequence of open covers is called a *development* for  $X$ . A regular developable space is called a *Moore space*. Bing [1] proved that every paracompact Moore space is metrizable.

According to Borges [3] a space  $X$  is a  $w\mathcal{A}$ -space if there is a sequence  $\mathcal{G}_1, \mathcal{G}_2, \dots$  of open covers of  $X$  such that, for each  $x$  in  $X$ , if  $x_n \in \text{st}(x, \mathcal{G}_n)$  for  $n = 1, 2, \dots$  then the sequence  $\langle x_n \rangle$  has a cluster point. Such a sequence of open covers is called a  *$w\mathcal{A}$ -sequence* for  $X$ . Clearly every countably compact space is a  $w\mathcal{A}$ -space, and in [3] Borges proved that every developable space and every  $M$ -space is a  $w\mathcal{A}$ -space. For the relationship between  $w\mathcal{A}$ -spaces, strict  $p$ -spaces, and  $p$ -spaces, see [6].

A space  $X$  is *subparacompact* if every open cover of  $X$  has a  $\sigma$ -discrete closed refinement. Every paracompact space is subparacompact [16], and in [8] Creede proved that every semi-stratifiable space is subparacompact. For further properties of subparacompact spaces see [5], [11], and [15].

A space  $X$  is  $\theta$ -refinable if for each open cover  $\mathcal{V}$  of  $X$  there is a sequence  $\mathcal{G}_1, \mathcal{G}_2, \dots$  of open refinements of  $\mathcal{V}$  such that, for each  $x$  in  $X$ , there is a  $n$  in  $N$  such that  $\text{ord}(x, \mathcal{G}_n)$  is finite. Such a sequence of open covers is called a  $\theta$ -refinement of  $\mathcal{V}$ . In [24] Wicke and Worrell state that every subparacompact space is  $\theta$ -refinable and that a countably compact  $T_1$  space is compact if and only if it is  $\theta$ -refinable.

2. Spaces with a  $G_\delta^*$ -diagonal. Recall that a space  $X$  has a  $G_\delta$ -diagonal if its diagonal  $\Delta = \{(x, x): x \text{ in } X\}$  is a  $G_\delta$ -subset of  $X \times X$ . The notion of a  $G_\delta$ -diagonal plays an important role in metrization theorems; see, for example, [2], [3], [7], [14], and [22].

Every semi-stratifiable Hausdorff space has a  $G_\delta$ -diagonal [8]. On the other hand the space  $[0, 1] \times [0, 1]$  with the lexicographic order is a compact perfectly normal space which fails to have a  $G_\delta$ -diagonal [14].

In [7] Ceder obtained this characterization of spaces with a  $G_\delta$ -diagonal.

**PROPOSITION 2.1.** (Ceder) *A space  $X$  has a  $G_\delta$ -diagonal if and only if there is a sequence  $\mathcal{G}_1, \mathcal{G}_2, \dots$  of open covers of  $X$  such that, for any two distinct points  $x$  and  $y$  of  $X$ , there is a  $n$  in  $N$  such that  $y \notin \text{st}(x, \mathcal{G}_n)$ .*

In light of this characterization of a  $G_\delta$ -diagonal and Borges' study of spaces with a  $\bar{G}_\delta$ -diagonal (see [3]), we introduce the following definition.

**DEFINITION 2.2.** A space  $X$  has a  $G_\delta^*$ -diagonal if there is a sequence  $\mathcal{G}_1, \mathcal{G}_2, \dots$  of open covers of  $X$  such that, for any two distinct points  $x$  and  $y$  of  $X$ , there is a  $n$  in  $N$  such that  $y \notin \text{st}(x, \mathcal{G}_n)^-$ . Such a sequence of open covers is called a  $G_\delta^*$ -sequence for  $X$ .

In [13] Kullman proved that every regular  $\theta$ -refinable space with a  $G_\delta$ -diagonal has a  $\bar{G}_\delta$ -diagonal. Since every space with a  $\bar{G}_\delta$ -diagonal has a  $G_\delta^*$ -diagonal, we have the following proposition.

**PROPOSITION 2.3.** *Every regular  $\theta$ -refinable space with a  $G_\delta$ -diagonal has a  $G_\delta^*$ -diagonal. In particular every regular semi-stratifiable space has a  $G_\delta^*$ -diagonal.*

The next result relates the  $G_\delta^*$ -diagonal property to the diagonal  $\Delta$ .

**PROPOSITION 2.4.** *Let  $X$  be a space, let  $\{V_n: n = 1, 2, \dots\}$  be a*

sequence of open subsets of  $X \times X$  containing  $\Delta$ , and suppose that  $\bigcap_{n=1}^{\infty} \bar{V}_n = \Delta$ . Then  $X$  has a  $G_\delta^*$ -diagonal. In particular, if  $X$  is Hausdorff and  $X \times X$  is perfectly normal then  $X$  has a  $G_\delta^*$ -diagonal.

*Proof.* For  $n = 1, 2, \dots$  let  $\mathcal{G}_n = \{G \subseteq X: G \text{ open, } G \times G \subseteq V_n\}$ . Since  $V_n$  is open and contains  $\Delta$ ,  $\mathcal{G}_n$  covers  $X$ . To show that  $\mathcal{G}_1, \mathcal{G}_2, \dots$  is a  $G_\delta^*$ -sequence for  $X$ , let  $x$  and  $y$  be distinct points of  $X$ . Choose  $n$  in  $N$  such that  $(x, y) \notin \bar{V}_n$ , and let  $U$  and  $W$  be open neighborhoods of  $x$  and  $y$  respectively such that  $(U \times W) \cap V_n = \emptyset$ . It follows that  $W \cap \text{st}(x, \mathcal{G}_n) = \emptyset$  and so  $y \notin \text{st}(x, \mathcal{G}_n)^-$ .

We now prove the main result in this section.

**THEOREM 2.5.** *Every  $w\Delta$ -space with a  $G_\delta^*$ -diagonal is developable.*

*Proof.* Let  $X$  be a space, let  $\mathcal{H}_1, \mathcal{H}_2, \dots$  be a  $w\Delta$ -sequence for  $X$ , and let  $\mathcal{K}_1, \mathcal{K}_2, \dots$  be a  $G_\delta^*$ -sequence for  $X$ . For each positive integer  $n$  let

$$\mathcal{G}_n = \left\{ G: G = \left( \bigcap_{i=1}^n H_i \right) \cap \left( \bigcap_{i=1}^n K_i \right), H_i \in \mathcal{H}_i, K_i \in \mathcal{K}_i, i = 1, \dots, n \right\}.$$

It is easy to check that  $\mathcal{G}_{n+1}$  is an open refinement of  $\mathcal{G}_n$  for all  $n$  in  $N$  and that  $\mathcal{G}_1, \mathcal{G}_2, \dots$  is a  $w\Delta$ -sequence and a  $G_\delta^*$ -sequence for  $X$ .

Suppose that  $\mathcal{G}_1, \mathcal{G}_2, \dots$  is not a development for  $X$ . Then there is a point  $x$ , a neighborhood  $W$  of  $x$ , and a sequence  $\langle x_n \rangle$  such that for all  $n$ ,  $x_n \in \text{st}(x, \mathcal{G}_n)$  and  $x_n \notin W$ . Since  $\mathcal{G}_1, \mathcal{G}_2, \dots$  is a  $w\Delta$ -sequence for  $X$ , the sequence  $\langle x_n \rangle$  has a cluster point  $p$ . Clearly  $p \notin W$  so  $p \neq x$ . Since  $\mathcal{G}_1, \mathcal{G}_2, \dots$  is a  $G_\delta^*$ -sequence for  $X$ , there is a positive integer  $k$  and a neighborhood  $V$  of  $p$  such that  $V \cap \text{st}(x, \mathcal{G}_k) = \emptyset$ . Now for  $n \geq k$ ,  $x_n \in \text{st}(x, \mathcal{G}_n) \subseteq \text{st}(x, \mathcal{G}_k)$  and so  $x_n \notin V$ . This contradicts the fact that  $p$  is a cluster point of  $\langle x_n \rangle$ . Thus  $\mathcal{G}_1, \mathcal{G}_2, \dots$  is a development for  $X$ .

**COROLLARY 2.6.** *The following are equivalent for a regular  $w\Delta$ -space  $X$ :*

- (a)  $X$  is a Moore space.
- (b)  $X$  is semi-stratifiable.
- (c)  $X$  is  $\theta$ -refinable and has a  $G_\delta$ -diagonal.
- (d)  $X$  has a  $G_\delta^*$ -diagonal.

*Proof.* The implication (a)  $\Rightarrow$  (b) is due to Creede [8]; (b)  $\Rightarrow$  (c) follows from results by Creede [8] and Wicke and Worrell [24]; (c)  $\Rightarrow$  (d) follows from Proposition 2.3; (d)  $\Rightarrow$  (a) follows from Theorem 2.5.

REMARK 2.7. The equivalence of (a) and (b) was first proved by Creede in [8], and the equivalence of (a) and (c) is due to Siwiec [23]. It is not known if every regular  $w\Delta$ -space with a  $G_s$ -diagonal is a Moore space. For a study of  $p$ -spaces with a  $G_s$ -diagonal, see [13].

COROLLARY 2.8. *The following are equivalent for a regular countably compact space  $X$ :*

- (a)  $X$  is metrizable.
- (b)  $X \times X \times X$  is completely normal.
- (c)  $X \times X$  is perfectly normal.
- (d)  $X$  has a  $G_s^*$ -diagonal.

*Proof.* Clearly (a)  $\Rightarrow$  (b); (b)  $\Rightarrow$  (c) follows from a theorem due to Katětov [12]; (c)  $\Rightarrow$  (d) follows from Proposition 2.4. To prove (d)  $\Rightarrow$  (a) observe that  $X$  is a Moore space (by Corollary 2.6) and recall that every countably compact Moore space is metrizable.

3. **Separating covers.** In 1938 Filippov [9] proved that every paracompact  $M$ -space with a point-countable base is metrizable. Filippov's theorem was generalized by Burke and Stoltenberg in [4], and recently Burke [6] obtained another generalization as follows.

BURKE'S THEOREM. *Every regular subparacompact  $w\Delta$ -space with a point-countable base is a Moore space.*

In another direction Nagata [20] proved a metrization theorem which not only generalizes Filippov's theorem but a result by Okuyama as well [22]. In order to state Nagata's theorem succinctly we use the following terminology due to Michael [17]. A cover  $\mathcal{V}$  of a set  $X$  is said to be *separating* if given distinct points  $x$  and  $y$  of  $X$ , there is a  $V$  in  $\mathcal{V}$  such that  $x \in V$ ,  $y \notin V$ .

NAGATA'S THEOREM. *Every paracompact  $M$ -space with a point-countable separating open cover is metrizable.*

In this section we use the techniques developed by Burke, Filippov, Nagata, and Stoltenberg, together with the results in §2, to obtain a generalization of the abovementioned theorems by Burke and Nagata.

In light of the usefulness of the concept of a  $\theta$ -base in the study of developable spaces (see [24]), we begin with the following definition.

**DEFINITION 3.1.** A  $\theta$ -separating cover of a space  $X$  is a sequence  $\mathcal{G}_1, \mathcal{G}_2, \dots$  of open collections such that, for any two distinct points  $x$  and  $y$  in  $X$ , there is a  $n$  in  $N$  such that

- (a)  $\text{ord}(x, \mathcal{G}_n)$  is finite;
- (b) there is a  $G$  in  $\mathcal{G}_n$  such that  $x \in G$  and  $y \notin G$ .

The relationship between a  $\theta$ -separating cover and a  $G_\delta$ -diagonal is given by the following two propositions.

**PROPOSITION 3.2.** *Let  $X$  be a space with a  $\theta$ -separating cover. If every closed subset of  $X$  is a  $G_\delta$  then  $X$  has a  $G_\delta$ -diagonal.*

*Proof.* Let  $\mathcal{G}_1, \mathcal{G}_2, \dots$  be a  $\theta$ -separating cover of  $X$ . For each pair of positive integers  $n$  and  $k$  let  $\mathcal{H}_{nk} = \{H: H \neq \phi, H = \bigcap_{i=1}^k G_i, G_1, \dots, G_k \text{ distinct elements of } \mathcal{G}_n\}$  and let  $F_{nk} = X - \bigcup \{H: H \in \mathcal{H}_{nk}\}$ . Now  $F_{nk}$  is a closed set and so  $F_{nk} = \bigcap_{j=1}^\infty W_{nkj}$ , where each  $W_{nkj}$  is open. For  $j = 1, 2, \dots$  let  $\mathcal{K}_{nkj} = \mathcal{H}_{nk} \cup \{W_{nkj}\}$ . Then each  $\mathcal{K}_{nkj}$  is an open cover of  $X$  and the sequence  $\{\mathcal{K}_{nkj}: n, k, j \text{ in } N\}$  exhibits the  $G_\delta$ -diagonal property for  $X$ .

**PROPOSITION 3.3.** *Every  $\theta$ -refinable space with a  $G_\delta$ -diagonal has a  $\theta$ -separating cover.*

*Proof.* Let  $X$  be a  $\theta$ -refinable space and let  $\mathcal{G}_1, \mathcal{G}_2, \dots$  be open covers of  $X$  exhibiting the  $G_\delta$ -diagonal property for  $X$ . For each  $n$  in  $N$  let  $\mathcal{H}_{n1}, \mathcal{H}_{n2}, \dots$  be a  $\theta$ -refinement of  $\mathcal{G}_n$ . Then

$$\{\mathcal{H}_{nk}: n = 1, 2, \dots, k = 1, 2, \dots\}$$

is a  $\theta$ -separating cover of  $X$ .

The following lemmas, due to Burke and Miscenko [19], play a key role in the proof of our theorem. For the sake of completeness we sketch the proof of Burke's result. (See Remark 1.9 in [6]).

**LEMMA 3.4.** (Burke) *Let  $X$  be a regular,  $\theta$ -refinable  $w\Delta$ -space. Then there is a sequence  $\mathcal{G}_1, \mathcal{G}_2, \dots$  of open covers of  $X$  such that for each  $x$  in  $X$ ,*

- (a)  $C_x = \bigcap_{n=1}^\infty \text{st}(x, \mathcal{G}_n)$  is compact;
- (b)  $\{\text{st}(x, \mathcal{G}_n): n = 1, 2, \dots\}$  is a base for  $C_x$ .

*Proof.* Let  $\mathcal{V}_1, \mathcal{V}_2, \dots$  be a  $w\Delta$ -sequence for  $X$ . By induction on  $n$  construct for each positive integer  $n$  a sequence  $\mathcal{W}_{n1}, \mathcal{W}_{n2}, \dots$  of open covers of  $X$  such that

- (1) for  $k = 1, 2, \dots, \{\bar{W}: W \text{ in } \mathcal{W}_{nk}\}$  refines  $\mathcal{V}_n$  and  $\mathcal{W}_{ij}, 1 \leq i \leq n-1, 1 \leq j \leq n-1$ ;

(2) for each  $x$  in  $X$  there is a  $k$  in  $N$  such that  $\text{ord}(x, \mathcal{W}_{nk})$  is finite.

For  $n = 1, 2, \dots$  let  $\mathcal{G}_n = \mathcal{W}_{n1}$ . Then the sequence  $\mathcal{G}_1, \mathcal{G}_2, \dots$  satisfies properties (a) and (b).

**LEMMA 3.5.** (Miščenko) *Let  $\mathcal{V}$  be a point-countable collection of subsets of a set  $X$  and let  $M$  be a subset of  $X$ . Then there are at most countably many finite minimal covers of  $M$  by elements of  $\mathcal{V}$ .*

We now state and prove the main result in this section.

**THEOREM 3.6.** *Let  $X$  be a regular,  $\theta$ -refinable  $w\Delta$ -space with a point-countable separating open cover. Then  $X$  is a Moore space.*

*Proof.* We are going to show that  $X$  has a  $\theta$ -separating cover and that every closed subset of  $X$  is a  $G_\delta$ . It follows by Proposition 3.2 that  $X$  has a  $G_\delta$ -diagonal and hence by Corollary 2.6  $X$  is a Moore space.

Let  $\mathcal{V}$  be a point-countable separating open cover of  $X$ . We assume that  $X \in \mathcal{V}$ , and hence for every subset  $M$  of  $X$  there is a finite subcollection of  $\mathcal{V}$  which covers  $M$ , namely  $\{X\}$ . Let  $\mathcal{G}_1, \mathcal{G}_2, \dots$  be open covers of  $X$  such that for each  $x$  in  $X$ ,

(a)  $C_x = \bigcap_{n=1}^{\infty} \text{st}(x, \mathcal{G}_n)$  is compact;

(b)  $\{\text{st}(x, \mathcal{G}_n) : n = 1, 2, \dots\}$  is a base for  $C_x$ .

For each  $n$  in  $N$  let  $\mathcal{H}_{n1}, \mathcal{H}_{n2}, \dots$  be a  $\theta$ -refinement of  $\mathcal{G}_n$ . Recall that

(c)  $\mathcal{H}_{nk}$  refines  $\mathcal{G}_n$ ,  $k = 1, 2, \dots$ ;

(d) for each  $x$  in  $X$  there is a  $k$  in  $N$  such that  $\text{ord}(x, \mathcal{H}_{nk})$  is finite.

$X$  has a  $\theta$ -separating cover. For each pair of positive integers  $n$  and  $k$  and for each  $H$  in  $\mathcal{H}_{nk}$  let  $H(n, k, 1), H(n, k, 2), \dots$  be all finite minimal covers of  $H$  by elements of  $\mathcal{V}$ , and let

$$\mathcal{K}_{nkj} = \{H \cap V : H \in \mathcal{H}_{nk}, V \in H(n, k, j)\}.$$

To show that  $\{\mathcal{K}_{nkj} : n, k, j \text{ in } N\}$  is a  $\theta$ -separating cover of  $X$ , let  $x$  and  $y$  be two distinct points of  $X$ . Choose  $V_1$  in  $\mathcal{V}$  such that  $x \in V_1$  and  $y \notin V_1$ , and let  $\{V_1, \dots, V_t\}$  be a finite cover of  $C_x$  by elements of  $\mathcal{V}$  such that  $x \notin V_i$  for  $i = 2, \dots, t$ . Now  $C_x \subseteq \bigcup_{i=1}^t V_i$  and so by (b) there is a  $n$  in  $N$  such that  $\text{st}(x, \mathcal{G}_n) \subseteq \bigcup_{i=1}^t V_i$ . Choose  $k$  in  $N$  such that  $\text{ord}(x, \mathcal{H}_{nk})$  is finite, and let  $H$  be some element of  $\mathcal{H}_{nk}$  such that  $x \in H$ . Since  $\mathcal{H}_{nk}$  refines  $\mathcal{G}_n$ ,  $H \subseteq \text{st}(x, \mathcal{G}_n)$

and so  $H \subseteq \bigcup_{i=1}^t V_i$ . Choose a minimal subcollection of  $\{V_1, \dots, V_i\}$  which covers  $H$  and label it  $H(n, k, j)$ . Note that  $V_1 \in H(n, k, j)$ . Thus  $(H \cap V_1) \in \mathcal{H}_{nkj}$ ,  $x \in (H \cap V_1)$ , and  $y \notin (H \cap V_1)$ . Finally, suppose  $H_1, \dots, H_r$  are all elements of  $\mathcal{H}_{nk}$  containing  $x$ . Since  $H_i(n, k, j)$  is finite for  $i = 1, \dots, r$  it follows that  $\text{ord}(x, \mathcal{H}_{nkj})$  is finite. This completes the proof that  $X$  has a  $\theta$ -separating cover.

*Every closed subset of  $X$  is a  $G_\delta$ .* Let  $M$  be a closed subset of  $X$ . For each pair of positive integers  $n$  and  $k$ , and for each  $H$  in  $\mathcal{H}_{nk}$  such that  $H \cap M \neq \emptyset$ , let  $H(n, k, j)$ ,  $j = 1, 2, \dots$  be all finite minimal covers of  $H \cap M$  by elements of  $\mathcal{V}$ . By repeatedly counting a cover if necessary, we may assume that  $H(n, k, j)$  exists for all  $j$  in  $N$ . For  $j = 1, 2, \dots$  let  $H^*(n, k, j)$  denote the union of all elements of  $H(n, k, j)$ , and let  $W_{nkj} = \bigcup \{H \cap (\bigcap_{i=1}^j H^*(n, k, i)) : H \in \mathcal{H}_{nk}, H \cap M \neq \emptyset\}$ . Clearly each  $W_{nkj}$  is open and contains  $M$ . To complete the proof that  $M$  is a  $G_\delta$  it suffices to show that if  $x \notin M$  then there exist  $n, k$ , and  $j$  such that  $x \notin W_{nkj}$ .

First suppose that  $C_x \cap M = \emptyset$ . Choose  $n$  in  $N$  such that  $\text{st}(x, \mathcal{G}_n) \cap M = \emptyset$ , and let  $k$  and  $j$  be any positive integers. Suppose  $x \in W_{nkj}$ . Then there is a  $H$  in  $\mathcal{H}_{nk}$  such that  $x \in H$  and  $H \cap M \neq \emptyset$ . Now  $\mathcal{H}_{nk}$  refines  $\mathcal{G}_n$  and so  $H \subseteq \text{st}(x, \mathcal{G}_n)$ . Hence  $\text{st}(x, \mathcal{G}_n) \cap M \neq \emptyset$  and this contradicts the choice of  $n$ .

Next suppose that  $C_x \cap M \neq \emptyset$ . Let  $\{V_1, \dots, V_t\}$  be a finite cover of  $C_x \cap M$  by elements of  $\mathcal{V}$  such that  $x \notin V_r$ ,  $r = 1, \dots, t$ . Choose  $n$  in  $N$  such that  $\text{st}(x, \mathcal{G}_n) \subseteq (\bigcup_{r=1}^t V_r) \cup (X - M)$ . Let  $k$  in  $N$  be such that  $\text{ord}(x, \mathcal{H}_{nk})$  is finite and let  $H_1, \dots, H_s$  be all elements of  $\mathcal{H}_{nk}$  which contain  $x$  and intersect  $M$ . For  $i = 1, \dots, s$ ,  $H_i \subseteq \text{st}(x, \mathcal{G}_n)$  and so  $H_i \cap M \subseteq \bigcup_{r=1}^t V_r$ . Select from  $\{V_1, \dots, V_t\}$  a minimal subcollection which covers  $H_i \cap M$  and label it  $H_i(n, k, j_i)$ . Now  $x \in H_i^*(n, k, j_i)$  and so if we take  $j = \max\{j_1, \dots, j_s\}$  then  $x \notin W_{nkj}$ .

**4.  $\alpha$ -spaces.** A space with a  $\sigma$ -closure preserving separating closed cover is called a  $\sigma^*$ -space. This definition was introduced by Nagata and Siwiec in [21].

**PROPOSITION 4.1.** *Every subparacompact space with a  $G_\delta$ -diagonal is a  $\sigma^*$ -space.*

*Proof.* Let  $X$  be a subparacompact space and let  $\mathcal{G}_1, \mathcal{G}_2, \dots$  be open covers of  $X$  exhibiting the  $G_\delta$ -diagonal property for  $X$ . For each  $n$  in  $N$  let  $\mathcal{F}_{n1}, \mathcal{F}_{n2}, \dots$  be a  $\sigma$ -discrete closed refinement of  $\mathcal{G}_n$ . Then  $\{\mathcal{F}_{nk} : n = 1, 2, \dots, k = 1, 2, \dots\}$  is a  $\sigma$ -closure preserving

separating closed cover of  $X$ .

In [6] Burke showed that a regular  $w\mathcal{A}$ -space is a Moore space if and only if it is a  $\sigma^*$ -space. His method of proof suggests introducing a new class of spaces which we call  $\alpha$ -spaces. We shall show that  $\sigma^*$ -spaces are  $\alpha$ -spaces and that a regular  $w\mathcal{A}$ -space is a Moore space if and only if it is an  $\alpha$ -space.

DEFINITION 4.2. A space  $X$  is an  $\alpha$ -space if there is a function  $g$  from  $N \times X$  into the topology of  $X$  such that for each  $x$  in  $X$ ,

(a)  $\bigcap_{n=1}^{\infty} g(n, x) = \{x\}$ ;

(b) if  $y \in g(n, x)$  then  $g(n, y) \subseteq g(n, x)$ .

Such a function is called an  $\alpha$ -function for  $X$ .

PROPOSITION 4.3. Every  $\sigma^*$ -space is an  $\alpha$ -space.

*Proof.* Let  $\mathcal{F}_1, \mathcal{F}_2, \dots$  be a  $\sigma$ -closure preserving separating closed cover of a  $\sigma^*$ -space  $X$ . For  $n$  in  $N$  and  $x$  in  $X$  let

$$g(n, x) = X - \bigcup \{F \in \mathcal{F}_n : x \notin F\}.$$

It is easy to check that the function  $g$  is an  $\alpha$ -function for  $X$ .

PROPOSITION 4.4. Every space with a  $\sigma$ -point finite separating open cover is an  $\alpha$ -space. In particular, every  $T_1$  space with a  $\sigma$ -point finite base is an  $\alpha$ -space.

*Proof.* Let  $\mathcal{G}_1, \mathcal{G}_2, \dots$  be a  $\sigma$ -point finite separating open cover of a space  $X$ . We may assume that  $X \in \mathcal{G}_n$  for all  $n$  in  $N$ . For  $n = 1, 2, \dots$  and  $x$  in  $X$  let  $g(n, x) = \bigcap \{G \text{ in } \mathcal{G}_n : x \text{ in } G\}$ . Then the function  $g$  is an  $\alpha$ -function for  $X$ .

The following characterization of semi-stratifiable spaces will be useful in proving the main theorem in this section.

LEMMA 4.5. The following are equivalent for a space  $X$ :

(a)  $X$  is semi-stratifiable.

(b) There is a function  $g$  from  $N \times X$  into the topology of  $X$  such that (1) for each  $x$  in  $X$ ,  $\bigcap_{n=1}^{\infty} g(n, x) = \{x\}$ ; (2) if  $x \in g(n, x_n)$  for  $n = 1, 2, \dots$  then the sequence  $\langle x_n \rangle$  converges to  $x$ .

(c) There is a function  $g$  from  $N \times X$  into the topology of  $X$  such that (1) for each  $x$  in  $X$  and  $n$  in  $N$ ,  $x \in g(n, x)$ ; (2) if  $x \in g(n, x_n)$  for  $n = 1, 2, \dots$  then  $x$  is a cluster point of the sequence  $\langle x_n \rangle$ .

*Proof.* The equivalence of (a) and (b) is due to Creede [8], and

(b)  $\Rightarrow$  (c) is obvious. To complete the proof we show that (c)  $\Rightarrow$  (b). Thus, let  $g$  be a function satisfying (c), and assume that  $g(n+1, x) \subseteq g(n, x)$  for all  $n$  in  $N$  and  $x$  in  $X$ .

To prove (1) of (b), first let  $y \in \bigcap_{n=1}^{\infty} g(n, x)$ . Then by (2) of (c),  $y$  is a cluster point of the sequence  $\{x, x, \dots\}$  and so  $y \in \{x\}^-$ . Next let  $y \in \{x\}^-$ . Then  $x \in g(n, y)$  for  $n = 1, 2, \dots$  so by (2) of (c) it follows that  $x$  is a cluster point of the sequence  $\{y, y, \dots\}$ . Thus  $y \in g(n, x)$  for  $n = 1, 2, \dots$  and so  $y \in \bigcap_{n=1}^{\infty} g(n, x)$ .

To prove (2) of (b), let  $x \in g(n, x_n)$ ,  $n = 1, 2, \dots$  and suppose that the sequence  $\langle x_n \rangle$  does not converge to  $x$ . Then there is a neighborhood  $W$  of  $x$  and a subsequence  $\langle x_{n_k} \rangle$  of  $\langle x_n \rangle$  such that  $x_{n_k} \notin W$  for all  $k$  in  $N$ . Now  $x \in g(n_k, x_{n_k}) \subseteq g(k, x_{n_k})$  for  $k = 1, 2, \dots$  so by (2) of (c),  $x$  is a cluster point of the sequence  $\langle x_{n_k} \rangle$ . But this is impossible, and so we conclude that  $\langle x_n \rangle$  converges to  $x$ .

**THEOREM 4.6.** *A regular  $w\Delta$ -space is a Moore space if and only if it is an  $\alpha$ -space.*

*Proof.* By Propositions 4.1 and 4.3 every Moore space is an  $\alpha$ -space. To complete the proof let  $X$  be a regular  $w\Delta$ -space which is also an  $\alpha$ -space and let us show that  $X$  is a Moore space. By Corollary 2.6 it suffices to show that  $X$  is semi-stratifiable.

Let  $\mathcal{S}_1, \mathcal{S}_2, \dots$  be a  $w\Delta$ -sequence for  $X$ , let  $g$  be an  $\alpha$ -function for  $X$ . We may assume that for  $x$  in  $X$  and  $n$  in  $N$ ,  $g(n+1, x) \subseteq g(n, x)$ . For  $x$  in  $X$  and  $n = 1, 2, \dots$  let  $h(n, x) = g(n, x) \cap \text{st}(x, \mathcal{S}_n)$ . We shall show that the function  $h$  satisfies (c) of Lemma 4.5.

Clearly (1) of (c) is satisfied. To check (2) let  $x \in h(n, x_n)$  for  $n = 1, 2, \dots$ . Then for  $n = 1, 2, \dots$ ,  $x \in \text{st}(x_n, \mathcal{S}_n)$  and so  $x_n \in \text{st}(x, \mathcal{S}_n)$ . Thus the sequence  $\langle x_n \rangle$  has a cluster point  $y$ . Suppose  $y \neq x$ . Now  $\{y\} = \bigcap_{n=1}^{\infty} g(n, y)$  and so there is a  $k$  in  $N$  such that  $x \notin g(k, y)$ . Since  $y$  is a cluster point of  $\langle x_n \rangle$  there is a  $m \geq k$  such that  $x_m \in g(k, y)$ . Since  $g$  is an  $\alpha$ -function for  $X$ ,  $x_m \in g(k, y)$  implies  $g(k, x_m) \subseteq g(k, y)$ . But  $x \in h(m, x_m) \subseteq g(m, x_m) \subseteq g(k, x_m)$  and so  $x \in g(k, y)$ , a contradiction. Thus  $x = y$  and  $x$  is a cluster point of  $\langle x_n \rangle$ .

**COROLLARY 4.7.** *Every regular  $w\Delta$ -space with a  $\sigma$ -point finite separating open cover is a Moore space.*

**COROLLARY 4.8.** *Every regular countably compact space with a  $\sigma$ -point finite separating open cover is metrizable.*

**5. A generalization of semi-stratifiable and  $w\Delta$ -spaces.** Let  $X$  be a space and let  $g$  be a function from  $N \times X$  into the topology of

$X$  such that for all  $x$  in  $X$  and  $n$  in  $N$ ,  $x \in g(n, x)$ . Consider the following properties of the function  $g$ .

(A) If  $x \in g(n, x_n)$  and  $y_n \in g(n, x_n)$  for  $n = 1, 2, \dots$  then  $x$  is a cluster point of the sequence  $\langle y_n \rangle$ .

(B) If  $x \in g(n, x_n)$  and  $y_n \in g(n, x_n)$  for  $n = 1, 2, \dots$  then the sequence  $\langle y_n \rangle$  has a cluster point.

(C) If  $x_n \in g(n, x)$  for  $n = 1, 2, \dots$  then  $x$  is a cluster point of the sequence  $\langle x_n \rangle$ .

(D) If  $x_n \in g(n, x)$  for  $n = 1, 2, \dots$  then the sequence  $\langle x_n \rangle$  has a cluster point.

(E) If  $x \in g(n, x_n)$  for  $n = 1, 2, \dots$  then  $x$  is a cluster point of the sequence  $\langle x_n \rangle$ .

(F) If  $x \in g(n, x_n)$  for  $n = 1, 2, \dots$  then the sequence  $\langle x_n \rangle$  has a cluster point.

In [10] Heath proved that developable spaces can be characterized in terms of a function  $g$  satisfying (A), and similarly  $w\Delta$ -spaces can be characterized in terms of a function  $g$  satisfying (B). Clearly 1<sup>st</sup> countable spaces are characterized by (C), and (D) is precisely the definition of a  $q$ -space [18]. Finally, as proved in §4, semi-stratifiable spaces are characterized by a function  $g$  satisfying (E). These observations suggest introducing a new class of spaces, based on (F), which generalizes semi-stratifiable and  $w\Delta$ -spaces.

**DEFINITION 5.1.** A space  $X$  is a  $\beta$ -space if there is a function  $g$  from  $N \times X$  into the topology of  $X$  such that

(a) for all  $x$  in  $X$  and  $n$  in  $N$ ,  $x \in g(n, x)$ ;

(b) if  $x \in g(n, x_n)$  for  $n = 1, 2, \dots$  then the sequence  $\langle x_n \rangle$  has a cluster point.

Such a function is called a  $\beta$ -function for  $X$ .

**THEOREM 5.2.** *The following are equivalent for a regular space  $X$ :*

(a)  $X$  is semi-stratifiable.

(b)  $X$  is a  $\beta$ -space with a  $G_\delta^*$ -diagonal.

(c)  $X$  is an  $\alpha$ -space and a  $\beta$ -space.

*Proof.* Clearly (a)  $\Rightarrow$  (b) and (a)  $\Rightarrow$  (c). To prove (b)  $\Rightarrow$  (a) let  $g$  be a  $\beta$ -function for  $X$  and let  $\mathcal{G}_1, \mathcal{G}_2, \dots$  be a  $G_\delta^*$ -sequence for  $X$ , where it is assumed that  $\mathcal{G}_{n+1}$  refines  $\mathcal{G}_n$  for all  $n$ . For  $x$  in  $X$  and  $n$  in  $N$  let  $h(n, x) = g(n, x) \cap \text{st}(x, \mathcal{G}_n)$ . Then  $h$  satisfies (c) of Lemma 4.5 and so  $X$  is semi-stratifiable.

To prove (c)  $\Rightarrow$  (a) let  $g$  be a  $\beta$ -function for  $X$  and let  $h$  be an  $\alpha$ -function for  $X$ , where  $h(n+1, x) \subseteq h(n, x)$  for all  $n$  in  $N$  and  $x$

in  $X$ . For  $x$  in  $X$  and  $n = 1, 2, \dots$  let  $k(n, x) = g(n, x) \cap h(n, x)$ . Then  $k$  satisfies (c) of Lemma 4.5 and so  $X$  is semi-stratifiable.

REMARK 5.3. The implication (d)  $\Rightarrow$  (a) of Corollary 2.6 and Theorem 4.6 can be proved using the above theorem together with Creede's result that every regular semi-stratifiable  $w\mathcal{A}$ -space is a Moore space.

6. Summary. The relationship between some of the classes of spaces considered in this paper can be summarized in a diagram as follows.

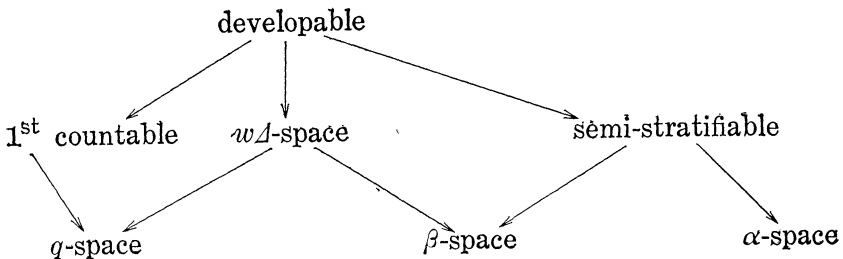


Fig. 1

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J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, <i>Uniquely representable semigroups on the two-cell</i> .....	565
Glen Eugene Bredon, <i>Some examples for the fixed point property</i> .....	571
William Lee Bynum, <i>Characterizations of uniform convexity</i> .....	577
Douglas Derry, <i>The convex hulls of the vertices of a polygon of order <math>n</math></i> ....	583
Edwin Duda and Jack Warren Smith, <i>Reflexive open mappings</i> .....	597
Y. K. Feng and M. V. Subba Rao, <i>On the density of <math>(k, r)</math> integers</i> .....	613
Irving Leonard Glicksberg and Ingemar Wik, <i>Multipliers of quotients of <math>L_1</math></i> .....	619
John William Green, <i>Separating certain plane-like spaces by Peano continua</i> .....	625
Lawrence Albert Harris, <i>A continuous form of Schwarz's lemma in normed linear spaces</i> .....	635
Richard Earl Hodel, <i>Moore spaces and <math>w</math> <math>\Delta</math>-spaces</i> .....	641
Lawrence Stanislaus Husch, Jr., <i>Homotopy groups of PL-embedding spaces. II</i> .....	653
Yoshinori Isomichi, <i>New concepts in the theory of topological space—supercondensed set, subcondensed set, and condensed set</i> .....	657
J. E. Kerlin, <i>On algebra actions on a group algebra</i> .....	669
Keizō Kikuchi, <i>Canonical domains and their geometry in <math>C^n</math></i> .....	681
Ralph David McWilliams, <i>On iterated <math>w^*</math>-sequential closure of cones</i> .....	697
C. Robert Miers, <i>Lie homomorphisms of operator algebras</i> .....	717
Louise Elizabeth Moser, <i>Elementary surgery along a torus knot</i> .....	737
Hiroshi Onose, <i>Oscillatory properties of solutions of even order differential equations</i> .....	747
Wellington Ham Ow, <i>Wiener's compactification and <math>\Phi</math>-bounded harmonic functions in the classification of harmonic spaces</i> .....	759
Zalman Rubinstein, <i>On the multivalence of a class of meromorphic functions</i> .....	771
Hans H. Storrer, <i>Rational extensions of modules</i> .....	785
Albert Robert Stralka, <i>The congruence extension property for compact topological lattices</i> .....	795
Robert Evert Stong, <i>On the cobordism of pairs</i> .....	803
Albert Leon Whiteman, <i>An infinite family of skew Hadamard matrices</i> .....	817
Lynn Roy Williams, <i>Generalized Hausdorff-Young inequalities and mixed norm spaces</i> .....	823