

Pacific Journal of Mathematics

CANONICAL DOMAINS AND THEIR GEOMETRY IN C^n

KEIZŌ KIKUCHI

CANONICAL DOMAINS AND THEIR GEOMETRY IN C^n

KEIZO KIKUCHI

In this paper we introduce some differential geometric properties of canonical domains of bounded domains in C^n , using our synthetic expression by matrix. In the proofs of the theorems, our formulas of matrix derivatives play the leading part.

In order to construct relatively invariant matrices, the author devised formulas of matrix derivatives and obtained some results ([2]). Here we use these formulas for the calculations on the argument of the theorems of geometry. The constructed matrix $\frac{1}{2} {}_2T_D(\bar{z}, z)$ (see [2]) becomes the *curvature tensor*, and $T_D^{-1}(\bar{z}, z)(\partial T_D(\bar{z}, z)/\partial z)$ becomes *Christoffel symbols* in the Kaehler manifold with the metric $ds_D = dz^* T_D(\bar{z}, z) dz$ where ${}_2T_D(\bar{z}, z) = (E_n \times T_D(\bar{z}, z))(\partial/\partial z^*)(T_D^{-1}(\bar{z}, z)(\partial T_D(\bar{z}, z)/\partial z))$ and $T_D(\bar{z}, z) = \partial^2 \log K_D(\bar{z}, z)/\partial z^* \partial z$. We study some differential geometric properties of canonical domains, that is, Bergman representative domains, m -representative domains, homogeneous domains, and our minimal domains of moment of inertia which are defined and investigated in § 2 ([1], [5], [7], [12]).

We calculate Christoffel symbols at the center of canonical domains and give the condition which a geodesic curve through the center of a representative domain satisfies in Theorems 3.4. In Theorems 3.7–12 and Corollaries 3.1–4, we discuss *scalar curvature* and *holomorphic sectional curvature*.

The author wishes to thank Professor S. Ozaki for helpful discussions in the preparation of the present paper.

1. Preliminaries. Let D be a domain in C^n which posses a Bergman kernel function $K_D(\bar{t}, z) \equiv \varphi^*(t)\varphi(z)$, $t, z \in D$, where $\varphi(z) \equiv (\varphi_1(z), \varphi_2(z), \dots)'$ and the marks ' and * denote the transposed and transposed conjugate matrices respectively. We consider a vector function $w(z) \equiv (w_1(z), \dots, w_n(z))'$ in D . If the function $w(z)$ is both holomorphic and locally one-to-one, i.e., $\det(dw/dz) \neq 0$, then the function defines a pseudo-conformal mapping of D onto another domain $\Delta \subset C^n$. Further, the inner product of two functions f, g belonging to a class \mathcal{L}_D^2 of all holomorphic functions $\zeta(z)$ of D which satisfy $\int_D |\zeta(z)|^2 dv_D \equiv Sp(\int_D \zeta(z)\zeta^*(z)dv_D) < \infty$, as follows:

$$(f, g)_D \equiv \int_D f(z)g^*(z)dv_D,$$

where dv_D denotes the Euclidean volume element on D . Moreover we

define a norm $\|f\|_D$ of $f(z)$ as

$$(1.1) \quad \|f\|_D^2 \equiv Sp(f, f)_D = \int_D |f(z)|_+^2 d\nu_D.$$

We shall define some notations for derivatives of matrix functions with respect to the vector variable $z = (z_1, \dots, z_n)'$:

$$(1.2) \quad \frac{\partial^{h+k} w(\bar{t}, z)}{\partial t^{*h} \partial z^k} = \left(\frac{\partial}{\partial t} \right)^{*h} \left(\frac{\partial}{\partial z} \right)^k \times w(\bar{t}, z),$$

where $(\partial/\partial t)^{*h}$ and $(\partial/\partial z)^k$ denote h -times and k -times Kronecker product of $(\partial/\partial t)^* = (\partial/\partial \bar{t}_1, \dots, \partial/\partial \bar{t}_n)'$ and $\partial/\partial z = (\partial/\partial z_1, \dots, \partial/\partial z_n)$ respectively. If $w(z)$ is a function of z only, the k th derivative is denoted by $d^k w(z)/dz^k$. In particular, if z and t are both fixed, then we shall write the derivatives merely $w_{t^{h*} z^k}$ or $\partial^{h+k} w / \partial t^{*h} \partial z^k$. Hereafter, sometimes we shall write $T_D(\bar{t}_0, t_0) = T_D$, $K_D(\bar{t}_0, t_0) = K_D$, $\partial^2 K_D(\bar{t}_0, t_0) / \partial t^{*} \partial z = \partial^2 K_D / \partial t^{*} \partial z = K_{t^* z}$, and so on. Further we denote the following formulas with respect to the matrix derivatives:

$$(1.3) \quad \frac{\partial F^{-1}}{\partial z} = -F^{-1} \frac{\partial F}{\partial z} (E_n \times F^{-1}),$$

(F is a regular $k \times k$ matrix function and E_n is an $n \times n$ unit matrix)

$$(1.4) \quad \frac{\partial(FG)}{\partial z} = \frac{\partial F}{\partial z} (E_n \times G) + F \frac{\partial G}{\partial z},$$

(F and G are $k \times l$, $l \times m$ matrices respectively)

$$(1.5) \quad \frac{\partial F}{\partial z} = \frac{\partial F}{\partial \zeta} \left(\frac{\partial \zeta}{\partial z} \times E_l \right) + \left(\frac{\partial \zeta^*}{\partial z} \times E_k \right) \left(E_{n^*} \times \frac{\partial F}{\partial \zeta^*} \right),$$

(F is a $k \times l$ matrix, z, ζ are $n \times 1$ vectors)

$$(1.6) \quad \frac{\partial(F \times G)}{\partial z} = \frac{\partial F}{\partial z} \times G + \left(F \times \frac{\partial G}{\partial z} \right) (\tilde{E}_{l_n} \times E_l),$$

(F, G are $k \times l$, $\mu \times \nu$ matrices respectively, and

$$\tilde{E}_{l_n} = \begin{pmatrix} e_{11} & \cdots & e_{l1} \\ & \ddots & \\ e_{1n} & \cdots & e_{ln} \end{pmatrix}$$

where e_{ij} are $l \times n$ matrices in which only (i, j) element equal to 1, and others 0). If $\zeta = \zeta(z)$ is a pseudo-conformal mapping of a domain D onto a domain Δ , then we have

$$(1.7) \quad K_D(\bar{t}, z) = \left(\det \frac{d\tau(t)}{dz} \right)^* K_\Delta(\bar{\tau}, \zeta) \det \frac{d\zeta(z)}{dz},$$

$$(1.8) \quad T_D(\bar{t}, z) = \left(\frac{d\tau(t)}{dt} \right)^* T_A(\bar{\tau}, \zeta) \frac{d\zeta(z)}{dz},$$

and for a matrix $P = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$ with the block subdivisions, it holds that

$$(1.9) \quad P^{-1} = \begin{pmatrix} K^{-1} + XZ^{-1}Y, & -XZ^{-1} \\ -Z^{-1}Y, & Z^{-1} \end{pmatrix},$$

where K, N are square matrices, and $X = K^{-1}L$, $Y = MK^{-1}$, $Z = N - MK^{-1}L$.

2. Moment of inertia and relative invariant matrix. For the holomorphic mappings $\zeta(z) = A(z - t_0) + (\text{higher powers})$ with respect to t_0 , we define the classes which satisfy respectively the following initial conditions at a fixed point $t_0 \in D$:

$$\begin{aligned} \mathcal{F}_{E;t_0}: A &= E, & \mathcal{F}_{|A|;t_0}: \det A &= 1, \\ \mathcal{F}_{|A^*A|;t_0}: \det A^*A &= 1, & \mathcal{F}_{SpA^*A;t_0}: \frac{SpA^*A}{n} &= 1 \\ (\mathcal{F}_{E;t_0} \subset \mathcal{F}_{|A|;t_0} \subset \mathcal{F}_{|A^*A|;t_0} \subset \mathcal{F}_{SpA^*A;t_0}). \end{aligned}$$

Bergman representative and minimal domains were considered for the classes $\mathcal{F}_{E;t_0}$ and $\mathcal{F}_{|A|;t_0}$, respectively. If we define the moment of inertia of Δ which is the image of D by $\zeta(z)$ as

$$(2.1) \quad mom(\Delta) \equiv \|\zeta\|_{\Delta}^2 = \int_{\Delta} |\zeta|^2 dv_{\Delta} = \int_D |\zeta \cdot \det \frac{d\zeta}{dz}|^2 dv_D,$$

then a minimal domain of moment of inertia with $\zeta(t_0)$ as center which minimizes $mom(\Delta)$ may be considered for the classes of the above four types. But now we treat for the class $\mathcal{F}_{E;t_0}$. First, we deal with the minimum problems following S. Bergman ([1], [5]). The following relations hold for any functions $\zeta(z) = A(z - t_0) + (\text{higher powers})$, using (1.9),

$$\begin{aligned} (2.2) \quad \|\zeta(z)\|_D^2 &= Sp \int_D \zeta \zeta^* dv_D \geq Sp(O, A) \begin{pmatrix} K_D & K_z \\ K_{t^*} & K_{t^*_z} \end{pmatrix}^{-1} \begin{pmatrix} O \\ A^* \end{pmatrix} \\ &= \frac{1}{K_D} Sp(AT_D^{-1}A^*), \end{aligned}$$

and minimizing function exists uniquely and is expressed as follows:

$$(2.3) \quad (O, A) \begin{pmatrix} K_D & K_z \\ K_{t^*} & K_{t^*_z} \end{pmatrix}^{-1} \begin{pmatrix} K_D(\bar{t}_0, z) \\ \partial K_D(\bar{t}_0, z) / \partial t^* \end{pmatrix} = \frac{K_D(\bar{t}_0, z)}{K_D} AT_D^{-1} \int_{t_0}^z T_D(\bar{t}_0, z) dz,$$

where A is an $n \times n$ matrix. If $\zeta(z) \in \mathcal{F}_{E;t_0}$, then $\zeta(z) \det(d\zeta(z)/dz)$ also

belongs to $\mathcal{F}_{E:t_0}$, hence the mapping $\zeta(z)$ which maps D onto a minimal domain of moment of inertia satisfies

$$(2.4) \quad \zeta(z) \det \frac{d\zeta(z)}{dz} = \frac{K_D(\bar{t}_0, z)}{K_D} T_D^{-1} \int_{t_0}^z T_D(\bar{t}_0, z) dz ,$$

and the moment of inertia of this minimal domain is $1/K_D \cdot (Sp T_D^{-1})$. (See [9]).

THEOREM 2.1. *A necessary and sufficient condition for a domain D to be a minimal domain of moment of inertia with t_0 as center is*

$$(2.5) \quad \frac{d}{dz} (K_D(\bar{t}_0, z) \int_{t_0}^z T_D(\bar{t}_0, z) dz) \equiv K_D T_D .$$

In fact, for the identity mapping $\zeta(z) = z$ of D , $\zeta(z) \det(d\zeta(z)/dz) = z$, therefore the necessary and sufficient condition is

$$z = \frac{K_D(\bar{t}_0, z)}{K_D} T_D^{-1} \int_{t_0}^z T_D(\bar{t}_0, z) dz .$$

THEOREM 2.2. *A domain D is a minimal domain of moment of inertia with t_0 as center, if the following condition is fulfilled:*

$$(2.6) \quad \partial^2 K_D(\bar{t}_0, z) / \partial t^* \partial z \equiv K_D T_D .$$

Proof. From the hypothesis, we have $\partial K_D(\bar{t}_0, z) / \partial t^* = K_D T_D \cdot (z - t_0)$, therefore $\partial K_D / \partial t^* = 0$. Hence, using the relation

$$K_D(\bar{t}_0, z) \int_{t_0}^z T_D(\bar{t}_0, z) dz = \partial K_D(\bar{t}_0, z) / \partial t^* - \frac{K_D(\bar{t}_0, z)}{K_D} \cdot \partial K_D / \partial t^* ,$$

we have

$$\frac{\partial}{\partial z} (K_D(\bar{t}_0, z) \int_{t_0}^z T_D(\bar{t}_0, z) dz) = K_D T_D ,$$

consequently the hypothesis of Theorem 2.1 is fulfilled.

COROLLARY 2.1. *Let D be a minimal domain with center at t_0 , then a necessary and sufficient condition for the domain D to be a minimal domain of moment of inertia with the same center t_0 is*

$$(2.7) \quad \partial^2 K_D(\bar{t}_0, z) / \partial t^* \partial z \equiv K_D T_D .$$

COROLLARY 2.2. *Let D be a representative domain with center at t_0 , then D is a minimal domain of moment of inertia if and only if*

$$(2.8) \quad \frac{d}{dz} (K_D(\bar{t}_0, z) \cdot (z - t_0)) \equiv K_D .$$

Proof. By the hypothesis we have $T_D(\bar{t}_0, z) \equiv T_D$, consequently $\int_{t_0}^z T_D(\bar{t}_0, z) dz = T_D \cdot (z - t_0)$. Substituting this into (2.5), we obtain (2.8).

COROLLARY 2.3. *Let D be a representative domain with center at t_0 , and simultaneously a minimal domain with the same center, then D is also a minimal domain of moment of inertia.*

Proof. We can prove easily from $\partial^2 K_D(\bar{t}_0, z)/\partial t^* \partial z \equiv K_D T_D$ which is a necessary and sufficient condition for a domain D to be a minimal domain with center at t_0 and simultaneously a representative domain with the same center, and (2.6).

Next, we introduce relative invariant matrices which play an important part in Riemannian geometry of a complex n -dimensional manifold.

LEMMA 2.1. *The following relation holds:*

$$\begin{aligned}
 (E_n \times T_D(\bar{z}, z)) \frac{\partial}{\partial z^*} \left(T_D^{-1}(\bar{z}, z) \frac{\partial T_D(\bar{z}, z)}{\partial z} \right) \\
 (2.9) \quad &= \frac{\partial^2 T_D(\bar{z}, z)}{\partial z^* \partial z} - \frac{\partial T_D(\bar{z}, z)}{\partial z^*} T_D^{-1}(\bar{z}, z) \frac{\partial T_D(\bar{z}, z)}{\partial z} \\
 &(\equiv {}_2T_D(\bar{z}, z)) ,
 \end{aligned}$$

and for any pseudo-conformal mapping $\zeta = \zeta(z)$ which maps D onto Δ , we have

$$(2.10) \quad {}_2T_D(\bar{z}, z) = (d\zeta(z)/dz)^{*2} {}_2T_\Delta(\bar{\zeta}, \zeta) (d\zeta(z)/dz)^2 ,$$

where the power means 2-times Kronecker product. (See [2]).

LEMMA 2.2. *Let ${}_p\psi_D(\bar{z}, z)$ be*

$$(2.11) \quad {}_p\psi_D(\bar{z}, z) \equiv \frac{1}{K^p} \left(\frac{\partial^2 (K^p T)}{\partial z^* \partial z} - \frac{\partial (K^p T)}{\partial z^*} (K^p T)^{-1} \frac{\partial (K^p T)}{\partial z} \right) ,$$

where $K^p = (K_D(\bar{z}, z))^p$ and $T = T_D(\bar{z}, z)$, then under any pseudo-conformal mapping we have

$$(2.12) \quad {}_p\psi_D(\bar{z}, z) = (d\zeta(z)/dz)^{*2} {}_p\psi_\Delta(\bar{\zeta}, \zeta) (d\zeta(z)/dz)^2 .$$

REMARK. For $p > 2$ we showed that ${}_p\psi_D(\bar{z}, z)$ are positive definite (see [2]), but in the following (Corollary 3.4), we shall show that, for $p > 1$, these quantities are also positive definite in a bounded domain by the properties of holomorphic sectional curvature.

In fact, Using the formulas (1.4)~(1.6), we can calculate as

follows:

$$\begin{aligned}\frac{\partial(K^p T)}{\partial z} &= pK^{p-1}\left(\frac{\partial K}{\partial z} \times T\right) + K^p \frac{\partial T}{\partial z}, \\ \frac{\partial^2(K^p T)}{\partial z^* \partial z} &= p\left[(p-1)K^{p-2}\frac{\partial K}{\partial z} \times \left(\frac{\partial K}{\partial z^*} \times T\right) \right. \\ &\quad \left. + K^{p-1}\left(\frac{\partial^2 K}{\partial z^* \partial z} \times T + \frac{\partial K}{\partial z^*} \times \frac{\partial T}{\partial z}\right)\right] \\ &\quad + pK^{p-1}\frac{\partial K}{\partial z} \times \frac{\partial T}{\partial z^*} + K^p \frac{\partial^2 T}{\partial z^* \partial z},\end{aligned}$$

therefore, we have

$$(2.13) \quad {}_p\psi_D(\bar{z}, z) = {}_2T_D(\bar{z}, z) + pT_D(\bar{z}, z) \times T_D(\bar{z}, z).$$

From this and (2.10), we obtain (2.12).

3. Curvature in canonical domains. We introduce a positive definite Kaehler metric on D which is invariant under any pseudo-conformal mapping of D

$$ds_D^2 = dz^* T_D(\bar{z}, z) dz,$$

and consider a real $2n$ -dimensional manifold V_{2n} of the variables $\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)'$ and let the metric be

$$\begin{aligned}(3.1) \quad ds_D^2 &= dz^* T_D(\bar{z}, z) dz \\ &= \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix}^* \begin{pmatrix} \frac{1}{2}T(\bar{z}, z), & 0 \\ 0, & \frac{1}{2}\bar{T}_D(\bar{z}, z) \end{pmatrix} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} \\ &= \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix}^* (g_{ij}) \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix},\end{aligned}$$

then we have $g_{\bar{\alpha}\beta} = \frac{1}{2}T_{\bar{\alpha}\beta} = \bar{g}_{\alpha\bar{\beta}}$, $g_{\alpha\bar{\beta}} = \frac{1}{2}T_{\alpha\bar{\beta}} = \bar{g}_{\bar{\alpha}\beta}$, $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$, where $T_{\bar{\alpha}\beta} = (\partial^2 \log K_D(\bar{z}, z) / \partial \bar{z}_\alpha \partial z_\beta)$, and $i, j = 1, \dots, n, \bar{1}, \dots, \bar{n}$; $\alpha, \beta = 1, \dots, n$. If we define a curve in V_{2n} by the functions

$$\begin{pmatrix} z(t) \\ \bar{z}(t) \end{pmatrix} \equiv (z_1(t), \dots, z_n(t), \bar{z}_1(t), \dots, \bar{z}_n(t))'$$

with respect to a parameter t , then the infinitesimal distance on this curve is given by $ds = \sqrt{dz^* T_D(\bar{z}(t), z(t)) dz}$, and the length of this curve joining two points $A_1 = \begin{pmatrix} z(t_1) \\ \bar{z}(t_1) \end{pmatrix}$ and $A_2 = \begin{pmatrix} z(t_2) \\ \bar{z}(t_2) \end{pmatrix}$ is

$$s = \int_{t_1}^{t_2} \sqrt{\frac{dz^*}{dt} T_D(\bar{z}, z) \frac{dz}{dt}} dt.$$

For the function $F \equiv (dz^*/ds)T_D(\bar{z}, z)(dz/ds) \equiv \dot{z}^*T_D(\bar{z}, z)\dot{z}$, we have $(\partial F/\partial z) = \dot{z}^*(\partial T_D(\bar{z}, z)/\partial z)(E_n \times \dot{z})$, therefore substituting this into Euler's equation we obtain

$$\begin{aligned} & \frac{d}{ds}(\dot{z}^*T_D(\bar{z}, z)) - \dot{z}^*\frac{\partial T_D(\bar{z}, z)}{\partial z}(E_n \times \dot{z}) \\ &= \ddot{z}^*T + \dot{z}^*\left(\frac{\partial T}{\partial z}\left(\frac{dz}{ds} \times E\right) + \left(\frac{dz^*}{ds} \times E\right)\frac{\partial T}{\partial z^*}\right) - \dot{z}^*\frac{\partial T}{\partial z}(E \times \dot{z}) \\ &= \ddot{z}^*T + (\dot{z}^* \times \dot{z}^*)\frac{\partial T}{\partial z^*} = 0, \end{aligned}$$

hence we have a differential equations of geodesic (see [6], [13])

$$\begin{aligned} (3.2) \quad & \ddot{z} + T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}(\dot{z} \times \dot{z}) = 0, \\ & \ddot{\bar{z}} + \bar{T}_D^{-1}(\bar{z}, z)\frac{\partial \bar{T}_D(\bar{z}, z)}{\partial \bar{z}}(\dot{\bar{z}} \times \dot{\bar{z}}) = 0. \end{aligned}$$

Consequently, the Christoffel symbol is expressed as

$$\begin{aligned} (3.3) \quad & T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z} = \begin{pmatrix} \Gamma_{11}^1, \dots, \Gamma_{1n}^1, \Gamma_{21}^1, \dots, \Gamma_{nn}^1 \\ \vdots \\ \Gamma_{11}^n, \dots, \Gamma_{1n}^n, \Gamma_{21}^n, \dots, \Gamma_{nn}^n \end{pmatrix} \\ & \bar{T}_D^{-1}(\bar{z}, z)\frac{\partial \bar{T}_D(\bar{z}, z)}{\partial \bar{z}} = \begin{pmatrix} \bar{\Gamma}_{11}^1, \dots, \bar{\Gamma}_{1n}^1, \bar{\Gamma}_{21}^1, \dots, \bar{\Gamma}_{nn}^1 \\ \vdots \\ \bar{\Gamma}_{11}^n, \dots, \bar{\Gamma}_{1n}^n, \bar{\Gamma}_{21}^n, \dots, \bar{\Gamma}_{nn}^n \end{pmatrix}. \end{aligned}$$

(See [4], [10], [13]).

Now, for any pseudo-conformal mapping $\zeta = \zeta(z)$, we can calculate as follows by virtue of the above mentioned formulas (1.4)~(1.6):

$$\begin{aligned} & T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z} \\ &= \left(\frac{d\zeta}{dz}\right)^{-1}T_A^{-1}(\bar{\zeta}, \zeta)\left(\frac{\partial T_A(\bar{\zeta}, \zeta)}{\partial \zeta}\left(\frac{d\zeta}{dz}\right)^2 + T_A(\bar{\zeta}, \zeta)\frac{d^2\zeta}{dz^2}\right) \\ &= \left(\frac{d\zeta}{dz}\right)^{-1}T_A^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_A(\bar{\zeta}, \zeta)}{\partial \zeta}\left(\frac{d\zeta}{dz}\right)^2 + \left(\frac{d\zeta}{dz}\right)^{-1}\frac{d^2\zeta}{dz^2}. \quad (\text{See [2]}). \end{aligned}$$

LEMMA 3.1. *For any pseudo-conformal mapping, we obtain the following relations with respect to the Christoffel symbol:*

$$(3.4) \quad \frac{d^2\zeta}{dz^2} = \frac{d\zeta}{dz}\left(T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}\right) - \left(T_A^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_A(\bar{\zeta}, \zeta)}{\partial \zeta}\right)\left(\frac{d\zeta}{dz}\right)^2,$$

$$(3.4') \quad \frac{d^2z}{d\zeta^2} = \frac{dz}{d\zeta}\left(T_A^{-1}(\bar{\zeta}, \zeta)\frac{\partial T_A(\bar{\zeta}, \zeta)}{\partial \zeta}\right) - \left(T_D^{-1}(\bar{z}, z)\frac{\partial T_D(\bar{z}, z)}{\partial z}\right)\left(\frac{dz}{d\zeta}\right)^2.$$

THEOREM 3.1. *The vector $\varepsilon_D(\bar{z}, z) \equiv \ddot{z} + T_D^{-1}(\bar{z}, z)(\partial T_D(\bar{z}, z)/\partial z)(\dot{z} \times \dot{z})$*

is a contravariant vector, and a geodesic curve in D is also a geodesic curve in Δ under any pseudo-conformal mapping $\Delta = \zeta(D)$.

Proof. We have

$$\begin{aligned}\dot{z} &= \frac{dz}{ds} = \frac{dz}{d\zeta} \dot{\zeta}, \\ \ddot{z} &= \frac{d^2 z}{ds^2} = \frac{d}{ds} \left(\frac{dz}{d\zeta} \dot{\zeta} \right) = \frac{d^2 z}{d\zeta^2} (\dot{\zeta} \times \dot{\zeta}) + \frac{dz}{d\zeta} \ddot{\zeta},\end{aligned}$$

hence, substituting (3.4') into this formula, we have

$$\begin{aligned}\ddot{z} &= \frac{dz}{d\zeta} \left(T_A^{-1}(\bar{\zeta}, \zeta) \frac{\partial T_A(\bar{\zeta}, \zeta)}{\partial \zeta} \right) (\dot{\zeta} \times \dot{\zeta}) \\ &\quad - \left(T_D^{-1}(\bar{z}, z) \frac{\partial T_D(\bar{z}, z)}{\partial z} \right) (\dot{z} \times \dot{z}) + \frac{dz}{d\zeta} \ddot{\zeta}.\end{aligned}$$

Therefore

$$\varepsilon_D(\bar{z}, z) = \frac{dz}{d\zeta} \left(T_A^{-1}(\bar{\zeta}, \zeta) \frac{\partial T_A(\bar{\zeta}, \zeta)}{\partial \zeta} \right) (\dot{\zeta} \times \dot{\zeta}) + \frac{dz}{d\zeta} \ddot{\zeta} = \frac{dz}{d\zeta} \varepsilon_A(\bar{\zeta}, \zeta).$$

Hence, $\varepsilon_D(\bar{z}, z)$ is a contravariant vector, and $\varepsilon_D(\bar{z}, z) = 0$ implies $\varepsilon_A(\bar{\zeta}, \zeta) = 0$.

Next, we consider a contravariant vector $\left(\frac{\lambda_D}{\lambda_D} \right)'$ which satisfies the following transformation law: $\lambda_A = (d\zeta/dz)\lambda_D$, $\bar{\lambda}_A = (d\bar{\zeta}/d\bar{z})\bar{\lambda}_D$. Then we have

$$\begin{aligned}d\lambda_A &= \frac{d^2 \zeta}{dz^2} (dz \times \lambda_D) + \frac{d\zeta}{dz} \frac{\partial \lambda_D}{\partial z} dz + \frac{d}{dz} \frac{\partial \lambda_D}{\partial \bar{z}} d\bar{z} \\ &= \frac{d^2 \zeta}{dz^2} (dz \times \lambda_D) + \frac{d\zeta}{dz} (d\lambda_D).\end{aligned}$$

Substituting (3.4) for $(d^2 \zeta/dz^2)$, we obtain

$$\begin{aligned}d\lambda_A &= \frac{d\zeta}{dz} T_D^{-1}(\bar{z}, z) \frac{\partial T_D(\bar{z}, z)}{\partial z} (dz \times \lambda_D) \\ &\quad - T_A^{-1}(\bar{\zeta}, \zeta) \frac{\partial T_A(\bar{\zeta}, \zeta)}{\partial \zeta} (d\zeta \times \lambda_A) + \frac{d\zeta}{dz} d\lambda_D,\end{aligned}$$

therefore we have the transformation expression of the covariant differential:

$$\begin{aligned}(3.5) \quad \delta \lambda_A &\equiv d\lambda_A + T_A^{-1}(\bar{\zeta}, \zeta) \frac{\partial T_A(\bar{\zeta}, \zeta)}{\partial \zeta} (d\zeta \times \lambda_A) \\ &= \frac{d\zeta}{dz} \left(d\lambda_D + T_D^{-1}(\bar{z}, z) \frac{\partial T_D(\bar{z}, z)}{\partial z} (dz \times \lambda_D) \right) = \frac{d\zeta}{dz} (\delta \lambda_D),\end{aligned}$$

$$(3.5)' \quad \delta \bar{\lambda}_d \equiv d\bar{\lambda}_d + \bar{T}_d^{-1}(\bar{\zeta}, \zeta) \frac{\partial \bar{T}_d(\bar{\zeta}, \zeta)}{\partial \bar{\zeta}} (d\bar{\zeta} \times \bar{\lambda}_d) = \frac{d\bar{\zeta}}{d\bar{z}} (\delta \bar{\lambda}_d),$$

and the covariant derivative of a vector $\left(\frac{\lambda_d}{\lambda_d}\right)'$ is given by

$$(3.6) \quad \nabla \lambda_d \equiv \left(\frac{\partial \lambda_d}{\partial z} + T_d^{-1}(\bar{z}, z) \frac{\partial T_d(\bar{z}, z)}{\partial z} (E \times \lambda_d), \frac{\partial \lambda_d}{\partial \bar{z}} \right),$$

$$(3.6)' \quad \nabla \bar{\lambda}_d \equiv \left(\frac{\partial \bar{\lambda}_d}{\partial z}, \frac{\partial \bar{\lambda}_d}{\partial \bar{z}} + \bar{T}_d^{-1}(\bar{z}, z) \frac{\partial \bar{T}_d(\bar{z}, z)}{\partial \bar{z}} (E \times \bar{\lambda}_d) \right).$$

Now, we have the conditions of the parallel displacement

$$\begin{aligned} \frac{\delta \lambda_d}{ds} &\equiv \frac{d\lambda_d}{ds} + T^{-1} \frac{\partial T}{\partial z} \left(\frac{dz}{ds} \times \lambda_d \right) = 0, \\ \frac{\delta \bar{\lambda}_d}{ds} &\equiv \frac{d\bar{\lambda}_d}{ds} + \bar{T}^{-1} \frac{\partial \bar{T}}{\partial \bar{z}} \left(\frac{d\bar{z}}{ds} \times \bar{\lambda}_d \right) = 0, \end{aligned}$$

for a contravariant vector $\left(\frac{\lambda_d}{\lambda_d}\right)$ on a curve, then substituting the tangent $\left(\frac{\dot{z}}{\dot{z}}\right)'$ of a curve for $\left(\frac{\lambda_d}{\lambda_d}\right)'$ we obtain a differential equation of geodesic (3.2). Therefore, a curve on which the tangent is displaced parallelly is a geodesic.

THEOREM 3.2. *At the center t_0 of any representative domain D , the Christoffel symbols with respect to the metric $ds_D^2 = dz^* T_D(\bar{z}, z) dz$ are all zero.*

Proof. A necessary and sufficient condition for a domain D to be a representative domain with t_0 as center is $T_D^{-1} T_D(\bar{t}_0, z) = E_n$, therefore $T_D^{-1}(\partial T_D / \partial z) = 0$.

THEOREM 3.3. *The Christoffel symbols at any point t_0 in a bounded domain with Kaehler metric $ds_D^2 = dz^* T_D(\bar{z}, z) dz$ become all zero by the Bergman representative function with respect to t_0*

$$(3.7) \quad \zeta(z) \equiv T_D^{-1} \int_{t_0}^z T_D(\bar{t}_0, z) dz + t_0.$$

Proof. Substituting $(d\zeta(t_0)/dz) = E$, $(d^2\zeta(t_0)/dz^2) = T_D^{-1}(\partial T_D / \partial z)$ into (3.4), we have $T_D^{-1}(\partial T_D / \partial z) = T_D^{-1}(\partial T_D / \partial z) - T_d^{-1}(\partial T_d / \partial \zeta)$, therefore $T_d^{-1}(\partial T_d / \partial \zeta) = 0$.

THEOREM 3.4. *Let t_0 be an arbitrary point in D which is bounded domain with the Kaehler metric $ds_D^2 = dz^* T_D(\bar{z}, z) dz$, and let the Bergman representative function with respect to t_0 be*

$$\zeta_D^{(1)}(z; t_0) \equiv T_D^{-1} \int_{t_0}^z T_D(\bar{t}_0, z) dz. \quad (\text{See [1], [2]}).$$

Then the point t_0 lies on geodesic (3.2) if and only if

$$d^2\zeta_D^{(1)}(t_0; t_0)/ds^2 = 0.$$

Proof. From Theorem 3.1 and Theorem 3.2, we have

$$d^2\zeta_D^{(1)}(t_0; t_0)/ds^2 = \varepsilon_D(\bar{z}, z).$$

THEOREM 3.5. *The Christoffel symbols at the center c_0 of any m -representative domain Δ ($m \geq 2$, see [5], [2]) with respect to $t_0\varepsilon D$ are equal to that at the point t_0 .*

Proof. For any m -representative function $\zeta(z)$ with respect to t_0 , we have $d\zeta(t_0)/dz = E$, $d^2\zeta(t_0)/dz^2 = 0$. Hence, by (3.4), we obtain

$$T_D(\bar{t}_0, t_0) \frac{\partial T_D(\bar{t}_0, t_0)}{\partial z} = T_D(\bar{c}_0, c_0) \frac{\partial T_D(\bar{c}_0, c_0)}{\partial \zeta}.$$

THEOREM 3.6 *At the center t_0 of a minimal domain of moment of inertia, if $\partial K_D(\bar{t}_0, t_0)/\partial z = 0$, then the Christoffel symbols are all zero.*

Proof. From Theorem 2.1, we have

$$\begin{aligned} \frac{d^2}{dz^2} \left(K_D(\bar{t}_0, z) \int_{t_0}^z T_D(\bar{t}_0, z) dz \right)_{z=t_0} \\ = K_D \frac{\partial T_D}{\partial z} + K_z \times T_D + T_D \times K_z = 0, \end{aligned}$$

therefore, $\partial T_D/\partial z = 0$.

REMARK. By theorem 3.4, we may locate the geodesic through a point t_0 , that is, doing coordinate transformation

$$\zeta(z) = T_D^{-1} \int_{t_0}^z T_D(\bar{t}_0, z) dz + t_0$$

at t_0 , the curve through the point t_0 on which $d^2\zeta(t_0)ds^2 = 0$ is geodesic.

Next, according to our method we express *Riemann-Christoffel tensor* as

$$\begin{aligned} (3.8) \quad \frac{\partial}{\partial z^*} \left(T_D^{-1}(\bar{z}, z) \frac{\partial T_D(\bar{z}, z)}{\partial z} \right) &= (E \times T_D^{-1}(\bar{z}, z))_2 T_D(\bar{z}, z) \\ &= \begin{pmatrix} R_{11\bar{1}}^1 \cdots R_{n\bar{n}\bar{1}}^1 \\ \vdots \\ R_{11\bar{n}}^n \cdots R_{n\bar{n}\bar{n}}^n \end{pmatrix} = - \begin{pmatrix} R_{11\bar{1}}^1 \cdots R_{n\bar{n}\bar{1}}^1 \\ \vdots \\ R_{11\bar{n}}^n \cdots R_{n\bar{n}\bar{n}}^n \end{pmatrix}. \end{aligned}$$

For any pseudo-conformal mapping $\zeta = \zeta(z)$, we have

$$(3.9) \quad \begin{aligned} & \frac{\partial}{\partial z^*} \left(T_D^{-1}(\bar{z}, z) \frac{\partial T_D(\bar{z}, z)}{\partial z} \right) \\ &= \left(\left(\frac{d\zeta}{dz} \right)^* \times \frac{dz}{d\zeta} \right) \frac{\partial}{\partial \zeta^*} \left(T_D^{-1}(\bar{\zeta}, \zeta) \frac{\partial T_D(\bar{\zeta}, \zeta)}{\partial \zeta} \right) \left(\frac{d\zeta}{dz} \right)^2, \end{aligned}$$

therefore it is a tensor of contravariant [degree 1 and covariant degree 3].

Further, we can express the *curvature tensor* as

$$(3.10) \quad \begin{aligned} & \frac{1}{2} (E \times T_D(\bar{z}, z)) \frac{\partial}{\partial z^*} \left(T_D^{-1}(\bar{z}, z) \frac{\partial T_D(\bar{z}, z)}{\partial z} \right) = \frac{1}{2} T_D(\bar{z}, z) \\ & \equiv \begin{pmatrix} R_{11\bar{1}\bar{1}}, & \dots, & R_{1n\bar{1}\bar{n}} \\ & \ddots & \\ R_{n1\bar{1}\bar{n}}, & \dots, & R_{nn\bar{n}\bar{n}} \end{pmatrix} = - \begin{pmatrix} R_{1\bar{1}1\bar{1}}, & \dots, & R_{1\bar{1}n\bar{n}} \\ & \ddots & \\ R_{n\bar{1}1\bar{n}}, & \dots, & R_{n\bar{1}n\bar{n}} \end{pmatrix}. \end{aligned}$$

And, we can express the contracted Christoffel symbols as

$$(3.11) \quad \left(Sp T^{-1} \frac{\partial T}{\partial z_1}, \dots, Sp T^{-1} \frac{\partial T}{\partial z_n} \right) \equiv (\Gamma_{1\alpha}^\alpha, \dots, \Gamma_{n\alpha}^\alpha).$$

By the rule $\partial \log(\det T) / \partial z_i = Sp T^{-1} (\partial T / \partial z_i)$, we obtain *Ricci tensor*

$$(3.12) \quad -\partial^2 \log(\det T) / \partial z^* \partial z \equiv \begin{pmatrix} R_{1\bar{1}}, & \dots, & R_{n\bar{1}} \\ & \ddots & \\ R_{1\bar{n}}, & \dots, & R_{n\bar{n}} \end{pmatrix}.$$

Therefore, the *scalar curvature* becomes

$$(3.13) \quad R_0 = -4 Sp \left(T_D^{-1}(\bar{z}, z) \frac{\partial^2 \log(\det T_D(\bar{z}, z))}{\partial z^* \partial z} \right),$$

which is invariant under any pseudo-conformal mapping.

THEOREM 3.7. *At any bounded domain D , $R_0 < 4n(n+1)$.*

Proof. It is known that both $M \equiv (n+1)T + (\partial^2 \log(\det T) / \partial z^* \partial z)$ and T^{-1} are positive definite Hermitian matrices (see [1], [3]), therefore

$$\frac{1}{\lambda_1} (\sum \rho_j), \quad \text{or} \quad \rho_n \left(\sum \frac{1}{\lambda_j} \right) \leq Sp(T^{-1}M) \leq \frac{1}{\lambda_n} (\sum \rho_j), \quad \text{or} \quad \rho_1 \left(\sum \frac{1}{\lambda_j} \right),$$

where $\lambda_1 \geq \dots \geq \lambda_n > 0$ and $\rho_1 \geq \dots \geq \rho_n > 0$ are eigenvalues of T and M , respectively. Thus we have

$$n(n+1) - \rho_1 Sp T^{-1},$$

or

$$n(n+1) - \frac{1}{\lambda_n} Sp M \leq \frac{R_0}{4} \leq n(n+1) - \rho_n Sp T^{-1}.$$

or

$$n(n+1) - \frac{SpM}{\lambda_1}.$$

THEOREM 3.8. *Let D be a homogeneous domain, then we have always*

$$(3.14) \quad R_0 = -4n.$$

Proof. At the homogeneous domain, it becomes

$$\frac{\partial^2 \log (\det T)}{\partial z^* \partial z} = T,$$

therefore we have $R_0 = -4Sp(T^{-1}T) = -4n$.

THEOREM 3.9. *In a manifold D with the metric $ds_D^2 = dz^* T_D(\bar{z}, z) dz$, if there exists a fixed point t_0 in D such that $I_D(\bar{z}, z) \leq I_D(\bar{t}_0, t_0)$ everywhere in D , and if $-4n \leq R_0$, then we must have $I_D(\bar{z}, z) \equiv I_D(\bar{t}_0, t_0)$ everywhere in D , and consequently we have $R_0 = -4n$, where $I_D(\bar{z}, z)$ is a real valued (invariant) function defined by $I_D(\bar{z}, z) \equiv K_D(\bar{z}, z)/\det T_D(\bar{z}, z)$.*

Proof. From $T = \partial^2 \log I / \partial z^* \partial z + \partial^2 \log (\det T) / \partial z^* \partial z$, we obtain

$$n = Sp \left(T^{-1} \frac{\partial^2 \log I}{\partial z^* \partial z} \right) - \frac{R_0}{4}.$$

Therefore, by Theorem of E. Hopf (see [13]), our proof is completed.

THEOREM 3.10. *In a bounded domain D , if there exists a fixed point t_0 in D such that $J_D(\bar{z}, z) \leq J_D(\bar{t}_0, t_0)$ everywhere in D , then we must have $J_D(\bar{z}, z) = J_D(\bar{t}_0, t_0)$ everywhere in D , and consequently $R_0 = 4n(n+1)$, where $J_D(\bar{z}, z) \equiv (K_D(\bar{z}, z))^{n+1} \det T_D(\bar{z}, z)$.*

Proof. From $(n+1)T + \partial^2 \log (\det T) / \partial z^* \partial z = \partial^2 \log J / \partial z^* \partial z$, we obtain

$$(n+1)n - \frac{R_0}{4} = Sp T^{-1} \frac{\partial^2 \log J}{\partial z^* \partial z}.$$

Since, by Theorem 3.7, we have $Sp T^{-1}(\partial^2 \log J / \partial z^* \partial z) > 0$ everywhere in D , then J is constant by theorem of E. Hopf. Consequently we obtain the following Ricci tensor: $(R_{\alpha\bar{\beta}}) = (n+1)T_D(\bar{z}, z)$. Thus we have $R_0 = 4n(n+1)$.

Next, a holomorphic sectional curvature $\kappa(z; u)$ with respect to a contravariant vector u which is invariant under any pseudo-conformal

mapping is expressed by our method as follows:

$$(3.15) \quad \kappa(z; u) = -2 \frac{(u \times u)^* {}_2T_D(\bar{z}, z)(u \times u)}{(u \times u)^*(T_D(\bar{z}, z) \times T_D(\bar{z}, z))(u \times u)}.$$

THEOREM 3.11. *If D is a homogenous domain with the metric $ds_D^2 = dz^* T dz$, then the holomorphic sectional curvature $\kappa(z; u)$ is constant everywhere in D .*

Proof. Since $\kappa(z; u)$ is invariant, then for arbitrary points z, t in D we have $\kappa(z; u) = \kappa(t; u)$ by a suitable holomorphic automorphism.

THEOREM 3.12. *In a manifold of constant holomorphic curvature κ , for the scalar curvature R_0 , we have*

$$(3.16) \quad R_0 = n(n+1)\kappa.$$

Proof. By the hypothesis, the curvature tensor becomes

$$(3.17) \quad R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{\kappa}{2}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}), \quad (\text{see [13]})$$

consequently we have $R_{\alpha\bar{\beta}} = (n+1)/2 \cdot \kappa g_{\alpha\bar{\beta}}$. Thus we have

$$R_0 = 2g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}} = n(n+1)\kappa.$$

COROLLARY 3.1. *The unit hypersphere $|z|^2 < 1$ is a manifold of constant holomorphic curvature κ and we have $\kappa = -4/(n+1)$. (See Theorem 4 in [10]).*

Proof. Using the formulas (1.3)~(1.6), we obtain

$$\begin{aligned} T_D(0, 0) &= (n+1)E, & \partial T_D(0, 0)/\partial z &= 0, \\ \partial^2 T_D(0, 0)/\partial z^* \partial z &= (n+1)(E^2 + \tilde{E}_{nn}). \end{aligned}$$

Then we have

$${}_2T_D(0, 0) = (n+1)(E^2 + \tilde{E}_{nn}),$$

and consequently $\kappa(0; u) = -4(n+1)(u^*u)^2/(n+1)^2(u^*u)^2 = -4/(n+1)$. Therefore, the holomorphic sectional curvature are all the same at origin. Consequently, by Theorem 3.11, we obtain the required results.

REMARK. Since a unit hypersphere is a homogeneous domain, then $R_0 = -4n$. Therefore, by Theorem 3.12, we can compute $\kappa = -4n/n(n+1) = -4/(n+1)$.

COROLLARY 3.2. In a polydisc $\{|z_j| < r_j, j = 1, \dots, n\}$, for the holomorphic sectional curvature κ , we have $-2 \leq \kappa \leq -2/n$. (See [10]).

Proof. We may calculate at origin as follows:

$$T_D(0, 0) = 2 \begin{pmatrix} \left(\frac{1}{r_1}\right)^2 & & 0 \\ & \ddots & \\ 0 & & \left(\frac{1}{r_n}\right)^2 \end{pmatrix}, \quad {}_2T_D(0, 0) = 4 \begin{pmatrix} \left(\frac{1}{r_1}\right)^4 & & & 0 \\ & \ddots & & \\ & & 0 & \left(\frac{1}{r_2}\right)^4 \\ & & & \ddots & \\ 0 & & & & \left(\frac{1}{r_n}\right)^4 \end{pmatrix}.$$

Thus, we have

$$\kappa = -2 \sum_j (|u_j|/r_j)^4 / (\sum_j (|u_j|/r_j)^2)^2,$$

and consequently $-2 \leq \kappa \leq -2/n$, ($n \geq 2$).

COROLLARY 3.3. In a complex spheres

$$\mathfrak{M}_{(n)} = \{z \mid |z'z| < 1, 1 - 2|z|^2 + |z'z|^2 > 0\},$$

for the holomorphic sectional curvature κ , we have

$$-\frac{2}{n} \left(2 - \frac{1}{n}\right) < \kappa < -\frac{2}{n}.$$

Proof. Since we have

$$T_D(\bar{z}, z) = \frac{2n}{K_0^2} [K_0(E - 2\bar{z}z') + 2(E - \bar{z}z')zz^*(E - \bar{z}z')]$$

where $K_0 = 1 - 2|z|^2 + |z'z|^2$, in the complex spheres (see [8]), then we have

$$T_D(0, 0) = 2nE,$$

$${}_2T_D(\bar{0}, 0) = 4n \left[E^2 - \begin{pmatrix} \boxed{\begin{matrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{matrix}} & & 0 \\ & \boxed{\begin{matrix} 0 & 1 & & \\ & & \ddots & \\ & & & 0 \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} 0 & & & \\ & \ddots & & \\ & & 0 & 1 \end{matrix}} \end{pmatrix} + \tilde{E}_{nn} \right]$$

Consequently,

$$\begin{aligned}\kappa(0; u) &= -2 \frac{4n[2(u^*u)^2 - (|u_1|^4 + \cdots + |u_n|^4)]}{4n^2(u^*u)^2} \\ &= \frac{-2}{n} \left[2 - \frac{|u_1|^4 + \cdots + |u_n|^4}{|u|^4} \right]\end{aligned}$$

where $u = (u_1, \dots, u_n)'$, hence we have the required result.

It is known that the holomorphic sectional curvature for a bounded domain in C^n is less than 2 ([1], [3]), therefore we have

COROLLARY 3.4. *Let D be a bounded domain with Kaehler metric $ds_D^2 = dz^* T dz$, then*

$$\begin{aligned}(3.18) \quad \gamma_D(\bar{z}, z) &\equiv \frac{1}{K} \left[\frac{\partial^2(KT)}{\partial z^* \partial z} - \frac{\partial(KT)}{\partial z^*} (KT)^{-1} \frac{\partial(KT)}{\partial z} \right] \\ &= {}_z T_D(\bar{z}, z) + T_D(\bar{z}, z) \times T_D(\bar{z}, z)\end{aligned}$$

is relative invariant under any pseudo-conformal mapping and positive definite.

Proof. From $\kappa < 2$, we have $(u \times u)^*({}_z T + T \times T)(u \times u) > 0$.

REFERENCES

1. S. Bergman, *Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes*, Mém. Sci. Math., **108** (1948), 27-42.
2. K. Kikuchi, *Various m -representative domains in several complex variables*, Pacific J. Math., **33** (1970).
3. S. Kobayashi, *Geometry of bounded domains*, Trans. of Amer. Math. Soc., **92** (1959), 267 ff.
4. A. Lichnerowicz, *Variétés complexes et tenseur de bergmann*, Ann. Inst. Fourier, Grenoble, vol. **15**, 2 (1965), 345-408.
5. M. Maschler, *Minimal domains and their Bergman kernel function*, Pacific J. Math., **6** (1956).
6. Y. Michiwaki, *On the theory of analytic functions of several complex variables*, Sci. Rep. Tokyo Kyoiku D., (A) **6** (1959).
7. J. Mitchell, *Area and moment of inertia theorems for circular domains in C^n* , Duke Math. J., **33** (1966).
8. K. Morita, *On the kernel functions for symmetric domains*, Sci. Rep. Tokyo Kyoiku D., (A) **5** (1956).
9. S. Ozaki, I. Ono and T. Umezawa, *General minimum problems and representative domains*, Sci. Rep. Tokyo Kyoiku D., (A) **5** (1955), 1-7.
10. Y. Tashiro, *On the holomorphic sectional curvature of the Bergman metric*, Sci. Rep. Tokyo Kyoiku D., (A) **8** (1965).
11. T. Tsuboi, *Bergman representative domains and minimal domains*, Jap. J. Math., **29** (1959).

12. T. Tsuboi and S. Matsuura, *Some canonical domains in C^n and moment of inertia theorems*, Duke Math. J., **36** (1969).
13. K. Yano and S. Bochner, *Curvature and Betti numbers*, Annals of Math. Studies, no. 32.

Received June 9, 1970.

KANAGAWA UNIVERSITY
YOKOHAMA, JAPAN

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, <i>Uniquely representable semigroups on the two-cell</i>	565
Glen Eugene Bredon, <i>Some examples for the fixed point property</i>	571
William Lee Bynum, <i>Characterizations of uniform convexity</i>	577
Douglas Derry, <i>The convex hulls of the vertices of a polygon of order n</i>	583
Edwin Duda and Jack Warren Smith, <i>Reflexive open mappings</i>	597
Y. K. Feng and M. V. Subba Rao, <i>On the density of (k, r) integers</i>	613
Irving Leonard Glicksberg and Ingemar Wik, <i>Multipliers of quotients of L_1</i>	619
John William Green, <i>Separating certain plane-like spaces by Peano continua</i>	625
Lawrence Albert Harris, <i>A continuous form of Schwarz's lemma in normed linear spaces</i>	635
Richard Earl Hodel, <i>Moore spaces and w Δ-spaces</i>	641
Lawrence Stanislaus Husch, Jr., <i>Homotopy groups of PL-embedding spaces. II</i>	653
Yoshinori Isomichi, <i>New concepts in the theory of topological space—supercondensed set, subcondensed set, and condensed set</i>	657
J. E. Kerlin, <i>On algebra actions on a group algebra</i>	669
Keizō Kikuchi, <i>Canonical domains and their geometry in C^n</i>	681
Ralph David McWilliams, <i>On iterated w^*-sequential closure of cones</i>	697
C. Robert Miers, <i>Lie homomorphisms of operator algebras</i>	717
Louise Elizabeth Moser, <i>Elementary surgery along a torus knot</i>	737
Hiroshi Onose, <i>Oscillatory properties of solutions of even order differential equations</i>	747
Wellington Ham Ow, <i>Wiener's compactification and Φ-bounded harmonic functions in the classification of harmonic spaces</i>	759
Zalman Rubinstein, <i>On the multivalence of a class of meromorphic functions</i>	771
Hans H. Storrer, <i>Rational extensions of modules</i>	785
Albert Robert Stralka, <i>The congruence extension property for compact topological lattices</i>	795
Robert Evert Stong, <i>On the cobordism of pairs</i>	803
Albert Leon Whiteman, <i>An infinite family of skew Hadamard matrices</i>	817
Lynn Roy Williams, <i>Generalized Hausdorff-Young inequalities and mixed norm spaces</i>	823