Pacific Journal of Mathematics

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Vol. 39, No. 1 May 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 1, 1971

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Throughout this paper rings are understood to be commutative with 1, and subrings are understood to have the same identity as their over-rings. Familiarity with the Utumi-Lambek concept of complete ring of quotients $Q(R)$, of a commutative ring R, is assumed. $Q(R)$ is commutative and it contains a copy of the classical ring of quotients of R (denoted $Q_{Cl}(R)$), obtained by localizing R at its set of nonzero-divisors. Any ring lying between R and $Q(R)$ is called a ring of quotients of R. R is π-regular if for $r \in R$ there exists $r' \in R$ and a positive integer n such that $r^n = (r^n)^2 r'$. This paper investigates the question: if $Q(R)$ is π-regular, under what conditions are all rings of quotients of R π-regular?

The characterization obtained is applied to the case of semiprime rings. Examples are given, followed by some results directed at the problem of characterizing internally those rings R for which $Q(R)$ is π-regular. The author is indebted to the referee for posing the latter question, and for his criticisms. The terminology and notation are consistent with Lambek's Lectures on Rings and Modules.

PROPOSITION 1. (Bourbaki-Storrer, [(6, 5.6), (1, p. 173, 16(d))]. *If R is a commutative ring then the following are equivalent*:

(1) *R is π-regular,*

(2) *$R/rad\ R$ is regular, where rad R is the prime radical of R,*

(3) *all prime ideals of R are maximal ideals.*

COROLLARY 2. *A semiprime π-regular ring is regular.*

Let R be a ring and let S be an over-ring of R. An element s of S is called integrally dependent on R if there exist elements $r_0, r_1, \cdots, r_{n-1}$ in R such that $s^n + r_{n-1}s^{n-1} + \cdots + r_1 s + r_0 = 0$. The set of all elements of S which are integrally dependent on R is a ring called the integral closure of R in S, and if this is all of S then S is called an integral extension of R.

PROPOSITION 3. [7, p. 259]. *Let R, S be rings, S an integral extension of R. If P is a prime ideal of S, P is maximal in S if and only if $P \cap R$ is a maximal ideal in R.*

DEFINITION 4. A ring is *classical* if it coincides with its classical ring of quotients. Equivalently, each of its elements is a unit or a zero-divisor.

LEMMA 5. *A π-regular ring is a classical ring.*

Proof. Let r be a nonzero-divisor in R, a π-regular ring. Then there exists $r' \in R$ and an integer n such that $r^n(1 - r^n r') = 0$. Since r does not divide zero, neither does r^n so $1 - r^n r' = 0$ which shows that r is a unit.

The main result.

PROPOSITION 6. *Let R be a commutative ring with complete ring of quotients $Q(R)$ which is π-regular. The following are equivalent:*

(1) *$Q(R)$ is integral over R,*
(2) *every ring of quotients of R is π-regular,*
(3) *every ring of quotients of R is classical,*
(4) *$R[q]$ is π-regular for all $q \in Q(R)$,*
(5) *$R[q]$ is classical for all $q \in Q(R)$,*
(6) *the units of $Q(R)$ are integral over R.*

Proof. Clearly $(2) \Rightarrow (4) \Rightarrow (5)$ and $(2) \Rightarrow (3) \Rightarrow (5)$. $(1) \Rightarrow (2)$. If S is a ring of quotients of R, then S is integral over R. Any prime ideal of S contracts to a prime ideal of R which is maximal in R by Proposition 1. Thus by Proposition 3 all prime ideals in S are maximal and by Proposition 1, S is π-regular.

$(5) \Rightarrow (6)$. Let q be a unit in $Q(R)$ with inverse q'. Since $R[q]$ is classical q is either a zero-divisor or a unit in $R[q]$. If it were a zero-divisor in $R[q]$ then it would be both a unit and a zero-divisor in $Q(R)$, an impossibility. Thus q' lies in $R[q]$, and $q' = r_n q^n + \cdots + r_1 q + r_0$ for some $r_i \in R$, $i = 0, 1, \cdots, n$. Now $1 = qq' = r_n q^{n+1} + \cdots + r_1 q^2 + r_0 q$. If one multiplies both sides of the equation by $(q')^{n+1}$ and transposes one obtains the equation $(q')^{n+1} - r_0(q')^n - r_1(q')^{n-1} - \cdots - r_{n-1}(q') - r_n = 0$ which shows that q' is integrally dependent on R. Since every unit is the inverse of a unit (6) is established.

$(6) \Rightarrow (1)$. Let $q \in Q(R)$ Since $Q(R)$ is π-regular there is a $q' \in Q(R)$ such that $q^n = (q^n)^2 q'$. Let $e = q^n q'$, $u = q^n + 1 - q^n q'$. One verifies immediately that $e = e^2$, that u is a unit with inverse $u^{-1} = q^n(q')^2 + 1 - q^n q'$ and that $q^n = ue$. Now e is integral over R, and by (6) u is, so q^n is integral over R, which implies in turn that q is integral over R.

PROPOSITION 7. [4, p. 42]. *Let R be a semiprime ring. Then $Q(R)$ is regular.*

PROPOSITION 8. *Let R be semiprime and let $Q(R)$ be its complete ring of quotients. Then the following are equivalent:*

(1) *$Q(R)$ is integral over R,*

(2) *all rings of quotients of R are regular,*

(3) *all rings of quotients of R are classical.*

Proof. $Q(R)$ is regular so by Proposition 6, $(2) \Rightarrow (3) \Rightarrow (1)$. $(1) \Rightarrow (2)$. Let S be a ring of quotients of R. $Q(R)$ is semiprime so S is as well. By Proposition 6, S is π-regular. Therefore by Corollary 2, S is a regular ring.

EXAMPLE 9. Boolean rings. A ring is Boolean if each element is idempotent. Thus a Boolean ring is regular. Rings of quotients of Boolean rings are discussed in [3, 2.4] where it is shown that a Boolean ring coincides with its complete ring of quotients if and only if it is complete when viewed as a partially ordered set. Furthermore the complete ring of quotients of a Boolean ring is Boolean. Thus if R is a non-complete Boolean algebra, $Q(R)$ is a proper extension of R, which clearly satisfies condition (1) of Proposition 8.

EXAMPLE 10. In Fine-Gillman-Lambek [2, 4.3] the rings $Q_L(X)$ and $Q_F(X)$ are introduced and it is shown that the former is the complete ring of quotients of the latter. To realize $Q_L(X)$ one considers the set of all locally constant continuous real-valued functions whose domains of definition are dense open subsets of a completely regular Hausdorff space X, and divides out by the equivalence relation which identifies two functions which agree on the intersection of their domains. $Q_F(X)$ is the subring determined by the functions with finite range. $Q_F(X)$ is regular. It is not difficult to see that the two rings differ if X is the real field in its usual topology.

Let $g \in Q_L(X)$ and suppose that $g^n + g^{n-1}f_{n-1} + \cdots + f_0 = 0$ for some $f_i \in Q_F(X)$, $i = 0, 1, \cdots, n-1$. We may assume that all the functions are defined on the domain D given by the intersection of their individual domains. Each f_i is defined on a finite clopen partition Π_i of D, on the elements of which it is fixed. Let Π be the common refinement of the Π_i. Then Π is finite and each f_i is fixed on the elements of Π. Since g must satisfy the above polynomial it can assume only a finite number of different values on a given element of Π. Thus g restricted to D has finite range and therefore lies in $Q_F(X)$. Thus the elements of $Q_L(X) - Q_F(X)$ are not integral over

$Q_F(X)$. Thus we have examples of regular rings for which the conditions of Proposition 8 fail.

Proposition 6 demands the π-regularity of $Q(R)$ thus raising the question: for which rings is the complete ring of quotients π-regular? In the Noetherian case the classical ring of quotients is Noetherian and it coincides with the complete ring of quotients. Thus [6, 5.5 and 5.7] the complete ring of quotients is π-regular if and only if it is Artinian. Furthermore Small [5] has shown that a Noetherian ring R has Artinian classical ring of quotients if and only if R satisfies the following 'regularity' condition: if $\bar{r}$ is not a zero-divisor in $R/\operatorname{rad} R$, then r is not a zero-divisor in R. We examine the question of $Q(R)$'s π-regularity in the light of this condition. By $\bar{R}$ and $\overline{Q(R)}$ we denote $R/\operatorname{rad} R$ and $Q(R)/\operatorname{rad} Q(R)$ respectively. The following diagram (with the obvious maps) is commutative

$$\begin{array}{ccc} R & \longrightarrow & Q(R) \\ \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \overline{Q(R)} \end{array}$$

and $\bar{R} \longrightarrow \overline{Q(R)}$ is a monomorphism since $\operatorname{rad}(Q(R)) \cap R = \operatorname{rad} R$.

LEMMA 11. *If $Q(R)$ is π-regular and* $\operatorname{rad} R$ *is nilpotent then R satisfies the regularity condition.*

Proof. Let $\bar{r}$ be a nonzero-divisor in $\bar{R}$. If $\bar{r}$ is a zero-divisor in $\overline{Q(R)}$, then there exists $s \in Q(R)\backslash \operatorname{rad} Q(R)$ such that $rs \in \operatorname{rad} Q(R)$. There is a dense ideal D in R such that $sD \subset R$. Suppose that $sD \subset R$. Since $\operatorname{rad} R$ is nilpotent, $(\operatorname{rad} R)^k = (0)$ for some integer k. Thus $s^k D^k = (0)$. But D^k is dense so $s^k = 0$, contradicting the fact that $s \notin \operatorname{rad} Q(R)$. Thus there exists $d \in D$ such that $sd \in R\backslash(\operatorname{rad} R)$. Now $r(sd) \in \operatorname{rad} R$ contradicting the fact that $\bar{r}$ is not a zero-divisor in $\bar{R}$. Thus $\bar{r}$ is a nonzero-divisor in $\overline{Q(R)}$. But $\overline{Q(R)}$ is regular by Proposition 1, so $\bar{r}$ is invertible in $\overline{Q(R)}$. Thus there is a $q \in Q(R)$ such that $rq - 1 \in \operatorname{rad} Q(R)$, from which it is easy to see that r is a unit in $Q(R)$, and therefore not a zero-divisor in R.

LEMMA 12. *If* $\operatorname{rad} R$ *is nilpotent then $\overline{Q(R)}$ is a ring of quotients of $\bar{R}$. Furthermore if R satisfies the regularity condition then $\overline{Q(R)}$ contains $Q_{Cl}(\bar{R})$.*

Proof. Let $\bar{q}$ be a nonzero element of $\overline{Q(R)}$. $qD \subset R$, for some dense ideal D of R. Suppose that $qD \subset \operatorname{rad} R$. There exists an integer k such that $(\operatorname{rad} R)^k = (0)$ so $q^k D^k = (0)$ yielding $q^k = 0$, a contradition. Thus there is a $d \in D$ such that $qd \in R \backslash \operatorname{rad} R$ yielding $\bar{q}\bar{d} \neq \bar{0}$ in $\bar{R}$, and $\overline{Q(R)}$ is a ring of quotients of $\bar{R}$. [4, p. 46 no. 5].

If the regularity condition holds and $\bar{r}$ is a nonzero-divisor in $\bar{R}$, then r is a nonzero-divisor in R and $rq = 1$ for some $q \in Q(R)$. But then $\bar{r}\bar{q} = \bar{1}$ showing the nonzero-divisors in $\bar{R}$ have inverses in $\overline{Q(R)}$. Thus $\overline{Q(R)} \supset Q_{Cl}(\bar{R})$.

PROPOSITION 13. *If* $\operatorname{rad} R$ *is nilpotent and* $Q_{Cl}(\bar{R}) = Q(\bar{R})$ *then* $Q(R)$ *is π-regular if and only if* R *satisfies the regularity condition.*

Proof. Lemma 11 gives one implication. If $Q_{Cl}(\bar{R}) = \overline{Q(R)}$ then by Lemma 12 $Q(\bar{R}) = \overline{Q(R)}$. But $Q(\bar{R})$ is regular by Proposition 7. Thus by Proposition 1, $Q(R)$ is π-regular.

The above proposition applies to the Noetherian case. More generally if R is commutative with maximum condition on annihilatar ideals then:

(a) $\operatorname{rad} R$ is nilpotent [3]

(b) $\bar{R}$ satisfies the maximum condition annihilator ideals [5, 1.16], and

(c) $Q(R) = Q_{Cl}(R)$, [4, p. 114, 5(g)].

By condition (b), condition (c) also holds for the ring $\bar{R}$. This together with condition (a) makes Proposition 13 meaningful for these rings as well.

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Received May 12, 1970. Based partially on a thesis supervised by J. Lambek at McGill University with N.R.C. support.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 39, No. 1 May, 1971