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**ON THE BRAUER GROUP OF  $\mathbb{Z}$**

ROBERT ALAN MORRIS

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Two dimensional Amitsur cohomology is computed for certain rings of quadratic algebraic integers. Together with computations of Picard groups, this yields information on the Brauer group  $B(S/Z)$ , for  $S$  quadratic algebraic integers, without resort to class field theory.

The classical Brauer group of central simple algebras over a field [10, X, Sec. 5] has been generalized to the Brauer group  $B(R)$  of central separable  $R$ -algebras over a commutative ring  $R$  [2]. One can prove, using class field theory, that the Brauer group  $B(Z)$  of the integers, is trivial. The proof is apparently well known but not in the literature, although it does appear in the dissertation of Fossum [9].

This paper is devoted to our attempt to establish this result using only an exact sequence of Chase and Rosenberg [7, p. 76]. We are able to show that if  $S$  is the integers of  $Q(\sqrt{m})$  for  $m = \pm 3, -1, 2$ , or 5, the subgroup  $B(S/Z)$  of  $B(Z)$  consisting of elements split by  $S$ , vanishes.

In § 2 we develop some technical results on norms which we use in § 3 to show that the Amitsur cohomology group  $H^2(S/Z, U)$  is zero whenever  $S$  is the ring of integers of a quadratic extension of the rationals. In § 4 we use a Mayer-Vietoris sequence of algebraic  $K$ -theory to show that the Picard group  $\text{Pic}(S \otimes_z S) = 0$  for  $S$  the integers of  $Q(\sqrt{m})$ ,  $m = \pm 3, -1, 2$ , or 5. In § 5 we use this result and an exact sequence of Chase and Rosenberg [7, p. 76] to show  $B(S/Z) = 0$  for these rings.

Dobbs [8] has results relating  $B(S/Z)$  to  $H^2(S/Z, U)$  which together with the triviality of  $B(Z)$  imply our results.

§ 2. Norms. If  $S$  is a commutative algebra over a commutative ring  $R$ ,  $S^n$  denotes  $S \otimes S \cdots \otimes S$ ,  $n$  times (here and throughout,  $\otimes$  means  $\otimes_R$ ), and  $\varepsilon_i: S^n \rightarrow S^{n+1}$ ,  $i = 0, \dots, n$ , is given by  $x_0 \otimes \cdots \otimes x_{n-1} \rightarrow x_0 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_{n-1}$ . These maps satisfy  $\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i$  for  $i \leq j$ . For any ring  $A$ ,  $U(A)$  denotes the group of units of  $A$ . All unexplained notation and terminology is as in [7].

**THEOREM 2.0.** *Let  $M/K$  be a Galois extension of commutative rings [6], with group  $G$ , and let  $F$  be an additive functor on a full subcategory  $\mathcal{C}$  of the category of commutative  $K$ -algebras, and suppose  $M$  and  $M \otimes_K M$  lie in  $\mathcal{C}$ . Then for any  $x$  in  $F(M)$ ,  $y = \sum_g \text{in } G Fg(x)$*

lies in  $\text{Ker}(F\varepsilon_0 - F\varepsilon_1)$ .

*Proof.* By Theorem 3.1 of [6] there are orthogonal idempotents  $e_g$  ( $g$  in  $G$ ), in  $M \otimes_K M$  with  $\sum_g e_g = 1$  and  $s \otimes 1 = \sum_g (1 \otimes g(s))e_g$ . In the above notation this becomes:  $\varepsilon_1(s) = \sum_g \varepsilon_0(g(s))e_g$  for all  $s$  in  $M$ .

Now  $\sum_g e_g = 1$  implies that  $M \otimes_K M = \Pi(M \otimes_K M)e_g$  as  $K$ -algebras. Thus, if  $\pi_g$  denotes the projection of the  $g^{\text{th}}$  component, we have  $\pi_g \varepsilon_1 = \pi_g \varepsilon_0 g$  as maps  $M \rightarrow (M \otimes_K M)e_g$ . Now  $y = \sum_{h \text{ in } G} Fh(x)$  is trivially invariant under  $Fg$  so we obtain  $F\pi_g F\varepsilon_1(y) = F\pi_g F\varepsilon_0 Fg(y) = F\pi_g F\varepsilon_0(y)$  for each  $g$  in  $G$ . By the additivity of  $F$ , this implies  $F\varepsilon_0(y) = F\varepsilon_1(y)$  as was to be shown.

Now let  $R$  be the ring of integers of an algebraic number field  $K$ . Let  $M$  be a finite galois field extension of  $K$  with group  $G$  and  $S$  its ring of integers and let  $M:K = n$ . For each  $i \geq 0$  there is a map  $n_i: U(S^{i+1}) \rightarrow U(S^i \otimes R)$  given by  $n_i(\sum x_0 \otimes \dots \otimes x_i) = \Pi_{g \text{ in } G} \sum x_0 \otimes \dots \otimes x_{i+1} \otimes g(x_i)$ .

Now  $S^{i+1}$  is projective, hence faithfully flat as an  $S^i \otimes R$  module. By [7, Lemma 3.8]  $S^i \otimes R = \text{Ker}(S^{i+1} \xrightarrow{\varepsilon_0 - \varepsilon_1} S^{i+1} \otimes_S s_{\otimes R} S^{i+1})$ , so applying Thm. 2.0 to  $M^{i+1}/(M^i \otimes_K K)$  (here  $M^j = M \otimes_K M \dots \otimes_K M$ ) noting that the natural map  $S^n \rightarrow M^n$  is injective for all  $n$  we see that the map  $n_i$  indeed has its image in  $S^i \otimes R$ .

**DEFINITION.** The  $i$ th norm map,  $N^i: U(S^{i+1}) \rightarrow U(S^i)$  is  $Cn_i$  where  $C: S^i \otimes R \rightarrow S^i$  is the natural isomorphism.  $N^i$  is easily seen to be an abelian group map.

**LEMMA 2.1.** If  $\varepsilon_j: U(S^{i+1}) \rightarrow U(S^{i+2})$  denote the maps defined at the beginning of the section, then  $N^{i+1}\varepsilon_j(x) = \varepsilon_j N^i(x)$  for  $0 \leq j < i + 1$  and  $N^{i+1}\varepsilon_{i+1}(x) = x^n$ , where  $n = M:K$ .

*Proof.* Clear

**PROPOSITION 2.2.** If  $d^i: U(S^{i+1}) \rightarrow U(S^{i+2})$  is the Amitsur coboundary (given by  $d^i(x) = \prod_{j=0}^{i+1} \varepsilon_j(x^{(-1)^j})$ ), then  $N^{i+1}d^i(x) = [d^{i-1}N^i(x)](x^n)^{(-1)^{i+1}}$ .

*Proof.*

$$\begin{aligned} N^{i+1}d^i(x) &= N^{i+1} \prod_{j=0}^{i+1} \varepsilon_j(x^{(-1)^j}) = \prod_{j=0}^{i+1} N^{i+1}\varepsilon_j(x^{(-1)^j}) \\ &= \left[ \prod_{j=0}^i \varepsilon_j N^i(x^{(-1)^j}) \right] (x^n)^{(-1)^{i+1}} \end{aligned}$$

by Lemma 2.1. The proposition then follows from the definition of  $d^{i-1}$ .

**COROLLARY 2.3.** *If for  $x$  in  $U(S^{i+1})$  we have  $d^i(x) = 1$ , then  $(x^n)^{(-1)^i} = d^{i-1}(N^i(x))$ . In particular,  $nH^i(S/R, U) = 0$  for  $i \geq 1$ .*

**REMARK.** The above are all closely parallel to results of Amitsur [1, Thm. 2.10] who defines a norm map via determinants whenever  $S/R$  is finitely generated and free. In the that case, our norm maps agree with Amitsur's [1, Lemma 5.2].

We are primarily interested in two-cocycles:

**COROLLARY 2.4.** *Let  $x$  in  $U(S^3)$  have  $d^2(x) = 1$ . Then  $N^1N^2(x)$  is in  $U(R) \cdot 1_S$ .*

*Proof.* By Corollary 2.3 with  $i = 2$ ,  $x^n = d^i(N^2(x))$  and so  $N^2(x^n) = N^2d^i(N^2(x)) = [d^iN^1N^2(x)]N^2(x^n)$  by Proposition 2.2 with  $i = 1$ . Hence  $d^0[N^1N^2(x)] = 1$  in  $S \otimes S$ . Since  $S$  is projective, hence faithfully flat, over  $R$ , it follows from Lemma 3.8 of [7] that  $N^1N^2(x)$  is in  $R \cdot 1_S$ ; say  $N^1N^2(x) = r \cdot 1_S$ . A priori  $r$  is a unit in  $S$ , but not obviously so in  $R$ . Let  $t$  be the inverse in  $S$  of  $r \cdot 1_S$  and let  $t$  satisfy the integral equation  $x^m + r_1x^{m-1} + \dots + r$  in  $R[x]$ . So

$$0 = (t^m + r_1t^{m-1} + \dots + r_m)r^m \cdot 1 = 1 + r_1r + \dots + r_mr^m.$$

Hence  $r$  is a unit in  $R$ , completing the proof.

Henceforth we will suppress the superscripts on norm maps.

Finally we give a technical lemma of general application:

**LEMMA 2.5.** *If  $R$  is any commutative ring and  $S$  a faithfully flat  $R$ -algebra, then  $v$  two cocycle  $x$  in  $U(S^3)$  lies in  $S \otimes S \otimes 1$  if and only if  $x$  is in  $1 \otimes S \otimes 1$ . In this case  $x$  is a coboundary.*

*Proof.* One implication is trivial.

If  $x$  is in  $S \otimes S \otimes 1$  we may write  $x = \varepsilon_2(a) = a \otimes 1$  for some  $a$  in  $S \otimes S$ . Then  $1 = d^2(x) = \varepsilon_0(x)\varepsilon_1(x^{-1})\varepsilon_2(x)\varepsilon_3(x^{-1})$ . Since  $x = \varepsilon_2(a)$ , it is clear that  $\varepsilon_2(x) = \varepsilon_3(x)$ , so that

$$1 = \varepsilon_0(x)\varepsilon_1(x^{-1}) = \varepsilon_0\varepsilon_2(a)\varepsilon_1\varepsilon_2(a^{-1}) = \varepsilon_3\varepsilon_0(a)\varepsilon_3\varepsilon_1(a^{-1}).$$

Since  $\varepsilon_3$  is a monomorphism, we have  $\varepsilon_0(a) = \varepsilon_1(a)$ . As in the previous result, an application of Lemma 3.8 of [7] shows that  $a$  is in  $1 \otimes S$  so that  $x = a \otimes 1$  is in  $1 \otimes S \otimes 1$ . We must have  $a = 1 \otimes u$  for some unit  $u$  of  $S$  and so  $x = 1 \otimes u \otimes 1 = d^1(1 \otimes u) = d^1(a)$ .

3. **The cohomology of quadratic integers.** In this section we use the results of the last section for explicit computations of cohomology groups. In this section  $R = Z$  and  $S$  is the ring of integers of a quadratic field extension,  $K$ , of the rationals,  $Q$ . Thus  $K = Q(\sqrt{m})$  for a square free integer  $m$ . The computations naturally divide themselves into the cases  $m \equiv 2$  or  $3$  and  $m \equiv 1 \pmod{4}$ .

**THEOREM 3.0.** *Let  $K = Q(\sqrt{m})$  with  $m \equiv 2$  or  $3 \pmod{4}$ . If  $S$  denotes the ring of integers of  $K$ , then  $H^2(S/Z, U) = 0$ .*

*Proof.* Let  $\rho = \sqrt{m}$ . Then  $\{1, \rho\}$  constitutes a basis of  $S$  over  $Z$  [12, Thm. 6-1-1]. For any  $x$  and  $y$  in  $Z$ , the nontrivial  $Q$ -automorphism takes  $x + y\rho$  to  $x - y\rho$ , so that  $N(x + y\rho) = (x + y\sqrt{m})(x - y\sqrt{m}) = x^2 - my^2$ .

Now  $S^i$  is free over  $S^{i-1}$  (acting on the first  $i - 1$  factors) with generators  $1_{S^{i-1}} \otimes 1$  and  $1_{S^{i-1}} \otimes \rho$ , so that  $N(x \otimes 1 + y \otimes \rho) = (x \otimes 1 + y \otimes \sqrt{m})(x \otimes 1 - y \otimes \sqrt{m}) = x^2 - my^2$  for  $x$  and  $y$  in  $S^{i-1}$ . For convenience, we call  $x \otimes 1 - y \otimes \rho$  the conjugate of  $x \otimes 1 + y \otimes \rho$  in  $S^i$ .

Suppose  $x$  in  $U(S^3)$  is a two cocycle and let  $y = N(x) = a \otimes 1 + b \otimes \rho$  with  $a$  and  $b$  in  $S$ . By Corollary 2.4,  $a^2 - mb^2 = N(y) = \pm 1$  in  $S$ . We treat the two cases separately, letting  $a = a_1 + a_2\rho$  and  $b = b_1 + b_2\rho$  with  $a_i, b_i$  in  $Z$ .

*Case 1.*  $N(y) = 1$ . Here one easily sees that  $y^{-1} = a \otimes 1 - b \otimes \rho$  the conjugate of  $y$ . Let  $M$  denote the ring homomorphism  $S \otimes S \rightarrow S$  defined by  $M(c \otimes d) = cd$  for  $c$  and  $d$  in  $S$ . Then the unit of  $S$ ,  $M(y) = a + b\rho$  has inverse  $M(y^{-1}) = a - b\rho$ . Explicitly

$$(1) \quad M(y) = a + b\rho = a_1 + mb_2 + (a_2 + b_1)\rho$$

and

$$(2) \quad M(y^{-1}) = a - b\rho = a_1 - mb_2 + (a_2 - b_1)\rho.$$

Now  $NM(y)$  is in  $U(Z)$ , so is  $\pm 1$ . If  $NM(y) = 1$  we see that  $M(y^{-1}) = M(y)^{-1}$  is the conjugate of  $M(y)$ , that is  $M(y)^{-1} = (a_1 + mb_2) - (a_2 + b_1)\rho$ . Using equation (2) we then have  $b_2 = a_2 = 0$ . Thus  $y = N(x) = a_1 \cdot 1 \otimes 1 + b_1 \cdot 1 \otimes \rho = \varepsilon_0(c)$  where  $c = a_1 + b_1\rho$  is in  $U(S)$  since  $y^{-1} = a \otimes 1 - b \otimes \rho = \varepsilon_0(a_1 - b_1\rho)$ .

Now by Corollary 2.3  $x^2 = d^i(N(x)) = d^i(\varepsilon_0(c)) = \varepsilon_0\varepsilon_0(c)\varepsilon_1\varepsilon_0(c^{-1})\varepsilon_2\varepsilon_0(c) = \varepsilon_1\varepsilon_0(c)\varepsilon_1\varepsilon_0(c^{-1})\varepsilon_2\varepsilon_0(c) = \varepsilon_2\varepsilon_0(c) = \varepsilon_0(c) \otimes 1 = N(x) \otimes 1$ . On the other hand, if we write  $x = \alpha \otimes 1 + \beta \otimes \rho$  with  $\alpha$  and  $\beta$  in  $S^2$ , then  $x^2 = (\alpha^2 + m\beta^2) \otimes 1 + 2\alpha\beta \otimes \rho$  and equating coefficients gives  $2\alpha\beta = 0$  and  $\alpha^2 + m\beta^2 = N(x) = \alpha^2 - m\beta^2$  (by the definition of  $N$ ). Hence  $m\beta^2 = 0$ . But

since the natural map of  $S^2$  into  $K^2$  is injective,  $S^2$  is torsion free with no nilpotents, so  $\beta = 0$ . Thus  $x = \alpha \otimes 1$  and so is a coboundary by Lemma 2.5.

In Case 1 there remains the possibility that  $NM(y) = -1$ . With the notation of the previous subcase we see that  $M(y)^{-1} = -(a_1 + mb_2) + (a_2 + b_1)\rho$ , the negative of the conjugate of  $M(y)$ . Equation (2) here leads to  $a_1 = b_1 = 0$  so that  $y = N(x) = a_2\rho \otimes 1 + b_2\rho \otimes \rho$ . Hence  $NN(x) = a_2^2\rho^2 + mb_2^2\rho^2 = a_2^2m + m^2b_2^2 = m(a_2^2 + mb_2^2)$ . By Corollary 2.4, this must be  $\pm 1_S$ . Since  $a_2, b_2$  and  $m$  are integers, this happens only if  $m = \pm 1$ . If  $m = 1$ ,  $K$  is not a proper extension (and in any case  $m$  is not congruent to 2 or 3 (mod 4)). We are thus, in Case 1, reduced to considering the Gaussian integers and must consider solutions of  $b_2^2 - a_2^2 = \pm 1$ . Thus in this subcase,  $\rho = i$ . Returning to equation (1), we have  $M(y) = -b_2 + a_2i$  and we have assumed  $-1 = NM(y) = b_2^2 - a_2^2 = (b_2 + a_2)(b_2 - a_2)$  in  $Z$ . The only solutions of this are  $b_2 = 0$  and  $a_2 = \pm 1$ . Thus by Corollary 2.3,  $x^2 = d^i(N(x)) = d^i(a_2i \otimes 1) = d^i(\varepsilon_1(a_2i)) = \varepsilon_0\varepsilon_1(a_2i)\varepsilon_1\varepsilon_1(a_2^{-1}i^{-1})\varepsilon_2\varepsilon_1(a_2i) = \varepsilon_0\varepsilon_1(a_2i)$  (since  $\varepsilon_1\varepsilon_1 = \varepsilon_2\varepsilon_1$ ) and so  $x^2 = \pm 1 \otimes i \otimes 1$ . But  $\pm 1 \otimes i \otimes 1$  is not a square in  $S^3$ , else after applying the ring homomorphism  $a \otimes b \otimes c \rightarrow abc$  of  $S^3$  to  $S$ , we would have that  $\pm i$ , and hence  $i$ , is a square in  $S$ .

Case 2.  $N(y) = -1$ . Here  $y^{-1} = -a \otimes 1 + b \otimes \rho$  and we obtain

$$(3) \quad M(y) = a + b\rho = a_1 + mb_2 + (a_2 + b_1)\rho$$

and

$$(4) \quad M(y^{-1}) = -a + b\rho = -a_1 + mb_2 + (b_1 - a_2)\rho.$$

Again  $NM(y) = \pm 1$  in  $Z$ . As in Case 1,  $NM(y) = 1$  implies  $M(y^{-1}) = M(y)^{-1}$  is the conjugate of  $M(y)$ , that is,  $M(y^{-1}) = (a_1 + mb_2) - (a_2 + b_1)\rho$ . Comparing coefficients with (4) gives  $a_1 = b_1 = 0$ . By computations similar to the second subcase of Case 1, we are reduced to considering only  $m = -1$ , ( $S$  the Gaussian integers) and  $a_2^2 + b_2^2 = 1$  in  $Z$ . This equation has the solutions  $a_2 = 0$  and  $b_2 = \pm 1$ ;  $a_2 = \pm 1, b_2 = 0$ .  $b_2 = 0$  and  $a_2 = \pm 1$  yields, parallel to Case 1,  $x^2 = d^i(N(x)) = d^i(y) = d^i(a_2i \otimes 1) = -a_2(i \otimes i \otimes i)$  which again cannot be a square in  $S^3$ .

In the subcase  $NM(y) = 1$  there remains the possibility  $a_2 = 0, b_2^2 = 1$ . Then again by Corollary 2.3,  $x^2 = d^i(N(x)) = d^i(y) = d^i(b_2i \otimes i) = b_2(1 \otimes i \otimes i)b_2(i \otimes 1 \otimes i)b_2(i \otimes i \otimes i) = -b_2(1 \otimes 1 \otimes 1) = \pm 1 \otimes 1 \otimes 1$ . That  $x$  is a coboundary then follows from Lemma 3.1 below, completing the subcase  $NM(y) = 1$ .

The subcase  $NM(y) = -1$ , by similar computations leads to  $b_2 = a_2 = 0$ . As in the first subcase of Case 1, an application of Corollary 2.3 and Lemma 2.5 shows that  $x$  is a coboundary, completing Case 2

and the proof, except for the following Lemma.

LEMMA 3.1. *Let  $S$  be the Gaussian integers and  $x$  in  $U(S^3)$  a two cocyle. If  $x^2 = \pm 1$  in  $S^3$  then  $x$  is a coboundary.*

*Proof.* Consider first  $x^2 = 1 \otimes 1 \otimes 1$ . The following are eight solutions in  $S^3$ :  $\pm 1 \otimes 1 \otimes 1, \pm 1 \otimes i \otimes i, \pm i \otimes 1 \otimes i,$  and  $\pm i \otimes i \otimes 1$ . We claim this exhausts the solutions of the equation in  $S^3$ . To see this note that if  $K = Q(i)$ , then distinct solutions in  $S^3$  are also distinct in  $K \otimes_Q K \otimes_Q K$ , since the natural map  $S \otimes S \otimes S \rightarrow K \otimes_Q K \otimes_Q K$  is monic. Since  $K/Q$  is galois,  $K \otimes_Q K \otimes_Q K$  is isomorphic to a direct product of copies of  $K$ . Comparing  $Q$  dimensions yields  $K \otimes_Q K \otimes_Q K \cong K \times K \times K \times K$ . Since the only solutions in  $K$  of  $x^2 = 1$  are  $\pm 1$ , it follows that there are exactly 16 solutions in  $K \otimes_Q K \otimes_Q K$ .

Let  $x_i$  denote the eight above mentioned distinct solutions which lie in  $S^3$  and let  $y = (1/2)(1 \otimes 1 \otimes 1 - i \otimes i \otimes 1 - i \otimes 1 \otimes i + 1 \otimes i \otimes i)$ . Then it can be seen that  $y^2 = 1$  and  $\{x_i, x_i y\}$  are solutions of  $x^2 = 1$  in  $K \otimes_Q K \otimes_Q K$ . We claim these are distinct and that the  $x_i y$  do not lie in  $S$ . For both claims it suffices, since the  $x_i$  are in  $U(S^3)$ , to show that  $y$  cannot lie in  $S^3$ . This follows easily from the fact that  $1 \otimes 1 \otimes 1, i \otimes i \otimes 1, i \otimes 1 \otimes i$  and  $1 \otimes i \otimes i$  are linearly independent over  $Z$  and that  $1/2$  does not lie in  $Z$ .

Thus the  $x_i$  exhaust the solutions in  $S^3$  of  $x^2 = 1$ . Now among these solutions a simple computation shows that the only cocycles are  $1 \otimes 1 \otimes 1$  and  $-1 \otimes 1 \otimes 1$  and these are, respectively  $d^1(1 \otimes 1)$  and  $d^1(-1 \otimes 1)$ . Similarly among the solutions of  $x^2 = -1 \otimes 1 \otimes 1$  only  $\pm i \otimes 1 \otimes 1, \pm i \otimes i \otimes i, \pm 1 \otimes i \otimes 1$  and  $1 \otimes 1 \otimes i$  lie in  $S^3$  (the remaining eight comprise the multiples of these by the element  $y$  given above and again cannot lie in  $S$ ). The only cocycles are  $\pm 1 \otimes i \otimes 1$  and these are coboundaries of  $1 \otimes i$  and  $i \otimes 1$  respectively. Thus the lemma, and so Theorem 3.0, is proved.

THEOREM 3.2. *Let  $K = Q(\sqrt{4k + 1})$ . If  $S$  denotes the integers of  $K$ , then  $H^2(S/Z, U) = 0$ .*

*Proof.* Let  $\rho = (1 + \sqrt{4k + 1})/2$ . Then  $\{1, \rho\}$  is a basis of  $S$  over  $Z$  [12, Thm. 6-1-1]. The nontrivial  $Q$ -automorphism of  $K$ , since it must preserve the roots of  $x^2 = 4k + 1$ , takes  $\sqrt{4k + 1}$  to  $-\sqrt{4k + 1}$  and so takes  $a + b\rho$  to  $a + b((1 - \sqrt{4k + 1})/2) = a + b(1 - \rho)$ . Hence,  $N(x) = a^2 - b^2\rho^2 + ab + b^2\rho$ . Since  $\rho^2 = \rho + k$ , we have  $N(x) = a^2 - b^2k + ab$ .

As in the previous theorem, the structure of  $S^i$  as  $S^{i-1}$ -algebra is analogous to the ring structure on  $S$ . That is  $1_{S^i}$  and  $1_{S^{i-1}} \otimes \rho$  are a basis and  $N(a \otimes 1 + b \otimes \rho) = a^2 + ab - b^2k$  for  $a, b$  in  $S^{i-1}$ .

Computations closely paralleling those of the previous theorem show

that if  $x$  is a two cocycle in  $U(S^3)$  then  $x$  is in  $S^2 \otimes 1$  and so by Lemma 2.5 is a coboundary. As before the computation divides itself into two cases,  $NN(x) = 1$  or  $NN(x) = -1$ . Various subcases lead either to the desired result or to an equation in integers of the form  $2 = a^2 - 4k$ . Since a square integer is never congruent modulo four to two, the theorem is proved.

4.  $\text{Pic}(S \otimes S)$ . Let  $R$  be an integral domain whose quotient field,  $K$ , has characteristic not 2. Let  $S$  be an integral quadratic extension of  $R$ , that is,  $S = R[\rho]$  where the minimal polynomial of  $\rho$  over  $R$  is  $p(x) = x^2 + ax + b$ . Let  $\bar{\rho}$  be the second (and distinct) root of  $p(x)$ . Note that  $S$  is an integral domain with quotient field  $K(\rho)$ , and that  $\bar{\rho}$  is in  $S$  as a consequence of the familiar formula  $\rho + \bar{\rho} = -a$ . The main theorem of this section characterizes the Picard group [5, Ch. II, Sec. 4]  $\text{Pic}(S \otimes_R S)$  of rank one projective  $S \otimes_R S$  modules in terms of the units of  $S$  and of  $S/(\rho - \bar{\rho})/S$ . Henceforth  $\otimes$  means  $\otimes_R$  and  $S'$  denotes  $S/(\rho - \bar{\rho})S$ .

LEMMA 4.0.  $S \otimes S \cong S \times_{S'} S$ . That is, in the notation of [3, IX Sec. 5, p. 478], there are maps  $h_1, h_2$  making

$$(1) \quad \begin{array}{ccc} S \otimes S & \xrightarrow{h_1} & S \\ h_2 \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

a cartesian square (here the unlabelled maps are the natural projections).

*Proof.* By assumption,  $S$  is free over  $R$  on 1 and  $\rho$ , so  $S \otimes S$  is free on 1 and  $1 \otimes \rho$  when regarded as an  $S$  module on the first factor. For  $s$  and  $t$  in  $S$ , define  $h_1(s \otimes 1 + t \otimes \rho) = s + t\rho$  and  $h_2(s \otimes 1 + t \otimes \rho) = s + t\bar{\rho}$ . Then  $h_1(a) - h_2(a) = t(\rho - \bar{\rho})$  for any  $a = s \otimes 1 + t \otimes \rho$  in  $S \otimes S$ . Conversely, suppose  $s_1 \equiv s_2 \pmod{(\rho - \bar{\rho})S}$ , i.e.,  $s_1 - s_2 = s_3(\rho - \bar{\rho})$  for some  $s_3$  in  $S$ . Then taking  $y = s_3$  and  $x = s_1 - s_3\rho$  gives  $s_1 = x + y\rho = h_1(x \otimes 1 + y \otimes \rho)$  and  $s_2 = x + y\bar{\rho} = h_2(x \otimes 1 + y \otimes \rho)$ . Thus  $\{(s_1, s_2) \text{ in } S \times S \mid s_1 \equiv s_2 \pmod{(\rho - \bar{\rho})S}\} = \{(h_1(a), h_2(a)) \mid a \text{ is in } S \otimes S\}$ . Since  $S$  is an integral domain, it follows that  $a \rightarrow (h_1(a), h_2(a))$  is a monomorphism of  $S \otimes S$  into  $S \times S$  so the square (1) satisfies the definition of cartesian.

REMARK. Let  $R$  be the ring of integers of an algebraic number field,  $K$ , with class number 1, and  $S$  the integers of a quadratic extension,  $L$  of  $K$ .  $S$  is finitely generated projective over  $R$  (cf. 12, p. 158).



If  $\{x_i, \phi_i\}$  is a projective coordinate system, the map  $f: S \rightarrow R$  given by  $f(x) = \sum \phi_i(x, x_i)$  is a split  $R$ -module epimorphism, so that  $S = R1 \oplus \ker F$ . Since  $R$  is a PID,  $\ker f$  is free and a simple rank argument (e.g. passing to  $L$ ) shows  $\ker f \cong R \cdot \rho$  for some  $\rho$  in  $S \subseteq L$ . Clearly such a  $\rho$  must satisfy a quadratic monic polynomial over  $R$ , so that  $S$  is quadratic over  $R$  in the above sense.

**THEOREM 4.1.** *Let  $S = R[\rho]$  be a commutative integral quadratic extension of an integral domain  $R$ , let  $\bar{\rho}$  be the conjugate of  $\rho$  and let  $S' = S/(\rho - \bar{\rho})S$ . Then the following sequence is exact:*

$$0 \rightarrow U(S \otimes S) \rightarrow U(S) \times U(S) \rightarrow U(S') \rightarrow \text{Pic}(S \otimes S) \rightarrow \text{Pic } S \times \text{Pic } S \rightarrow \text{Pic } S'.$$

*Proof.* In view of Lemma 4.0 the above sequence is given by Theorem 5.3 [3, IX Sec. 5, p. 481].

**REMARKS.** The maps of the above sequence are those of the Mayer-Vietoris sequence of [3, VII Sec. 4]. In particular,  $U(S \otimes S) \rightarrow U(S) \times U(S)$  is given by  $u \rightarrow (h_1(u), h_2(u)^{-1})$  where  $h_i$  are the maps in Lemma 4.0, and  $U(S) \times U(S) \rightarrow U(S')$  is given by  $(s, t) \rightarrow \pi(s)\pi(t)$  where  $\pi: S \rightarrow S'$  is the natural projection. Clearly the image of  $U(S) \times U(S) \rightarrow U(S')$  is the same as the image of  $\pi$  restricted to  $U(S)$ .

**COROLLARY 4.2.** *With  $R$  and  $S$  as in Theorem 4.1,  $\text{Pic}(S \otimes S) = 0$  if and only if  $\text{Pic } S = 0$  and the natural projection  $U(S) \rightarrow U(S')$  is surjective.*

*Proof.* The  $R$ -algebra map  $\epsilon_1: S \rightarrow S \otimes S$  given by  $x \rightarrow x \otimes 1$  is split by the map  $M: x \otimes y \rightarrow xy$ . Hence  $\text{Pic } S \xrightarrow{\text{Pic } \epsilon_1} \text{Pic}(S \otimes S) \xrightarrow{\text{Pic } M} \text{Pic } S$  is identity, so that  $\text{Pic } \epsilon_1$  is a monomorphism, i.e.,  $\text{Pic } S \subseteq \text{Pic}(S \otimes S)$ . The corollary is then immediate from Theorem 4.1 and remarks following it.

Now let  $K = Q(\sqrt{m})$  be a quadratic field extension of the rationals, and  $S$  be its ring of integers. As in § 3,  $S = Z[\rho]$  where  $\rho = \sqrt{m}$  or  $(1 + \sqrt{m})/2$  according to whether  $m \equiv 2$  or  $3$  or  $m \equiv 1 \pmod{4}$ . We can easily compute  $S'$ :

**LEMMA 4.3.** *If  $m \equiv 1 \pmod{4}$ , then  $S' \cong Z/mZ$ .*

*Proof.* Let  $m = 4k + 1$  so that  $\rho + 2k = (\sqrt{m})\rho$ , and write  $x + y\rho = x - 2ky + y(\rho + 2k) = x - 2ky + y\sqrt{m}\rho = x - 2ky + y\rho(\rho - \bar{\rho})$  where  $x, y$  lie in  $Z$ . Hence  $x + y\rho \equiv x - 2ky \pmod{(\rho - \bar{\rho})S}$ . Moreover,  $m = \sqrt{m}\sqrt{m} = (\rho - \bar{\rho})^2 \equiv 0 \pmod{S(\rho - \bar{\rho})}$ . Thus if, for an integer  $a$ ,  $\bar{a}$  denotes the coset of  $a \pmod{m}$ , we see that  $x + y\rho \rightarrow \overline{x - 2ky}$  is a ring map of  $S$  onto  $Z/mZ$  whose kernel,  $J = \{x + y\rho \mid x - 2ky = am\}$  is con-

tained in  $(\rho - \bar{\rho})S$ . Conversely, since  $\rho - \bar{\rho} = \sqrt{m} = -1 - 2\rho$  we have  $\rho - \bar{\rho} \rightarrow -1 - 4k = -m$ , so that  $(\rho - \bar{\rho})S$  is contained in  $J$ . Thus  $S' \cong Z/mZ$ .

LEMMA 4.4. *If  $m \not\equiv 1 \pmod{4}$  then  $S' \cong T = Z/2mZ + Z/2Z(\sqrt{m})$  (where this ring has the obvious multiplication).*

*Proof.* In this case  $\rho = \sqrt{m}$  and  $\rho - \bar{\rho} = 2\sqrt{m}$ . Let  $\sim$  and  $\wedge$  denote reduction mod  $2m$  and  $2$  respectively. Then for  $x, y$  in  $Z$ ,  $x + y\rho \rightarrow \tilde{x} + \hat{y}\sqrt{m}$  is a ring map whose kernel is  $\{2ma + 2b\rho \mid a, b \text{ are in } Z\}$ . Since  $2ma + 2b\rho = 2\sqrt{m}(\sqrt{m}a + b) = (\rho - \bar{\rho})(\sqrt{m}a + b)$ , this kernel is just  $(\rho - \bar{\rho})S$  and the lemma is proved.

Now  $S'$  is finite in either case; It follows from Proposition 5 of [5, Ch. 2, Sec. 5, No. 4] that any semi-local ring has trivial Picard group, hence  $\text{Pic}(S') = 0$  under the hypotheses of Lemmas 4.3 or 4.4. Suppose that  $\text{Pic } S = 0$  and let  $\pi: U(S) \rightarrow U(S')$  denote the map induced by the projection  $S \rightarrow S'$ . Then employing the remarks following Theorem 4.1 the exact sequence of that theorem becomes in this case

$$(2) \quad 0 \rightarrow \text{Im } \pi \rightarrow U(S') \rightarrow \text{Pic}(S \otimes S) \rightarrow 0.$$

THEOREM 4.5. *Let  $K = Q(\sqrt{m})$  be a quadratic extension of the rational numbers,  $Q$ , with  $m$  a square free integer. If  $S$  denotes the integers of  $K$ , then  $\text{Pic}(S \otimes S) = 0$  for  $m = \pm 3, -1, 2$ , and  $5$  but for no other value of  $m$ .*

*Proof.* For the given values of  $m$ ,  $S$  is a euclidean domain [12, Propn. 6-4-1] hence a PID, or equivalently [cf. 5, Sec. 5, No. 7]  $\text{Pic } S = 0$ . Referring to Lemmas 4.3 and 4.4 we may easily verify the following table by direct calculation

$m$	$S'$	$U(S')$
2	$Z/4Z + Z/2Z\sqrt{2}$	$\{\pm \bar{1}, \pm \bar{1} + \sqrt{2}\}$
3	$Z/2Z + Z/2z\sqrt{3}$	$\{\pm \bar{1}, \pm \bar{2} + \sqrt{3}\}$
5	$Z/5Z$	$\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

where  $\bar{\phantom{x}}$  denotes the coset mod 4, 6, or 5 respectively.

Now by the Dirichlet units theorem [12, Sec. 6-3],  $U(S) = \{\pm \varepsilon^i \mid i \text{ in } Z\}$  where the fundamental unit,  $\varepsilon$ , is  $1 + \sqrt{2}$ ,  $2 + \sqrt{3}$ , or  $(1 + \sqrt{5})/2$  respectively [11, "Tables"]. Referring to Lemmas 4.3 and 4.4 for the definition of  $\pi$  we find in case  $m = 2$  that  $\pi(\varepsilon) = \bar{1} + \sqrt{2}$ ,  $\pi(-\varepsilon) = -\bar{1} - \sqrt{2} = -\bar{1} + \sqrt{2}$  in  $S'$ . In all cases  $\pi(-1) = -\bar{1}$  and  $\pi(1) = \bar{1}$ . Since  $\pi$  is (the restriction of) a ring map, we see that  $\pi$  is onto when  $m = 2$ . Similarly when  $m = 3$ ,  $\pi(\varepsilon) = \bar{2} + \sqrt{3}$  and  $\pi(-\varepsilon) = -\bar{2} - \sqrt{3} = -\bar{2} + \sqrt{3}$  and when  $m = 5$   $\pi(\varepsilon) = -\bar{2} = \bar{3}$

which generates the cyclic group of units of  $S' = \mathbb{Z}/5\mathbb{Z}$ . Thus also in these cases  $\pi$  is onto.

If  $m = -3$  then  $U(S') = U(\mathbb{Z}/3\mathbb{Z}) = \{\pm 1\}$ .  $\pi$  is again onto because it is the restriction of a ring map. If  $m = -1$  then  $U(S') = \{1, \sqrt{-1}\}$ . By definition  $\pi(\sqrt{-1}) = \sqrt{-1}$  so that the fact that  $\pi$  is the restriction of a ring map again implies  $\pi$  is onto. That  $\text{Pic}(S \otimes S) = 0$  for the given  $m$  now follows from Corollary 4.2.

Now suppose  $m$  is not one of the listed integers. By Corollary 4.2 we need only consider integers  $m$  for which  $S$  is a *PID*. If  $m \leq -5$ , the Units Theorem shows  $U(S) = \pm 1$ . Now  $S'$  contains  $\mathbb{Z}/m\mathbb{Z}$  or  $\mathbb{Z}/2m\mathbb{Z}$  according to whether  $m \equiv 1 \pmod{4}$  or not. Let  $m = -p_1 p_2 \cdots p_r$  with  $p_i$  distinct primes, and consider first  $m \equiv 1 \pmod{4}$ . Then  $\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r\mathbb{Z}$  with  $p_i$  odd primes. There being only two units in  $S$ , if  $\pi$  is to be onto we must clearly have  $r = 1$  and  $p_r = 3$ , so  $\pi$  is not onto. Similarly, if  $-5 < m \equiv 3 \pmod{4}$ ,  $\mathbb{Z}/2m\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r\mathbb{Z}$  which has the same units as  $\mathbb{Z}/m\mathbb{Z}$  and, as above  $\pi$  is not onto. If  $m \equiv 2$  we take  $p_1 = 2$ , so that  $\mathbb{Z}/2m\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r\mathbb{Z}$ . Again, if  $\pi$  is to be onto there can be no factors other than  $p_1$ , since  $\mathbb{Z}/4\mathbb{Z}$  has 2 units, so that for no  $m \leq -5$  can  $\pi$  be onto.

Consider now  $m > 5$ . For any unit  $a + b\rho$  in  $S$  we have that the norm

$$N(a + b\rho) = (a + b\rho)(a + b\bar{\rho})$$

is a unit in  $\mathbb{Z}$ , so

$$\pm 1 = (a + b\rho)(a + b\bar{\rho}) \equiv (a + b\rho)^2 \pmod{(\rho - \bar{\rho})S}.$$

Squaring shows that for any unit  $v$  in  $S' = S/(\rho - \bar{\rho})S$  we have  $v^4 = 1$ . Now the Units Theorem shows that  $U(S)$  is the direct product of the cyclic group  $\langle -1 \rangle$  of order two, generated by  $-1$  with an infinite cyclic group  $\langle \epsilon \rangle$  for some unit  $\epsilon$ , called the *fundamental unit*. It then follows that  $\text{Im } \pi \in U(S')$  is a group of exponent dividing four, generated by two elements, one, namely  $\pi(-1)$ , of order at most two. In particular  $\text{Im } \pi$  has at most eight elements.

Suppose first that  $m = 2p_1 \cdots p_r$  with  $p_i$  distinct odd primes. Then  $S' \cong \mathbb{Z}/2m\mathbb{Z} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r\mathbb{Z}$ . If this ring is to have at most eight units we must clearly have  $p_i \leq 5$ . Indeed  $m = 6$  or  $m = 10$  are the only possibilities, since  $m = 30$  produces more than eight units. However, if  $m = 6$  or  $10$ ,  $S$  is not a *PID* [11, "Tables"] so by Corollary 4.2 we can not have  $\text{Pic}(S \otimes S) = 0$ . Thus in all possible remaining cases,  $n = 2k$  implies  $\pi$  is not onto and again Corollary 4.2 shows  $\text{Pic}(S \otimes S) \neq 0$ .

Consider next  $m \equiv 3 \pmod{4}$  and write  $m = p_1 \cdots p_r$  as the product of distinct odd primes. Then

$$S' \cong Z/2mZ = Z/2Z \times Z/p_1Z \times \cdots \times Z/p_rZ.$$

In order to have at most eight units we must have each  $p_i \leq 7$ . But some  $p_i = 7$  would entail a unit of order three which can not happen. Since  $m > 5$ , we see that  $\pi$  is onto possibly only if  $m = 15$ . But in this case  $S$  is not a *PID* [11, "Tables"] so again we can not have  $\text{Pic}(S \otimes S) = 0$ .

Finally there remains the case  $m \equiv 1 \pmod{4}$ . If  $m = p_1 p_2 \cdots p_r$ , then the units of  $S = Z/mZ = Z/p_1Z \times \cdots \times Z/p_rZ$  are the same as those of  $Z/2mZ = Z/2Z \times Z/p_1Z \times \cdots \times Z/p_rZ$  so the same argument as above for  $m \equiv 3$  shows that  $\text{Pic}(S \otimes S) = 0$  only for the listed values of  $m \equiv 1 \pmod{4}$ , completing the proof.

5.  $B(S/Z)$ . All notation is as in [7].

**THEOREM 5.0.** *Let  $K = Q(\sqrt{m})$  with  $m$  a square free integer and  $Q$  the rationals. Let  $S$  be the ring of integers of  $K$ . Then the split Brauer group  $B(S/Z)$  is zero when  $m = -3, -1, 2, 3$  or  $5$ .*

*Proof.* In each case  $S$  is euclidean [12, Propn. 6-4-1] hence a *PID*. Thus as remarked in § 4,  $\text{Pic } S = 0$ , so that  $H^0(S/R, \text{Pic})$ , being a subgroup of  $\text{Pic } S$ , is zero. By Theorem 4.3,  $\text{Pic}(S \otimes S) = 0$ , hence  $H^1(S/Z, \text{Pic})$ , which is a homomorphic image of a subgroup of  $\text{Pic}(S \otimes S)$ , is zero. It then follows from Theorem 7.6 of [7] that  $B(S/Z) \cong H^2(S/Z, U)$  and the result follows from Theorems 3.0 and 3.2.

Using the global class field theory, one can prove that in fact  $B(S/Z) \cong B(Z) = 0$  [9]. Dobbs [8] has exploited this fact to obtain an improvement of our Theorems 3.0 and 3.2. Of course the conclusion of Theorem 4.5 is more than is needed to show  $B(S/Z) = 0$ . It would suffice to prove directly that  $H^1(S/Z, \text{Pic}) = 0$  or that the map  $H^1(S/Z, \text{Pic}) \rightarrow H^3(S/Z, U)$  given in Theorem 7.6 of [7] is a monomorphism. However,  $H^1(S/Z, \text{Pic})$  does not seem amenable to computation at the present time.

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