Pacific Journal of Mathematics

ON CERTAIN POSET AND SEMILATTICE HOMOMORPHISMS

CARLOS JOHNSON, JR.

Vol. 39, No. 3 July 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 3, 1971

ON CERTAIN POSET AND SEMILATTICE HOMOMORPHISMS

C. S. JOHNSON, JR.

In this paper a coordinatizing semigroup is used to define and characterize certain homomorphisms on a bounded poset or semilattice. These homomorphisms are determined by their kernels and in the semilattice case the ideals which occur as such kernels are characterized.

1. Introduction. In [4] B. J. Thorne characterized certain congruence relations on a bounded lattice by looking at AP homomorphisms on a coordinatizing Baer semigroup. We intend to carry out a similar procedure for bounded posets and semilattices. It will turn out that one of our semilattice results gives Thorne's central result as a corollary.

Our notation will be that of [4]. If S is a semigroup with 0 and $A \subseteqq S$ we define $L(A) = \{x \in S; xa = 0 \text{ for all } a \in A\}$, $R(A) = \{x \in S; ax = 0 \text{ for all } a \in A\}$, $LR(A) = L(R(A))$, $RL(A) = R(L(A))$, and so forth. If $x \in S$ we write $L(\{x\}) = L(x)$ and $R(\{x\}) = R(x)$. We define $\mathscr{L}(S) = \{L(x); x \in S\}$ and $\mathscr{R}(S) = \{R(x); x \in S\}$ and say that S *coordinatizes* a poset P in case $P \cong \mathscr{L}(S)$ when $\mathscr{L}(S)$ is partially ordered by set inclusion.

The coordinatization machinery which we will use is developed in [2]. The following is a summary of the relevant material.

DEFINITION 1.1. A semigroup S with 0 and 1 will be called a *pre-Baer semigroup* in case, for each $x \in S$, there exist elements $x^r, x^l \in S$ such that $LR(x) = L(x^r)$ and $RL(x) = R(x^l)$.

Recall that a map ϕ of a poset P into itself is *residuated* if the inverse image of a principal ideal is again a principal ideal or, equivalently, if ϕ is isotone and there is another isotone map ϕ^+ (called a *residual* map) of P into itself such that $x\phi^+\phi \leqq x \leqq x\phi\phi^+$ for all $x \in P$.

LEMMA 1.2. *If S is a pre-Baer semigroup and $z \in S$, then $\phi_z: \mathscr{L}(S) \to \mathscr{L}(S)$ given by $LR(x)\phi_z = LR(xz)$ is residuated with $\phi_z^+: \mathscr{L}(S) \to \mathscr{L}(S)$ given by $L(x)\phi_z^+ = L(zx)$ as its residual.*

If P is a bounded poset we use $S(P)$ to denote the semigroup of residuated maps on P.

THEOREM 1.3. *Every bounded poset can be coordinatized by a pre-*

Baer semigroup. In particular, if P is a bounded poset, then $S(P)$ is a pre-Baer semigroup which coordinatizes P. If S is any other pre-Baer semigroup which coordinatizes P, then $z \mapsto \phi_z$ is a homomorphism, with kernel 0, *of S into $S(P)$ and the image of S in $S(P)$ is a pre-Baer semigroup which coordinatizes P.*

DEFINITION 1.4. A pre-Baer semigroup S is a *right Baer semigroup* in case for each $x \in S$ there exists an idempotent $x^r \in S$ such that $R(x) = x^r S$, i.e., such that $xy = 0 \Leftrightarrow y = x^r y$. S is a *left Baer semigroup* in case for each $x \in S$ there exists an idempotent $x^l \in S$ such that $L(x) = Sx^l$.

THEOREM 1.5. *Every right* (resp., left) *Baer semigroup coordinatizes a bounded join* (resp., meet) *semilattice. Conversely, every bounded join* (resp., meet) *semilattice can be coordinatized by a right* (resp., left) *Baer semigroup. In particular, if P is a bounded join* (resp., meet) *semilattice, then $S(P)$ is a right* (resp., left) *Baer semigroup which coordinatizes P. If S is any other right* (resp., left) *Baer semi-group which coordinatizes P then the image of S in $S(P)$ under the homomorphism $z \mapsto \phi_z$ is a right* (resp., left) *Baer semigroup.*

REMARK. If S is a right Baer semigroup the join operation in $\mathscr{L}(S)$ is given by $LR(x) \vee LR(y) = L(y^r(xy^r)^l) = LR(x)\phi_{y^r}\phi^+_{y^r}$. If S is a left Baer semigroup the meet operation in $\mathscr{L}(S)$ is given by $L(x) \cap L(y) = LR((y^l x)^l y^l) = L(x)\phi^+_{x^l}\phi_{y^l}$.

2. Homomorphisms preserving r and l.

DEFINITION 2.1. A homomorphism ϕ of a pre-Baer semigroup S onto a semigroup T is called *r-preserving* in case, for each $x \in S$, $LR(x\phi) = L(x^r\phi)$ for some choice of x^r. (Recall x^r is such that $LR(x) = L(x^r)$.) ϕ is *l-preserving* in case, for each $x \in S$, $RL(x\phi) = R(x^l\phi)$ for some choice of x^l. (Recall x^l is such that $RL(x) = R(x^l)$.) Notice that if ϕ is r-and l- preserving, then T is a pre-Baer semigroup.

LEMMA 2.2. *Let ϕ be a homomorphism of a pre-Baer semigroup S onto a semigroup T.*

(i) *If ϕ is r-preserving, then $\Phi: \mathscr{L}(S) \to \mathscr{L}(T)$ given by $LR(x)\Phi = LR(x\phi)$ is well defined and isotone.*

(ii) *If ϕ is l-preserving, then $\Phi: \mathscr{L}(S) \to \mathscr{L}(T)$ given by $L(x)\Phi = L(x\phi)$ is well defined and isotone.*

Proof. (i). Suppose that ϕ is r-preserving and that $LR(x) \subseteq LR(y)$. Choose y^r so that $LR(y\phi) = L(y^r\phi)$. Then we have $LR(x) \subseteq LR(y) \Rightarrow x \in LR(x) \subseteq L(y^r) \Rightarrow xy^r = 0 \Rightarrow x\phi y^r\phi = 0 \Rightarrow x\phi \in L(y^r\phi) = LR(y\phi) \Rightarrow LR(x\phi) \subseteq$

$LR(y\phi)$. This shows that Φ is well defined and isotone. Finally, $LR(x)\Phi = L(x^r\phi) \in \mathscr{L}(T)$.

(ii). Suppose that ϕ is l-preserving and that $L(x) \subseteqq L(y)$. Choose x^l so that $RL(x\phi) = R(x^l\phi)$. Then we have $L(x) \subseteqq L(y) \Rightarrow RL(y) \subseteqq RL(x) \Rightarrow y \in RL(y) \subseteqq R(x^l) \Rightarrow x^l y = 0 \Rightarrow x^l\phi y\phi = 0 \Rightarrow y\phi \in R(x^l\phi) = RL(x\phi) \Rightarrow RL(y\phi) \subseteqq RL(x\phi) \Rightarrow L(x\phi) \subseteqq L(y\phi)$. This makes Φ well defined and isotone.

REMARK. Notice that, in part (i) of the lemma, $L(x)\Phi = LR(x^l)\Phi = LR(x^l\phi)$. Hence $L(x)\Phi = L(x\phi)$ for all $x \in S$ iff ϕ is l-preserving. Similarly, in part (ii), $LR(x)\Phi = L(x^r\phi)$ and it is clear that $LR(x)\Phi = LR(x\phi)$ for all $x \in S$ iff ϕ is r-preserving. If ϕ is r-and l-preserving, then the mappings in parts (i) and (ii) of the lemma coincide.

If S is a pre-Baer semigroup and $\phi: S \twoheadrightarrow T$ (i.e., from S onto T) an r-preserving homomorphism, then the map defined in part (i) of Lemma 2.2 induces an equivalence relation $\equiv$ on $\mathscr{L}(S)$ by the rule $LR(x) \equiv LR(y)$ iff $LR(x)\Phi = LR(y)\Phi$ iff $LR(x\phi) = LR(y\phi)$. It is this equivalence relation we wish to examine.

DEFINITION 2.3. If S is a pre-Baer semigroup and $\phi: S \twoheadrightarrow T$ an r-preserving homomorphism, then the equivalence relation on $\mathscr{L}(S)$ just described will be called the *equivalence relation on $\mathscr{L}(S)$ induced by ϕ*.

DEFINITION 2.4. An equivalence relation $\equiv$ on $\mathscr{L}(S)$ where S is a pre-Baer semigroup is *S-compatible* in case $LR(x) \equiv LR(y) \Rightarrow LR(x)\phi_z \equiv LR(y)\phi_z$ for all $z \in S$. It is *S^+-compatible* in case $LR(x) \equiv LR(y) \Rightarrow LR(x)\phi_z^+ \equiv LR(y)\phi_z^+$ for all $z \in S$.

DEFINITION 2.5. An equivalence relation $\equiv$ on a poset P is *ordered* if $P/\equiv$ is partially ordered by the rule $[x] \leqq [y] \Leftrightarrow$ there exist elements $x_1 \in [x]$ and $y_1 \in [y]$ such that $x_1 \leqq y_1$.

REMARK. Congruence relations on lattices and semilattices are ordered.

LEMMA 2.6. *If $\equiv$ is an equivalence relation on $\mathscr{L}(S)$, S a pre-Baer semigroup, and $\mathscr{L}(S)/\equiv$ is partially ordered in such a way that $LR(x) \subseteqq LR(y) \Rightarrow [LR(x)] \leqq [LR(y)]$, then the following are equivalent.*

(a) $[LR(x)\phi_{z^r}] = [0] \Rightarrow [LR(x)] \leqq [0\phi_{z^r}^+]$, *for all* $x \in S$.

(b) $[LR(x)] = [0] \Rightarrow [LR(x)\phi_{z^r}^+] = [0\phi_{z^r}^+]$, *for all* $x \in S$.

(b′) $LR(x) \equiv 0 \Rightarrow LR(x)\phi_{z^r}^+ \equiv 0\phi_{z^r}^+ = LR(z)$, *for all* $x \in S$.

Proof. (b) $\Leftrightarrow$ (b′). This is only a difference in notation.

(a) $\Rightarrow$ (b). Suppose $[LR(x)] = [0]$. Since $LR(x)\phi_{z^r}^+\phi_{z^r} \subseteqq LR(x)$, we have $[LR(x)\phi_{z^r}^+\phi_{z^r}] = [0]$. Now by (a), $[LR(x)\phi_{z^r}^+] \leqq [0\phi_{z^r}^+]$. The reverse inequality holds since $0\phi_{z^r}^+ \subseteqq LR(x)\phi_{z^r}^+$.

(b) $\Rightarrow$ (a). If $[LR(x)\phi_{z^r}] = [0]$, we have by (b) that $[LR(x)\phi_{z^r}\phi_{z^r}^+] = [0\phi_{z^r}^+]$. Now $LR(x) \subseteqq LR(x)\phi_{z^r}\phi_{z^r}^+$ gives $[LR(x)] \leqq [LR(x)\phi_{z^r}\phi_{z^r}^+] = [0\phi_{z^r}^+]$.

THEOREM 2.7. *If S is a pre-Baer semigroup and $\phi: S \twoheadrightarrow T$ an r-preserving homomorphism, the equivalence relation $\equiv$ on $\mathscr{L}(S)$ induced by ϕ has the following properties*:

(i) *For each $z \in S$, z^r can be chosen so that $LR(x) \equiv 0 \Rightarrow LR(x)\phi_{z^r}^+ \equiv 0\phi_{z^r}^+$ for all $x \in S$.*

(ii) *$\equiv$ is ordered.*

(iii) *$\equiv$ is S-compatible.*

In part (i) any z^r such that $L(z^r\phi) = LR(z\phi)$ suffices.

Proof. Recall that $LR(x) \equiv LR(y) \Leftrightarrow LR(x\phi) = LR(y\phi)$.

(i). $\mathscr{L}(S)/\equiv$ is partially ordered by $[LR(x)] \leqq [LR(y)] \Leftrightarrow LR(x\phi) \subseteqq LR(y\phi)$. Choose z^r so that $L(z^r\phi) = LR(z\phi)$. Since $LR(x) \subseteqq LR(y) \Rightarrow LR(x\phi) \subseteqq LR(y\phi)$ by Lemma 2.2, we can apply Lemma 2.6. Since $LR(x)\phi_{z^r} \equiv 0 \Rightarrow LR(xz^r\phi) = 0 \Rightarrow x\phi z^r\phi = 0 \Rightarrow x\phi \in L(z^r\phi) \Rightarrow LR(x\phi) \subseteqq LR(z\phi)$ for all $x \in S$, part (a) of Lemma 2.6 is satisfied and part (b) is what we are trying to prove.

(ii). It will suffice to show that $LR(x\phi) \subseteqq LR(y\phi) \Rightarrow$ there exists $y_1 \in S$ such that $LR(x) \subseteqq LR(y_1)$ and $LR(y_1\phi) = LR(y\phi)$. If $LR(x\phi) \subseteqq LR(y\phi) = L(y^r\phi)$, we have $x\phi y^r\phi = 0 \Rightarrow LR(xy^r\phi) = 0 \Rightarrow LR(xy^r) = 0$. By (i), $LR(xy^r)\phi_{y^r}^+ \equiv 0\phi_{y^r}^+ = LR(y)$. Since $LR(xy^r)\phi_{y^r}^+ = L(y^r(xy^r)^r) = LR((y^r(xy^r)^r)^l)$, this says that $LR((y^r(xy^r)^r)^l\phi) = LR(y\phi)$. Letting $y_1 = (y^r(xy^r)^r)^l$ finishes the proof since $x \in L(y^r(xy^r)^r) \Rightarrow LR(x) \subseteqq L(y^r(xy^r)^r) = LR((y^r(xy^r)^r)^l) = LR(y_1)$.

(iii). $LR(x) \equiv LR(y) \Rightarrow LR(x\phi) = LR(y\phi) \Rightarrow LR(x\phi z\phi) = LR(y\phi z\phi) \Rightarrow LR(xz\phi) = LR(yz\phi) \Rightarrow LR(x)\phi_z \equiv LR(y)\phi_z$.

The equivalence relation in Theorem 2.7 has another nice property. It is determined by its kernel.

THEOREM 2.8. *Let $\equiv$ be the equivalence relation of Theorem 2.7. The following are equivalent.*

(a) $LR(x) \equiv LR(y)$.

(b) *If $L(x^r\phi) = LR(x\phi)$ and $L(y^r\phi) = LR(y\phi)$, then $LR(x)\phi_{y^r} \equiv 0$ and $LR(y)\phi_{x^r} \equiv 0$.*

Proof. (a) $\Rightarrow$ (b). Since $\equiv$ is S-compatible, $LR(x) \equiv LR(y) \Rightarrow LR(x)\phi_{y^r} \equiv LR(y)\phi_{y^r} = 0$. Similarly $LR(y)\phi_{x^r} \equiv 0$.

(b) $\Rightarrow$ (a). Part (b) of Lemma 2.6 is satisfied by Theorem 2.7, so by part (a) of Lemma 2.6, $LR(x)\phi_{y^r} \equiv 0 \Rightarrow [LR(x)] \leqq [0\phi_{y^r}^+] = [LR(y)]$. Similarly $LR(y)\phi_{x^r} \equiv 0 \Rightarrow [LR(y)] \leqq [LR(x)]$. Thus $[LR(x)] = [LR(y)]$.

We now wish to show that any equivalence relation on $\mathscr{L}(S)$ having the three properties of Theorem 2.7 is induced on $\mathscr{L}(S)$ by some r-preserving homomorphism.

LEMMA 2.9. *Let S be a pre-Baer semigroup and let $\equiv$ be an S-compatible equivalence relation on $\mathscr{L}(S)$. For each $z \in S$ define Φ_z: $\mathscr{L}(S)/\equiv \to \mathscr{L}(S)/\equiv$ by $[LR(x)]\Phi_z = [LR(x)\phi_z] = [LR(xz)]$. Φ_z is well defined because of S-compatibility. Let S' denote the semigroup generated by $\{\Phi_z; z \in S\}$ under composition. The map $z \mapsto \Phi_z$ is a homomorphism of S onto S' and if $\equiv$ also possesses properties* (i) *and* (ii) *of Theorem 2.7, this homomorphism is r-preserving.*

Proof. It is a clear that $z \mapsto \Phi_z$ is a homomorphism of S onto S'. Let $z \in S$ and choose z^r to satisfy part (i) of Theorem 2.7. $\Phi_z\Phi_{z^r} = 0$ since $zz^r = 0$ so we have $LR(\Phi_z) \subseteqq L(\Phi_{z^r})$. To show that $L(\Phi_{z^r}) \subseteqq LR(\Phi_z)$ we suppose that $\Phi_x \in L(\Phi_{z^r})$ and show that $\Phi_y \in R(\Phi_z)$ implies $\Phi_x\Phi_y = 0$. Since $\Phi_{xz^r} = 0$ we have $[LR(1)]\Phi_{xz^r} = [LR(xz^r)] = 0$ and by Lemma 2.6, which applies since we are assuming part (i) of Theorem 2.7, $[LR(x)] \leqq [LR(z)]$. Since $\equiv$ is ordered, the elements of S' are isotone maps and we have $[LR(xy)] = [LR(x)]\Phi_y \leqq [LR(z)]\Phi_y = [LR(zy)] = [LR(1)]\Phi_{zy} = [0]$. Now $[LR(1)]\Phi_{xy} = [0]$ implies $\Phi_{xy} = \Phi_x\Phi_y = 0$.

REMARK. If an S-compatible equivalence relation $\equiv$ possesses properties (i) and (ii) of Theorem 2.7, and if we denote the kernel of $z \mapsto \Phi_z$ by I, then $z \mapsto \Phi_z$ is the homomorphism studied by R. S. Pierce in [3]. To prove this we must show that $\Phi_x = \Phi_y \Leftrightarrow axb \in I$ iff $ayb \in I$. Suppose $\Phi_x = \Phi_y$. Then $axb \in I \Leftrightarrow \Phi_{axb} = \Phi_a\Phi_x\Phi_b = 0 \Leftrightarrow \Phi_{ayb} = \Phi_a\Phi_y\Phi_b = 0 \Leftrightarrow ayb \in I$. Now suppose $axb \in I$ iff $ayb \in I$. Then $\Phi_{zxw} = 0$ iff $\Phi_{zyw} = 0 \Rightarrow [LR(zxw)] = [0]$ iff $[LR(zyw)] = [0] \Rightarrow [LR(zx)\phi_w] = [0]$ iff $[LR(zy)\Phi_w] = [0]$. Setting $w = (zx)^r$, where $(zx)^r$ is chosen as in part (i) of Theorem 2.7, and using part (a) of Lemma 2.6 we have $[LR(zy)] \leqq [L((zx)^r)] = [LR(zx)]$. Similarly we have $[LR(zx)] \leqq [LR(zy)]$. Thus $[LR(zx)] = [LR(zy)]$ for all $z \in S$, but this just says that $\Phi_x = \Phi_y$.

THEOREM 2.10. *Let S be a pre-Baer semigroup and let $\equiv$ be an equivalence relation on $\mathscr{L}(S)$ which possesses properties* (i), (ii), *and* (iii) *of Theorem 2.7. Then $\equiv$ is induced on $\mathscr{L}(S)$ by the r-preserving homomorphism $z \mapsto \Phi_z$ described in Lemma 2.9. Furthermore, $z \mapsto \Phi_z$ is the largest r-preserving homomorphism* (considered as a congruence relation on S) *which induces* $\equiv$.

Proof. Consider the r-preserving homomorphism $z \mapsto \Phi_z$ of Lemma 2.9. We wish to show that $LR(\Phi_x) = LR(\Phi_y)$ iff $LR(x) \equiv LR(y)$. Let $LR(\Phi_x) = LR(\Phi_y)$ and choose y^r as in part (i) of Theorem 2.7. Then

$R(\Phi_x) = R(\Phi_y)$ and we have $\Phi_x\Phi_{y^r} = 0$ since $\Phi_y\Phi_{y^r} = 0$. $\Phi_{xy^r} = 0$ means $[LR(x^r y)] = [0]$ and by Lemma 2.6 $[LR(x)] \leqq [LR(y)]$. Similarly we get $[LR(y)] \leqq [LR(x)]$ and thus $LR(x) \equiv LR(y)$. Conversely, suppose $LR(x) \equiv LR(y)$. Choose x^r and y^r such that $L(\Phi_{x^r}) = LR(\Phi_x)$ and $L(\Phi_{y^r}) \equiv LR(\Phi_y)$. By S-compatibility we have $LR(xy^r) \equiv LR(yy^r) = 0$ and $LR(yx^r) \equiv LR(xx^r) = 0$. This means $\Phi_{xy^r} = \Phi_{yx^r} = 0$. Now $\Phi_x \in L(\Phi_{y^r}) = LR(\Phi_y)$ gives $LR(\Phi_x) \subseteqq LR(\Phi_y)$ and $\Phi_y \in L(\Phi_{x^r}) = LR(\Phi_x)$ gives $LR(\Phi_y) \subseteqq LR(\Phi_x)$.

Finally, suppose ϕ is another r-preserving homomorphism which induces $\equiv$. Then $x\phi = y\phi \Rightarrow zx\phi = zy\phi$ for all $z \in S \Rightarrow LR(zx\phi) = LR(zy\phi)$ for all $z \in S \Rightarrow LR(z)\phi_x \equiv LR(z)\phi_y$ for all $z \in S \Rightarrow \Phi_x = \Phi_y$.

REMARK. The r-preserving homomorphisms which induce $\equiv$ all have the same kernel since, if ϕ is such a homomorphism, $x\phi = 0 \Leftrightarrow LR(x\phi) = 0 \Leftrightarrow LR(x) \equiv 0$.

THEOREM 2.11. *Let S be a pre-Baer semigroup and $\phi: S \twoheadrightarrow T$ an r-preserving homomorphism. Let $\Phi: \mathscr{L}(S) \rightarrow \mathscr{L}(T)$ be the map described in Lemma 2.2 (i), i.e., $LR(x)\Phi = LR(x\phi)$. The following are equivalent.*

(a) $\ker \phi \in \mathscr{L}(S)$.

(b) $\ker \Phi$ *is a principal ideal.*

(c) $\Phi: \mathscr{L}(S) \rightarrow \mathscr{L}(T)$ *is residuated.*

Proof. (a) $\Leftrightarrow$ (b). This follows from the observation that $x \in \ker \phi \Leftrightarrow x\phi = 0 \Leftrightarrow LR(x\phi) = 0 \Leftrightarrow LR(x) \in \ker \Phi$.

(c) $\Rightarrow$ (b). This is clear.

(a) $\Rightarrow$ (c). Suppose $\ker \phi = LR(w)$. Define $\Phi^+: \mathscr{L}(T) \rightarrow \mathscr{L}(S)$ by $L(x\phi)\Phi^+ = L(xw^r)$. Φ^+ is well defined and isotone since when $L(x\phi) \subseteqq L(y\phi)$ we have $z \in L(xw^r) \Rightarrow zxw^r = 0 \Rightarrow zx \in \ker \phi \Rightarrow z\phi x\phi = 0 \Rightarrow z\phi \in L(x\phi) \subseteqq L(y\phi) \Rightarrow z\phi y\phi = 0 \Rightarrow zy \in \ker \phi \Rightarrow zyw^r = 0 \Rightarrow z \in L(yw^r)$, which says that $L(xw^r) \subseteqq L(yw^r)$. Choose x^r so that $L(x^r\phi) = LR(x\phi)$. Now since $x \in L(x^r w^r)$ we have $LR(x) \subseteqq L(x^r w^r) = L(x^r\phi)\Phi^+ = LR(x\phi)\Phi^+ = LR(x)\Phi\Phi^+$. Now all that remains is to show that $L(x\phi)\Phi^+\Phi \subseteqq L(x\phi)$. Since $(xw^r)^l xw^r = 0 \Rightarrow (xw^r)^l x \in \ker \phi \Rightarrow (xw^r)^l \phi x\phi = 0 \Rightarrow (xw^r)^l \phi \in L(x\phi)$ we have $L(x\phi)\Phi^+\Phi = L(xw^r)\Phi = LR((xw^r)^l\phi) \subseteqq L(x\phi)$.

If S is a pre-Baer semigroup and $z \in S$, notice that $\mathscr{R}(S)$ is dual isomorphic to $\mathscr{L}(S)$ and the residuated map on $\mathscr{R}(S)$ given by $RL(x) \mapsto RL(zx)$, considered as a map on $\mathscr{L}(S)$, is ϕ_z^+. (See Lemma 1.2.) Bearing this in mind and applying left-right duality to the results obtained thus far, we find that every l-preserving homomorphism on a pre-Baer semigroup S induces on $\mathscr{L}(S)$ an ordered S^+-compatible equivalence relation $\equiv$ with the property that, for each $z \in S$, z^l can be chosen so that $LR(x) \equiv 1 \Rightarrow LR(x)\phi_{y^l} \equiv 1\phi_{z^l}$ for all $x \in S$. Furthermore, every such equivalence relation on $\mathscr{L}(S)$ is induced by some l-preserving homomorphism on S. We now have

THEOREM 2.12. *Let ϕ be an r-and l-preserving homomorphism on a pre-Baer semigroup S. The ordered equivalence relation on $\mathscr{L}(S)$ induced by ϕ is S- and S^+-compatible. Furthermore, every S- and S^+-compatible ordered equivalence relation on $\mathscr{L}(S)$ is induced by some r- and l-preserving homomorphism on S.*

Proof. This follows from previous results and the remarks preceding the theorem if we make the following observation: If an ordered equivalence relation $\equiv$ on $\mathscr{L}(S)$ is S- and S^+-compatible, then $\Phi_z: \mathscr{L}(S)/\equiv \to \mathscr{L}(S)/\equiv$ given by $[LR(x)]\Phi_z = [LR(x)\phi_z]$ is residuated with residual $\Phi_z^+: \mathscr{L}(S)/\equiv \to \mathscr{L}(S)/\equiv$ given by $[LR(x)]\Phi_z^+ = [LR(x)\Phi_z^+]$. Since residuated maps uniquely determine their residuals and vice versa, the r-preserving homomorphism $z \mapsto \Phi_z$ (considered as a congruence on S) coincides with the l-preserving congruence on S associated with the anti-homomorphism $z \mapsto \Phi_z^+$.

3. RAP and LAP homomorphisms.

DEFINITION 3.1. If S is a right Baer semigroup, a semigroup homomorphism $\phi: S \twoheadrightarrow T$ is *right annihilator preserving* or *RAP* in case $R(x\phi) = R(x)\phi$. Notice that $R(x)\phi = (x^r\phi)T$. Dually, if S is a left Baer semigroup, ϕ is *left annihilator preserving* or *LAP* in case $L(x\phi) = L(x)\phi$. Finally, ϕ is *annihilator preserving* or *AP* if it is both *RAP* and *LAP*.

REMARK. Any *RAP* homomorphism is r-preserving since $LR(x\phi) = L((x^r\phi)T) = L(x^r\phi)$. Dually, any *LAP* homomorphism is l-preserving.

LEMMA 3.2. *In a right Baer semigroup S we have*

(i) $LR(x) \vee LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} = LR(y) \vee LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r}$.

(ii) $LR(zy) \vee LR(xy) = LR(zx^ry) \vee LR(xy)$.

(iii) $LR(x) \vee LR(y) \vee LR(xy^r) = LR(y) \vee LR(xy^r)$.

Proof. It is shown in [2] that, in a right Baer semigroup S, $R(x) \cap R(y) \in \mathscr{R}(S)$ and that the join operation in $\mathscr{L}(S)$ is given by $LR(x) \vee LR(y) = L(R(x) \cap R(y))$.

(i). It is enough to show that $R(x) \cap R(xy^r) \cap R(yx^r) = R(y) \cap R(xy^r) \cap R(yx^r)$. If $z \in R(x) \cap R(xy^r) \cap R(yx^r)$, then $z = x^rz$ and $yz = yx^rz = 0$ so $z \in R(y) \cap R(xy^r) \cap R(yx^r)$. The other inclusion follows by symmetry.

(ii). It is enough to show $R(zy) \cap R(xy) = R(zx^ry) \cap R(xy)$. This follows from the observation that if $xyw = 0$, then $yw = x^r yw$ so that $zyw = 0 \Leftrightarrow zx^r yw = 0$.

(iii). It is enough to show that $R(y) \cap R(xy) \subseteq R(x) \cap R(y) \cap R(xy^r)$. If $yw = 0$, then $w = y^r w$ so that $xy^r w = 0 \Rightarrow xw = 0$.

LEMMA 3.3. *If S is a right Baer semigroup and $\equiv$ is an S-compatible equivalence relation on $\mathscr{L}(S)$, the following are equivalent.*

(a) $\equiv$ *is a join congruence.*

(b) $LR(x) \vee LR(z) = LR(y) \vee LR(z)$, $LR(z) \equiv 0 \Rightarrow LR(x) \equiv LR(y)$.

Proof. (a) $\Rightarrow$ (b). Since $LR(z) \equiv 0$, we have $LR(x) = LR(x) \vee 0 \equiv LR(x) \vee LR(z) = LR(y) \vee LR(z) \equiv LR(y) \vee 0 = LR(y)$.

(b) $\Rightarrow$ (a). Suppose $LR(x) \equiv LR(y)$. If $LR(z) \in \mathscr{L}(S)$, we have, using Lemma 3.2, that $LR(x) \vee LR(z) \vee LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} = LR(y) \vee LR(z) \vee LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r}$. To show that $LR(x) \vee LR(z) \equiv LR(y) \vee LR(z)$ it will suffice, by (b), to show $LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \equiv 0$. Since $\equiv$ is S-compatible we have $LR(x)\phi_{y^r} \equiv LR(y)\phi_{y^r} = 0 = LR(x)\phi_{x^r} \equiv LR(y)\phi_{x^r}$. Using (b), $LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \vee LR(y)\phi_{x^r} = LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r}$ and $LR(y)\phi_{x^r} \equiv 0 \Rightarrow LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \equiv LR(x)\phi_{y^r} \equiv 0$.

THEOREM 3.4. *Let S be a right Baer semigroup and $\phi: S \twoheadrightarrow T$ an RAP homomorphism. Then the equivalence relation $\equiv$ induced on $\mathscr{L}(S)$ by ϕ* (recall $LR(x) \equiv LR(y)$ iff $LR(x\phi) = LR(y\phi)$) *is an S-compatible join congruence.*

Proof. S-compatibility was proven in Theorem 2.7. By Lemma 3.3 it is sufficient to show that $LR(x) \vee LR(z) = LR(y) \vee LR(z)$ and $LR(z\phi) = 0 \Rightarrow LR(x\phi) = LR(y\phi)$. Now $LR(z\phi) = 0$ means that $R(z\phi) = (z^r\phi)T = T$, so $1\phi = z^r\phi 1\phi = z^r\phi$. Since $LR(xz^r) = (LR(x) \vee LR(z))\phi_{z^r} = (LR(y) \vee LR(z))\phi_{z^r} = LR(yz^r)$, we have $LR(x\phi) = LR(xz^r\phi) = LR(yz^r\phi) = LR(y\phi)$.

An S-compatible join congruence is determined by its kernel in the following manner.

THEOREM 3.5. *If S is a right Baer semigroup and $\equiv$ is an S-compatible join congruence on $\mathscr{L}(S)$, the following are equivalent.*

(a) $LR(x) \equiv LR(y)$.

(b) $LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \equiv 0$.

(c) *There is an* $LR(z) \equiv 0$ *such that* $LR(x) \vee LR(z) = LR(y) \vee LR(z)$.

Proof. (a) $\Rightarrow$ (b). If $LR(x) \equiv LR(y)$, then $LR(x)\phi_{y^r} \equiv LR(y)\phi_{y^r} = 0 = LR(x)\phi_{x^r} \equiv LR(y)\phi_{x^r}$ and hence $LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \equiv 0$.

(b) $\Rightarrow$ (c). Follows from part (i) of Lemma 3.2.
(c) $\Rightarrow$ (a). Follows from Lemma 3.3.

COROLLARY 3.6. *An S-compatible join congruence* $\equiv$ *has the property that, for each* $z \in S$, *any choice of* z^r *gives* $LR(x) \equiv 0 \Rightarrow LR(x)\phi^+_{z^r} \equiv 0\phi^+_{z^r}$ *for all* $x \in S$.

Proof. Since a join congruence is ordered, it is sufficient by Lemma 2.6 to show that $LR(xz^r) \equiv 0 \Rightarrow [LR(x)] \leqq [LR(z)]$. Since by part (iii) of Lemma 3.2 we have $LR(x) \vee LR(z) \vee LR(xz^r) = LR(z) \vee LR(xz^r)$, it follows from the theorem that when $LR(xz^r) \equiv 0$, $LR(x) \vee LR(z) \equiv LR(z)$. Since $\equiv$ is a join congruence, this says that $[LR(x)] \leqq [LR(z)]$.

THEOREM 3.7. *If S is a right Baer semigroup and* $\equiv$ *is an S-compatible join congruence on* $\mathscr{L}(S)$, *then the homomorphism* $z \mapsto \Phi_z$ *described in Lemma* 2.9 *is RAP.*

Proof. We wish to show that $R(\Phi_x) = \Phi_{x^r}S'$ or, in other words, that $\Phi_x\Phi_y = 0 \Leftrightarrow \Phi_y = \Phi_{x^r}\Phi_y$. Notice that $\Phi_x\Phi_y = 0 \Leftrightarrow [1]\Phi_x\Phi_y = [0] \Leftrightarrow [LR(xy)] = [0] \Leftrightarrow LR(xy) \equiv 0$ and that $\Phi_y = \Phi_{x^r}\Phi_y \Leftrightarrow LR(zy) \equiv LR(zx^ry)$ for all $z \in S$. Since it is clear that $\Phi_y = \Phi_{x^r}\Phi_y \Rightarrow \Phi_x\Phi_y = 0$, we will be done if we can show that $LR(xy) \equiv 0 \Rightarrow LR(zy) \equiv LR(zx^ry)$ for all $z \in S$. Since $LR(zy) \vee LR(xy) = LR(zx^ry) \vee LR(xy)$ by part (ii) of Lemma 3.2, $LR(xy) \equiv 0$ implies by Theorem 3.5 that $LR(zy) \equiv LR(zx^ry)$ for all $z \in S$.

COROLLARY 3.8. *If S is a right Baer semigroup, then any S-compatible join congruence* $\equiv$ *on* $\mathscr{L}(S)$ *is induced by an RAP homomorphism on S.*

Proof. Since, by Corollary 3.6, $\equiv$ has property (i) of Theorem 2.7, the proof of Theorem 2.10 applies and says that $\equiv$ is induced on $\mathscr{L}(S)$ by the homomorphism $z \mapsto \Phi_z$ on S. By Theorem 3.7, $z \mapsto \Phi_z$ is RAP.

COROLLARY 3.9. *If S is a right Baer semigroup, then every S- and* S^+*-compatible join congruence on* $\mathscr{L}(S)$ *is induced by an RAP and l-preserving homomorphism on S.*

Proof. This follows from Corollary 3.8 and from Theorem 2.12 and its proof.

COROLLARY 3.10. *If S is a left Baer semigroup, then any* S^+-

compatible meet congruence on $\mathscr{L}(S)$ is induced by an LAP homomorphism on S.

Proof. This is the dual of Corollary 3.8. (See the remarks preceding Theorem 2.12.)

COROLLARY 3.11 (Thorne). *If S is a Baer semigroup, then every S- and S^+-compatible congruence on $\mathscr{L}(S)$ is induced by an AP homomorphism on S.*

Proof. This follows from Corollaries 3.8 and 3.10 and from Theorem 2.12 and its proof.

4. Kernels of S-compatible join congruences.

THEOREM 4.1. *Let I be an ideal of a join semilattice $L = \mathscr{L}(S)$, S a right Baer semigroup. The following are equivalent.*

(a) *I is the kernel of an S-compatible join congruence.*

(b) *$I\phi_z \subseteqq I$ for each $z \in S$.*

Proof. (a) $\Rightarrow$ (b). If $LR(x) \in I$, then $LR(x) \equiv 0$ and by S-compatibility $LR(x)\phi_z \equiv 0\phi_z = 0$, i.e., $LR(x)\phi_z \in I$.

(b) $\Rightarrow$ (a). Suppose $I\phi_z \subseteqq I$ for each $z \in S$. Define $LR(x) \equiv LR(y)$ iff $LR(x) \vee LR(w) = LR(y) \vee LR(w)$ for some $LR(w) \in I$. It is easy to see that $\equiv$ is a join congruence. If $LR(x) \equiv LR(y)$, then $LR(x) \vee LR(w) = LR(y) \vee LR(w)$ with $LR(w) \in I$ and since ϕ_z, being a residuated map, preserves join we have $LR(x)\phi_z \vee LP(w)\phi_z = LR(y)\phi_z \vee LR(w)\phi_z$. Since $LR(w)\phi_z \in I$ it follows that $LR(x)\phi_z \equiv LR(y)\phi_z$. Clearly $\equiv$ has I as its kernel.

LEMMA 4.2. *In any semigroup S with 0, if R(w) is a two-sided ideal, for some $w \in S$, then LR(w) is a two-sided ideal. Hence, if S is a pre-Baer semigroup, LR(w) is two-sided if and only if R(w) is two-sided.*

Proof. Suppose $R(w)$ is two-sided. $LR(w)$ is already a left ideal so we must show that it is a right ideal. Let $x \in LR(w)$, $y \in S$, and $z \in R(w)$. We need $xyz = 0$. But $yz \in R(w)$ since $R(w)$ is two-sided and hence $xyz = 0$. The second assertion follows from the first and its dual.

Theorem 4.1 characterized kernels of S-compatible join congruences. We now look at principal ideals which occur as kernels of S-compatible join congruences.

THEOREM 4.3. *Let S be a right Baer semigroup. The following are equivalent.*

(a) $[0, LR(w)]$ *is the kernel of an S-compatible join congruence on $\mathscr{L}(S)$.*

(b) $LR(w)$ *is the kernel of an RAP homomorphism on S.*

(c) $LR(w)\phi_x \subseteqq LR(w)$ *for all $x \in S$.*

(d) $xw^r = w^r x w^r$ *for all $x \in S$ and for any choice of w^r.*

(e) $LR(w)$ *is a two-sided ideal.*

(f) $R(w)$ *is a two-sided ideal.*

Proof. (a) $\Leftrightarrow$ (b). Since every RAP homomorphism ϕ on S induces an S-compatible join congruence $\equiv$ on $\mathscr{L}(S)$ by the rule $LR(x) \equiv LR(y)$ iff $LR(x\phi) = LR(y\phi)$ and since every S-compatible join congruence arises in this manner for some ϕ, it suffices to notice that $x \in \ker \phi \Leftrightarrow x\phi = 0 \Leftrightarrow LR(x\phi) = 0 \Leftrightarrow LR(x) \equiv 0$.

(a) $\Leftrightarrow$ (c). Use Theorem 4.1.

(e) $\Leftrightarrow$ (f). Use Lemma 3.2.

(d) $\Leftrightarrow$ (f). This follows from the dual of Theorem 1 of [1].

(b) $\Rightarrow$ (e). This is obvious.

(d) $\Rightarrow$ (b). $x \mapsto xw^r$ is a homomorphism of S onto Sw^r and it is *RAP* since $yw^r \in R(xw^r) \Leftrightarrow xw^r yw^r = 0 \Leftrightarrow yw^r = w^r yw^r = x^r w^r yw^r \Leftrightarrow yw^r \in (x^r w^r)(Sw^r) = (R(x))w^r$.

REMARK. By Theorem 2.11, the kernel of an S-compatible join congruence $\equiv$ is a principal ideal if and only if $\equiv$ is residuated in the sense that the canonical join homomorphism taking $\mathscr{L}(S)$ onto $\mathscr{L}(S)/\equiv$ is a residuated map.

In light of Theorem 4.1 we make the following definition.

DEFINITION 4.4. An ideal I of a join semilattice $L = \mathscr{L}(S)$, S a right Baer semigroup, is called *S-compatible* in case $I\phi_z \subseteqq I$ for all $z \in S$.

THEOREM 4.5. *Let S be a right Baer semigroup and let $L = \mathscr{L}(S)$. The set $I_S(L)$ of S-compatible ideals of L forms a subcomplete sublattice of $I(L)$, the lattice of ideals of L. $I_S(L)$ is isomorphic to the lattice of S-compatible join congruences on $\mathscr{L}(S)$.*

Proof. If $\{I_i\}$ is a family of S-compatible ideals of $\mathscr{L}(S)$ it is clear that $\bigcap_i\{I_i\}$ is an S-compatible ideal. Suppose $LR(x) \in \bigvee_i\{I_i\}$. Then there exist

$$LR(y_1) \in I_{i_1},\ LR(y_2) \in I_{i_2},\ \cdots,\ LR(y_n) \in I_{i_n}$$

such that

$$LR(x) \subseteqq LR(y_1) \vee LR(y_2) \vee \cdots \vee LR(y_n).$$

Hence

$$\begin{aligned} LR(x)\phi_z &\subseteqq (LR(y_1) \vee LR(y_2) \vee \cdots \vee LR(y_n))\phi_z \\ &= LR(y_1)\phi_z \vee LR(y_2)\phi_z \vee \cdots \vee LR(y_n)\phi_z \end{aligned}$$

and since $LR(y_k)\phi_z \subseteqq I_{i_k}$ $(k = 1, 2, \cdots, n)$ we have $LR(x)\phi_z \in \bigvee_i\{I_i\}$. Thus $\bigvee_i\{I_i\}$ is S-compatible and we have proven the first part of the theorem. Now, if $I \in I_S(L)$ let Θ_I denote the unique S-compatible join congruence with kernel I. In light of Theorem 3.5 it is clear that $I \subseteqq J \Leftrightarrow \Theta_I \subseteqq \Theta_J$.

THEOREM 4.6. *Let S be a right Baer semigroup in which, for each $x \in S$, $LR(x^l) = LR(x^l x^l)$ for some choice of x^l. Then $I_S(L)$ is distributive and obeys the following infinite distributive law*:

$$I \cap (\bigvee_i\{J_i\}) = \bigvee_i\{I \cap J_i\} .$$

Proof. It will suffice to show $I \cap (\bigvee_i\{J_i\}) \subseteqq \bigvee_i\{I \cap J_i\}$. Suppose $L(x) = LR(x^l) \in I$ and $LR(x^l) \in \bigvee_i\{J_i\}$. Then $LR(x^l) \subseteqq LR(y_1) \vee LR(y_2) \vee \cdots \vee LR(y_n)$ where $LR(y_k) \in J_{i_k}$ $(k = 1, 2, \cdots, n)$. Now $LR(x^l) = LR(x^l)\phi_{x^l} \subseteqq LR(y_1)\phi_{x^l} \vee LR(y_2)\phi_{x^l} \vee \cdots \vee LR(y_n)\phi_{x^l}$. For $k = 1, 2, \cdots, n$ we have $LR(y_k)\phi_{x^l} \in J_{i_k}$ by S-compatibility and $LR(y_h)\phi_{x^l} = LR(y_k x_l) \subseteqq LR(x^l) \in I$. Thus $LR(y_k)\phi_{x^l} \in I \cap J_{i_k}$ for $k = 1, 2, \cdots, n$. Thus $LR(x^l) \in \bigvee_i\{I \cap J_i\}$.

REMARK. Theorem 4.6 applies, in particular, when S is a Baer semigroup. In that case x^l is taken to be an idempotent generating $L(x)$. The $LR(x^l) = LR(x^l x^l)$ condition could also be taken care of by requiring, in the definition of pre-Baer semigroup, that x^r and x^l be idempotents. (It is pointed out in [2] that all our results involving pre-Baer semigroups remain valid if x^r and x^l are required to be idempotents.)

THEOREM 4.7. *Let S be a right Baer semigroup in which, for each $x \in S$, $LR(x^l) = LR(x^l x^l)$ for some choice of x^l. Let $L = \mathscr{L}(S)$. $I_S(L)$ is pseudo complemented since it is complete and obeys the infinite distributive law of Theorem 4.6. If $I \in I_S(L)$, its pseudo complement I^* is given by $I^* = \{LR(x); LR(x) \subseteqq L(J)\}$, where J is the kernel of any RAP homomorphism which induces the S-compatible join congruence with kernel I, i.e., $y \in J \Leftrightarrow LR(y) \in I$.*

Proof. I^* is an ideal since $LR(x), LR(y) \subseteqq L(J) \Rightarrow J \subseteqq R(x) \cap$

$R(y) \Rightarrow LR(x) \vee LR(y) = L(R(x) \cap R(y)) \subseteqq L(J)$. Suppose $LR(x) \in I^*$ and $y \in S$. Then $z \in J \Rightarrow yz \in J \Rightarrow xyz = 0 \Rightarrow xy \in L(J) \Rightarrow LR(xy) \subseteqq L(J) \Rightarrow LR(x)\phi_y = LR(xy) \in I^*$. Thus I^* is S-compatible. Now suppose $L(x) \in I \cap I^*$. Then $L(x) = LR(x^l) \in I \Rightarrow x^l \in J$ and $LR(x^l) \in I^* \Rightarrow x^l \in LR(x^l) \subseteqq L(J)$. Thus $x^l x^l = 0$ and $L(x) = LR(x^l) = LR(x^l x^l) = 0$. Therefore $I \cap I^* = 0$. Finally, suppose $I \cap K = 0$, with $K \in I_S(L)$. Let $LR(x) \in K$, $y \in J$. Then $LR(y) \in I \Rightarrow LR(xy) \subseteqq LR(y) \in I$ and $LR(x) \in K \Rightarrow LR(x)\phi_y = LR(xy) \in K$. Thus $LR(xy) \in I \cap K = 0 \Rightarrow xy = 0 \Rightarrow x \in L(J) \Rightarrow LR(x) \subseteqq L(J) \Rightarrow LR(x) \in I^*$. Therefore $K \subseteqq I^*$.

References

1. T. S. Blyth and M. F. Janowitz, *On decreasing Baer semigroups*, Bull. Soc. Roy. Sci., Liège **38** (1969), 414-423.

2. C. S. Johnson, Jr., *Semigroups coordinatizing posets and semilattices*, J. London Math. Soc., to appear.

3. R. S. Pierce, *Homomorphisms of semi-groups*, Ann. of Math., **59** (1954), 287-291.

4. B. J. Thorne, *A-P congruences on Baer semigroups*, Pacific J. Math., **28** (1969). 681-698.

Recieved June 25, 1970. The results presented here were part of a thesis submitted to the University of Massachusetts. The research was supported by an NSF Traineeship. The author is grateful to Professor M. F. Janowitz for his generous help and guidance.

BOWLING GREEN STATE UNIVERSITY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 39, No. 3 July, 1971