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PROJECTIVE LATTICES AND BOUNDED HOMOMORPHISMS

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The main purpose of this paper is to prove that a finitely generated lattice is projective iff it is imbeddable in a free lattice. This result appears as a consequence of a more general theorem, in which a sufficient condition for projectivity is given in terms of the notion (due to Ralph McKenzie) of bounded homomorphism.

In [1, Theorems 4.1, 4.4] Baker and Hales completely describe the distributive projective lattices and obtain as a corollary the fact that a finite distributive lattice is projective iff it is imbeddable in a free lattice. This last result has been improved by McKenzie, who finds in [6, proof of Theorem 6.3] that for any finite lattice L , L is projective iff it is imbeddable in a free lattice. McKenzie's proof uses some ideas due to B. Jónsson. To extend the theorem to finitely generated lattices we sharpen arguments of [6]. As stated above, we use the notion of bounded homomorphism; this idea is defined by McKenzie in [6], and it plays an important role in that paper. Theorem 3.4 below was first announced in the author's abstract [4].

1. **Preliminaries.** We regard a lattice as an algebraic structure $\langle L, +, \cdot \rangle$ in which the sum (join) and product (meet) satisfy the usual equational axioms. It will not cause confusion to refer to a lattice by naming its universe. We denote by \leq the ordering of the lattice L , that is, the partial ordering naturally associated with L ($x < y$ means $x \leq y$ and $x \neq y$). If the greatest lower bound (least upper bound) of a subset U of L exists in L , it is denoted $\bigwedge U$ ($\bigvee U$).

The notation and terminology used for maps is largely standard. By an epimorphism of a lattice L into a lattice M we mean a homomorphism of L onto M . For any sets L, M, N , and any maps $f: L \rightarrow M$ and $g: M \rightarrow N$, the composite map (of L into N) is denoted $g \circ f$. (In the arrow notation, maps—whether homomorphisms or not—which are onto may be indicated by the use of a double-headed arrow, \rightarrow .)

A chain is a lattice whose ordering is a linear ordering. A chain is bounded iff it has a least element and a greatest element. Any ordinal α may be viewed as the chain whose ordering is the natural ordering of α . The set of all natural numbers is denoted ω .

We regard Boolean algebras as lattices (thus, zero, one, and complementation are not primitive operations).

The terms of the language of lattice theory are built up from individual variables v_0, v_1, \dots and the binary operation symbols \vee and \wedge (interpreted in lattices as $+$ and \cdot , respectively). The notion of *length* of a term is assumed familiar. Let τ be a term of lattice theory with variables among v_0, \dots, v_n ; let L be a lattice and $x_0, \dots, x_n \in L$. Then by $\tau[L, x_0, \dots, x_n]$ we mean the denotation of τ in L under the assignment $v_i \rightarrow x_i$ ($i \leq n$). If τ is a term of lattice theory, L and M are lattices, $x_0, \dots, x_n \in L$, and f is a homomorphism of L into M , then $f(\tau[L, x_0, \dots, x_n]) = \tau[M, fx_0, \dots, fx_n]$.

The free lattices are especially important for our present work. For any nonempty set X , we let $FL(X)$ denote some fixed lattice freely generated by X . Recall that if X_0, X_1 are disjoint nonempty subsets of X , and if x_i is in the sublattice of $FL(X)$ generated by X_i ($i = 0, 1$), then $x_0 \cdot x_1 < x_0 < x_0 + x_1$ in $FL(X)$.

Whitman's famous solution to the word problem for lattices, in [7], provides a characterization of the free lattice as follows. Suppose L is a lattice generated by $X \neq \emptyset$. Then L is freely generated by X (that is, $L \cong FL(X)$) iff all of the following hold in L : (W0) for all $x, x' \in X$, if $x \leq x'$ then $x = x'$; (W1) for all $x \in X$ and all $a, b \in L$, if $a \cdot b \leq x$ then $a \leq x$ or $b \leq x$, and if $x \leq a + b$ then $x \leq a$ or $x \leq b$; (W2) for all $a, b, c, d \in L$, if $a \cdot b \leq c + d$ then $a \leq c + d$ or $b \leq c + d$ or $a \cdot b \leq c$ or $a \cdot b \leq d$. The latter two properties (which we refer to as Whitman's (W1) and (W2)) are frequently used below. Note that (W2) makes no reference to a generating set; for any lattice L , there is no ambiguity in saying that Whitman's (W2) holds (or does not hold) in L .

We also use the following theorem, derived by Jónsson from a result of Whitman [7]:

THEOREM 1.1. (Jónsson [3, Lemma 2.6, p. 262]) *In any free lattice $FL(X)$, the following hold: for all $u, a, b, c \in FL(X)$, if $u = a \cdot b = a \cdot c$ then $u = a \cdot (b + c)$; if $u = a + b = a + c$ then $u = a + b \cdot c$.*

Finally we recall some basic facts about linear sums of lattices. Let $\langle E, \leq \rangle$ be a linearly ordered structure and let $\langle L_e : e \in E \rangle$ be a system of lattices such that $e \neq e'$ implies $L_e \cap L_{e'} = \emptyset$. Then the linear sum $\Sigma_E L_e$ is the lattice L completely determined by the following: $L = \bigcup \{L_e : e \in E\}$; for each $e \in E$, L_e is a sublattice of L ; and whenever $e \neq e'$, $x \in L_e$, $y \in L_{e'}$, then $x < y$ in L iff $e < e'$ in E . Roughly speaking, L is constructed simply by stacking up the L_e in accordance with the ordering of E .

A lattice is linearly indecomposable iff it is not the linear sum of two lattices. For any lattice L there are a linearly ordered struc-

ture E (unique up to isomorphism) and a system of lattices $\langle L_e: e \in E \rangle$ (with range $\{L_e: e \in E\}$ uniquely determined by L) such that each L_e is linearly indecomposable and $L = \Sigma_E L_e$; the L_e are called the linear components of L .

2. Projectivity and bounded homomorphisms.

DEFINITION 2.1. A lattice L is *projective* (in the category of all lattices and lattice homomorphisms) iff for any lattices M, N , and any lattice homomorphisms $h: L \rightarrow N$ and $g: M \rightarrow N$ (g onto), there is a homomorphism $f: L \rightarrow M$ such that $g \circ f = h$.

It is well-known that there are simpler descriptions of projectivity than 2.1; in particular, we have:

NOTE 2.2. For any lattice L the following three conditions are equivalent:

- (1) L is projective;
- (2) for any lattice M and any epimorphism $f: M \rightarrow L$, there is a homomorphism $g: L \rightarrow M$ such that $f \circ g$ is the identity map on L ;
- (3) there are a free lattice $FL(X)$, an epimorphism $f: FL(X) \rightarrow L$, and a homomorphism $g: L \rightarrow FL(X)$ such that $f \circ g$ is the identity map on L .

We shall use formulation (3) in this paper; (2) is used in [1].

Note that every projective lattice is isomorphic to a sublattice of a free lattice (the map g of (3) clearly must be one-to-one). Also, every free lattice is obviously projective. Baker and Hales [1, Theorem 3.1, p. 473] prove that a countable lattice is projective iff each of its linear components is projective.

In [2, Theorem 6, p. 271] Galvin and Jónsson show that a distributive lattice L is imbeddable in a free lattice iff L is countable and each linear component of L is one of the following: a one-element lattice, an eight-element Boolean algebra, or an isomorphic image of the direct product of a countable chain and a two-element chain. Using this result, Baker and Hales [1, Theorem 4.1, p. 474] characterize the distributive projective lattices as follows: a distributive lattice L is projective iff L is countable and each linear component of L is one of the following: a one-element lattice, an eight-element Boolean algebra, or an isomorphic image of the direct product of a countable bounded chain and a two-element chain.

As to non-projective lattices, the above remarks readily yield many examples. Thus, all the non-distributive modular lattices are non-projective¹, for they are not imbeddable in free lattices (recall

¹ This fact was pointed out by the referee.

that the five-element lattice with three mutually incomparable elements is imbeddable in every non-distributive modular lattice; apply Jónsson's Theorem 1.1). And, as observed in [1], the above-mentioned results on distributive lattices show that the direct product $2 \times \omega$ is a distributive non-projective lattice imbeddable in $FL(\omega)$ (and hence imbeddable in $FL(3)$, by Whitman [8, Theorem 6, p. 109]).

DEFINITION 2.3 (McKenzie [6, Definition 5.2]) Suppose L, M are lattices and f is a homomorphism of L into M . We say f is *upper bounded* iff for each $b \in M$, $\{a \in L: fa \leq b\}$ either is empty or has a greatest element; f is *lower bounded* iff for each $b \in M$, $\{a \in L: b \leq fa\}$ either is empty or has a least element. We say f is *bounded* iff it is both upper and lower bounded.

N. B. These notions are defined with respect to the entire codomain M of f , not merely with respect to the range of f . The intended codomain will be specified below in the rare cases where there is ambiguity.

NOTE 2.4. Suppose L, M are lattices and f is a homomorphism of L into M ; suppose that f , viewed as a homomorphism of L into M , is bounded. Then if N is any sublattice of M which includes the range of f , f is also bounded as a homomorphism of L into N . (Trivial.)

Usually we shall deal with epimorphisms. If f is an epimorphism of L into M , clearly f is upper iff for each $b \in M$, $\{a \in L: fa = b\}$ has a greatest element; similarly for the other two notions of 2.3.

DEFINITION 2.5. Let \mathcal{B} be the class of all lattices which are bounded epimorphic images of free lattices. ($L \in \mathcal{B}$ iff there are a free lattice $FL(X)$ and an epimorphism $f: FL(X) \rightarrow L$ such that f is bounded.)

Trivially every free lattice is in \mathcal{B} . It follows readily from McKenzie [6, Lemma 5.2] that every finitely generated sublattice of a free lattice is in \mathcal{B} (see also the proof of Theorem 3.4 below).

An element of \mathcal{B} need not be imbeddable in a free lattice. Thus, according to [6, remarks following Theorem 5.1], every finite Boolean algebra is in \mathcal{B} ; but, by the Galvin-Jónsson result stated above, a Boolean algebra with more than eight elements is not imbeddable in a free lattice. Those elements of \mathcal{B} which are not imbeddable in free lattices are, of course, not projective; we shall see that all other elements of \mathcal{B} are projective (Theorem 3.3).

In Corollary 5.3 of [6], McKenzie shows that the two properties

of free lattices described in Jónsson's Theorem 1.1 carry over to the lattices of \mathcal{B} ; it follows that every non-distributive modular lattice fails to be in \mathcal{B} .

Recall from above that Baker and Hales have shown that if L is countable and is a linear sum of projective lattices, then L is projective. The situation is quite different for \mathcal{B} , as we shall see in Lemma 2.7. For the moment we remark that every countably infinite chain is projective, but is not in \mathcal{B} .

LEMMA 2.6. *Suppose f is a homomorphism of a free lattice $FL(X)$ into a lattice L . Then the following hold:*

- (1) *if f is upper bounded then for each $b \in L$, $\{x \in X: fx \leq b\}$ is finite;*
- (2) *if f is lower bounded then for each $b \in L$, $\{x \in X: b \leq fx\}$ is finite.*

Proof. We prove (1) ((2) is similar). Suppose $\{x \in X: fx \leq b\}$ is infinite. Let z be any element of $\{a \in FL(X): fa \leq b\}$; let X' be a finite subset of X such that z is in the sublattice of $FL(X)$ generated by X' ; and choose $x_0 \in \{x \in X: fx \leq b\} \sim X'$. Then $f(z + x_0) \leq b$, and $z < z + x_0$ in $FL(X)$ (see § 1). Thus f is not upper bounded.

LEMMA 2.7. *Suppose $L = \Sigma_E L_e$, where E is any infinite linearly ordered structure and the L_e are any lattices. Then $L \notin \mathcal{B}$.*

Proof. Say $f: FL(X) \rightarrow L$. We show that f cannot be bounded. For every nonempty subset S of E , $\bigcup\{L_e: e \in S\}$ is a sublattice of L ; hence f must map an element of X into every L_e . For each $e \in E$, choose $x_e \in X$ such that $fx_e \in L_e$. The x_e are distinct. Let $d \in E$. For each $e \in E$, fx_e is comparable to fx_d in L ; hence either $\{x \in X: fx \leq fx_d\}$ or $\{x \in X: fx_d \leq fx\}$ is infinite. Thus, by 2.3 and 2.6, f is not bounded.

3. Main results. Lemma 3.1 and Theorem 3.3 below are closely based on Lemma 5.2 of McKenzie's paper [6]; 3.1 generalizes that lemma.

LEMMA 3.1. *Suppose L is a lattice generated (not necessarily freely) by a set X , and suppose f is a homomorphism of L into a free lattice $FL(Y)$. Then the following hold:*

- (1) *if for each $b \in FL(Y)$, $\{x \in X: fx \leq b\}$ is finite, then f is upper bounded;*
- (2) *if for each $b \in FL(Y)$, $\{x \in X: b \leq fx\}$ is finite, then f is lower bounded.*

Proof. We prove (1); a dual argument works for (2). Assume the hypothesis of (1). Let T be the set of all elements b of $FL(Y)$ such that $\{a \in L: fa \leq b\}$ either is empty or has a greatest element. As in [6, Lemma 5.2], we show inductively that $T = FL(Y)$; we use the fact that Whitman's (W1) and (W2) hold in free lattices (see § 1).

First we show that $Y \subseteq T$. Suppose $y \in Y$ and $\{a \in L: fa \leq y\} \neq \emptyset$. Since L is generated by X , repeated application of Whitman's (W1) to a relation $fa \leq y$ ($a \in L$) yields an $x \in X$ such that $fx \leq y$. This, together with the hypothesis of (1), shows that $a_0 = \mathbf{V}\{x \in X: fx \leq y\}$ exists in L and that $fa_0 \leq y$. We claim that for all $a \in L$, if $fa \leq y$ then $a \leq a_0$. To see this, let S be the set of $a \in L$ for which the claim is true. Obviously $X \subseteq S$; and if $a, a' \in S$ then, trivially, $a + a' \in S$. If $a, a' \in S$ and $fa \cdot fa' \leq y$, then by Whitman's (W1), $fa \leq y$ or $fa' \leq y$, so that $a \leq a_0$ or $a' \leq a_0$; hence $a \cdot a' \leq a_0$. Therefore $S = L$, as claimed. Thus a_0 is the greatest element of $\{a \in L: fa \leq y\}$, and $y \in T$.

It is easy to see that T is closed under product. If $b_0, b_1 \in T$ and $\{a \in L: fa \leq b_0 \cdot b_1\}$ is nonempty, then both $\{a \in L: fa \leq b_0\}$ and $\{a \in L: fa \leq b_1\}$ are nonempty, hence have largest elements a_0, a_1 , respectively. Clearly $a_0 \cdot a_1$ is the largest element of $\{a \in L: fa \leq b_0 \cdot b_1\}$.

Finally we show that T is closed under sum. Suppose $b_0, b_1 \in T$ and $\{a \in L: fa \leq b_0 + b_1\} \neq \emptyset$. For $i \in \{0, 1\}$, in case $\{a \in L: fa \leq b_i\} \neq \emptyset$, let a_i be its largest element. Next consider the set $\{x \in X: fx \leq b_0 + b_1\}$; if this set is empty then there is a term τ of lattice theory of some minimum length > 1 such that for some $x_0, \dots, x_n \in X$, $f(\tau[L, x_0, \dots, x_n]) \leq b_0 + b_1$; τ must have the form $\tau_0 \wedge \tau_1$, so that $\tau_0[FL(Y), fx_0, \dots, fx_n] \cdot \tau_1[FL(Y), fx_0, \dots, fx_n] \leq b_0 + b_1$; but $\tau_j[FL(Y), fx_0, \dots, fx_n] \not\leq b_0 + b_1$ for $j \in \{0, 1\}$; hence, by Whitman's (W2) in $FL(Y)$, for $i = 0$ or $i = 1$, $f(\tau[L, x_0, \dots, x_n]) \leq b_i$. Therefore, we see that at least one of the three elements $a_0, a_1, \mathbf{V}\{x \in X: fx \leq b_0 + b_1\}$ is defined in L ; let a_2 be the sum of those that are defined.

Clearly $fa_2 \leq b_0 + b_1$. Now we claim that for all $a \in L$, if $fa \leq b_0 + b_1$ then $a \leq a_2$. Let S be the set of $a \in L$ for which this is true. Obviously $X \subseteq S$; and if $a, a' \in S$ then $a + a' \in S$. If $a, a' \in S$ and $fa \cdot fa' \leq b_0 + b_1$, then by Whitman's (W2) in $FL(Y)$, we have at least one of the following: $fa \leq b_0 + b_1$, $fa' \leq b_0 + b_1$, $fa \cdot fa' \leq b_1$, $fa \cdot fa' \leq b_0$; in the first case $a \leq a_2$ by assumption $a \in S$, so $a \cdot a' \leq a_2$; in the last case a_1 must be defined and $a \cdot a' \leq a_1 \leq a_2$; the other cases are similar, so $a \cdot a' \in S$. Therefore $S = L$, as claimed. It follows that $b_0 + b_1 \in T$.

Thus, $T = FL(Y)$, that is, f is upper bounded.

COROLLARY 3.2. *Suppose f is a homomorphism of a free lattice $FL(X)$ into a free lattice $FL(Y)$. Then f is bounded iff for each*

$b \in FL(Y)$, $\{x \in X: fx \text{ is comparable to } b\}$ is finite. (Immediate from 2.6 and 3.1.)

THEOREM 3.3. *Suppose $L \in \mathcal{B}$ and Whitman's (W2) holds in L . Then L is projective.*

Proof. We are given $f: FL(X) \rightarrow L$, f a bounded homomorphism. Define α and β , maps of L into $FL(X)$, as follows: for each $b \in L$, αb is the greatest element of $\{a \in FL(X): fa \leq b\}$, βb is the least element of $\{a \in FL(X): fa \leq b\}$. Certain properties of α and β are immediate. Thus, for each $b \in L$, $f\beta b = f\alpha b = b$ and $\beta b \leq \alpha b$. Also, β preserves sum and α preserves product; that is for any $b_0, b_1 \in L$, $\beta(b_0 + b_1) = \beta b_0 + \beta b_1$ and $\alpha(b_0 \cdot b_1) = \alpha b_0 \cdot \alpha b_1$. And both α and β are order-preserving.

We claim an additional property for β . Let $b_0, b_1 \in L$. By Lemma 2.6, the set $S = \{x \in X: b_0 \cdot b_1 \leq fx\}$ is finite. Our claim is that $\beta(b_0 \cdot b_1) = (\bigwedge \{x \in X: b_0 \cdot b_1 \leq fx\}) \cdot \beta b_0 \cdot \beta b_1$ if $S \neq \emptyset$, and $\beta(b_0 \cdot b_1) = \beta b_0 \cdot \beta b_1$ if $S = \emptyset$. Let a_0 denote the right-hand element; that is, $a_0 = (\bigwedge S) \cdot \beta b_0 \cdot \beta b_1$ if $S \neq \emptyset$, $a_0 = \beta b_0 \cdot \beta b_1$ if $S = \emptyset$. Clearly $fa_0 = b_0 \cdot b_1$. We show that for all $a \in FL(X)$, if $b_0 \cdot b_1 \leq fa$ then $a_0 \leq a$. Let T be the set of $a \in FL(X)$ for which this is true. Obviously $X \subseteq T$ and T is closed under product. If $a, a' \in T$ and $b_0 \cdot b_1 \leq fa + fa'$, then, by Whitman's (W2) in L , we have at least one of the following: $b_0 \leq fa + fa'$, $b_1 \leq fa + fa'$, $b_0 \cdot b_1 \leq fa$, $b_0 \cdot b_1 \leq fa'$; in the first case have $a + a' \geq \beta b_0 \geq a_0$; in the third case $a_0 \leq a$ by assumption $a \in T$, so $a_0 \leq a + a'$; the other cases are similar, so that $a + a' \in T$. Thus $T = FL(X)$, and $a_0 = \beta(b_0 \cdot b_1)$, as desired.

Now let the endomorphism $h: FL(X) \rightarrow FL(X)$ be the extension of the map $x \rightarrow \alpha fx$, $x \in X$. We claim that for each $a \in FL(X)$, $\beta fa \leq ha \leq \alpha fa$ (so that $pha = fa$). The property is obvious for $x \in X$. Proceeding inductively, suppose $\beta fa_i \leq ha_i \leq \alpha fa_i$ for $i \in \{0, 1\}$; then, using the properties of α and β established above, we have $\beta f(a_0 + a_1) = \beta(fa_0 + fa_1) = \beta fa_0 + \beta fa_1 \leq ha_0 + ha_1 = h(a_0 + a_1) \leq \alpha fa_0 + \alpha fa_1 \leq \alpha(fa_0 + fa_1)$; similarly, $\beta f(a_0 \cdot a_1) \leq h(a_0 \cdot a_1) \leq \alpha f(a_0 \cdot a_1)$.

Define the map $g: L \rightarrow FL(X)$ by $g = h \circ \beta$. We show that g is a homomorphism of L into $FL(X)$. Since β and h preserve sum, so does g . Now for $b_0, b_1 \in L$, we must show that $h\beta(b_0 \cdot b_1) = h\beta b_0 \cdot h\beta b_1$; it suffices to show $h(\beta b_0 \cdot \beta b_1) \leq h\beta(b_0 \cdot b_1)$. If $\beta(b_0 \cdot b_1) = \beta b_0 \cdot \beta b_1$ this is trivial; thus we may assume by above that $\{x \in X: b_0 \cdot b_1 \leq fx\} \neq \emptyset$ and that $h\beta(b_0 \cdot b_1) = (\bigwedge \{hx: x \in X \text{ and } b_0 \cdot b_1 \leq fx\}) \cdot h\beta b_0 \cdot h\beta b_1$. Therefore, it suffices to show that $h(\beta b_0 \cdot \beta b_1) \leq hx$ whenever $x \in X$ and $b_0 \cdot b_1 \leq fx$. But for any such x , $f(\beta b_0 \cdot \beta b_1) = f\beta b_0 \cdot f\beta b_1 = b_0 \cdot b_1 \leq fx$, so that $\beta b_0 \cdot \beta b_1 \leq \alpha fx = hx$ (see definition of h); thus, using the claim of the preceding paragraph, we have $h(\beta b_0 \cdot \beta b_1) \leq h\alpha fx \leq \alpha fhx = \alpha fx =$

hx , as desired. Thus g is a homomorphism of L into $FL(X)$.

To prove L projective it remains only to show that for each $b \in L$, $fgb = b$. In fact, $fgb = fh\beta b = f\beta b = b$, as desired (the middle equality holds by our claim concerning h). This completes the proof of 3.3.

THEOREM 3.4. *A finitely generated lattice is projective iff it is imbeddable in a free lattice.*

Proof. We already know that a projective lattice is imbeddable in a free lattice. Now suppose that L is a sublattice of $FL(Y)$, L finitely generated. For some sufficiently large finite X , there is an epimorphism f of $FL(X)$ onto L . By Corollary 3.2, f , viewed as a homomorphism into $FL(Y)$, is bounded; by Note 2.4, f is bounded as an epimorphism onto L . Thus $L \in \mathcal{B}$; and L inherits Whitman's (W2) from $FL(Y)$. It follows from 3.3 that L is projective.

Notice that, by Theorem 3.3 and the earlier remarks, for a lattice L of \mathcal{B} , L is projective iff L is imbeddable in a free lattice iff Whitman's (W2) holds in L .

We have a fair amount of information on the relationship between \mathcal{B} and the class of projective lattices. Our specific examples above include lattices which are in both classes, in neither, in one class but not the other. From [1] we have the example $2 \times \omega$, a denumerable distributive non-projective lattice imbeddable in $FL(3)$; it is now clear from Theorem 3.3 that $2 \times \omega \notin \mathcal{B}$. In [5, Figure 5B, p. 49] we display a denumerable non-modular sublattice of $FL(3)$ which also is non-projective and not in \mathcal{B} . We sketch a proof of the following additional fact:

THEOREM 3.5. *Suppose L is a distributive lattice, $L \in \mathcal{B}$, and Whitman's (W2) holds in L . Then L is a finite projective lattice.*

Proof. We know that L is projective by Theorem 3.3. By Lemma 2.7, L is a linear sum of just finitely many linear components. The Galvin-Jónsson result mentioned earlier implies that any infinite linear component of L must be isomorphic to the direct product of a countable chain and a two-element chain. An argument similar to that of Lemma 2.7 now shows that there is no infinite linear component of L . Thus L is a finite projective lattice.

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