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SPACE**

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Let C_Y denote the Yeh-Wiener space, i.e., the space of all real-valued continuous functions $f(x, y)$ on $I^2 \equiv [0, 1] \times [0, 1]$ such that $f(0, y) = f(x, 0) \equiv 0$. Yeh has defined a Gaussian probability measure on C_Y such that the mean of the process

$$m(x, y) \equiv \int_{C_Y} f(x, y) d_Y f = 0$$

and the covariance

$$R(s, t, x, y) \equiv \int_{C_Y} f(s, t) f(x, y) d_Y f = (1/2) \min(s, x) \min(t, y).$$

Consider now a linear transformation of C_Y onto C_Y of the form

$$(1.1) \quad \begin{aligned} T: f(x, y) &\rightarrow g(x, y) \\ &= f(x, y) + \int_{I^2} K(x, y, s, t) f(s, t) ds dt, \end{aligned}$$

which is often called a Fredholm transformation. The main purpose of this paper is to find the corresponding Radon-Nikodym derivative thus showing how the Yeh-Wiener integrals transform under the transformation.

The transformations considered here contain the Volterra transformation

$$T_1[f(x, y)] = f(x, y) + \int_0^y \int_0^x K(x, y, s, t) f(s, t) ds dt$$

as a special case.

Such transformations in Wiener space have been studied a great deal by Cameron and Martin [1], Woodward [9], Segal [5], [6], and Shepp [7], and the results have proved very useful in the evaluation of various Wiener integrals.

The transformation theorems in this paper are based on stochastic integrals called the generalized Paley-Wiener-Zygmund (P.W.Z.) integrals given in [3] and [4]. For a function $h(x, y) \in L^2(I^2)$ and $f(x, y) \in C_Y$, the generalized P.W.Z. integral is defined to be

$$(1.2) \quad \int_{I^2} h f d^* f \equiv \lim_{n \rightarrow \infty} \int_{I^2} (h f)_n df,$$

$$\begin{aligned}
(2.3) \quad & \frac{\partial^2}{\partial y \partial x} K(x, y, s, t) \equiv H(x, y, s, t) = \lim_{(u, v) \rightarrow (x^-, y^-)} \frac{\partial^2}{\partial v \partial u} K(u, v, s, t) \\
& \frac{\partial}{\partial x} \phi(x, s, t) \equiv A(x, s, t) = \lim_{u \rightarrow x^-} \frac{\partial}{\partial u} \phi(u, s, t) , \\
& \frac{\partial}{\partial y} \psi(y, s, t) \equiv B(y, s, t) = \lim_{v \rightarrow y^-} \frac{\partial}{\partial v} \psi(v, s, t) ,
\end{aligned}$$

and assume that there exists an integrable function $M(s, t)$ such that for all $(s, t) \in I^2$

$$(2.4) \quad \left. \begin{aligned} & \sup_{(x, y) \in I^2} |H(x, y, s, t)| \\ & \text{var}_{(x, y) \in I^2} H(x, y, s, t) \\ & \text{var}_{x \in I} H(x, 1, s, t) \\ & \text{var}_{y \in I} H(1, y, s, t) \\ & \text{var}_{x \in I} A(x, s, t) \\ & \text{var}_{y \in I} B(y, s, t) \end{aligned} \right\} \leq M(s, t) .$$

Also assume that

$$(2.5) \quad |A(x, s, t)|, |B(y, s, t)| \leq \beta, \text{ a constant, and}$$

$$(2.6) \quad D(K) \neq 0 .$$

Then for any Yeh-Wiener measurable functional $F(f)$, we have under the Fredholm transformation (1.1)

$$(2.7) \quad \int_{C_Y} F(g) d_Y g \stackrel{*}{=} |D(K)| \int_{C_Y} F(Tf) \cdot \exp \{-\Phi(f)\} d_Y f ,$$

where

$$\begin{aligned}
(2.8) \quad \Phi(f) = & \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \right]_{I^2} K(x, y, s, t) f(s, t) ds dt \Big]^2 dx dy \\
& + 2 \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \right]_{I^2} K(x, y, s, t) f(s, t) ds dt \Big] d^* f(x, y) ,
\end{aligned}$$

and “ $\stackrel{*}{=}$ ” indicates the existence of one side implies that of the other and the equality.

THEOREM II. *Let $h(x, y) \in L^2$ on I^2 , $K(x, y, s, t)$ and $F(f)$ as in Theorem I. Then under the (nonlinear) transformation*

$$\begin{aligned}
(2.9) \quad L: f(x, y) \rightarrow g(x, y) = & f(x, y) + f_0(x, y) \\
& + \int_{I^2} K(x, y, x, s, t) f(s, t) ds dt ,
\end{aligned}$$

where $f_0(x, y) = \int_0^y \int_0^x h(s, t) ds dt$, we have

$$\int_{c_Y} F(g) d_Y g \stackrel{*}{=} |D(K)| \int_{c_Y} F(Lf) \cdot \exp \{-\Psi(f)\} d_Y f ,$$

where

$$\begin{aligned} \Psi(f) = & \int_{I^2} \left[h(x, y) + \frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds dt \right]^2 dx dy \\ & + 2 \int_{I^2} \left[h(x, y) + \frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds dt \right] d^* f(x, y) . \end{aligned}$$

3. Definitions. Corresponding to each continuous function $f(x, y)$ on I^2 , the (n^{th}) quasi-polyhedric function $f_{(n)}(x, y)$ of $f(x, y)$ is defined to be

$$(3.1) \quad f_{(n)}(x, y) = a_{ij}xy + b_{ij}x + c_{ij}y + d_{ij}$$

on each square $Q_{ij} \equiv [(i-1)/n, i/n] \times [(j-1)/n, j/n]$, $i, j = 1, 2, \dots, n$, where a_{ij} , b_{ij} , c_{ij} , and d_{ij} are so chosen that $f_{(n)}(x, y)$ and $f(x, y)$ agree at the vertices $(i/n, j/n)$, $(i/n, (j-1)/n)$, $((i-1)/n, j/n)$ and $((i-1)/n, (j-1)/n)$.

REMARKS. (i). Since the function $f_{(n)}(x, y)$ is linear horizontally and vertically in each square Q_{ij} , we see that $f_{(n)}(x, y)$ is continuous on I^2 . Furthermore the sequence $\{f_{(n)}(x, y)\}$ converges to $f(x, y)$ uniformly on I^2 as $n \rightarrow \infty$. Evaluating a_{ij} , b_{ij} , c_{ij} , and d_{ij} explicitly, and then combining terms, we have on each square Q_{ij} ,

$$\begin{aligned} f_{(n)}(x, y) = & f\left(\frac{i}{n}, \frac{j}{n}\right)[n^2xy - n(j-1)x - n(i-1)y + (i-1)(j-1)] \\ & + f\left(\frac{i-1}{n}, \frac{j}{n}\right)[-n^2xy + n(j-1)x + niy - i(j-1)] \\ (3.2) \quad & + f\left(\frac{i}{n}, \frac{j-1}{n}\right)[-n^2xy + njx + n(i-1)y - j(i-1)] \\ & + f\left(\frac{i-1}{n}, \frac{j-1}{n}\right)[n^2xy - njx - niy + ij] . \end{aligned}$$

(ii) If $K(x, y, s, t)$ is continuous on I^4 , then for each $(s, t) \in I^2$ we can think of $K(x, y, s, t)$ as a function in x, y and so we have the (n^{th}) quasi-polyhedric function $K_{(n)}(x, y, s, t)$ in x, y , namely for $(x, y) \in Q_{ij}$; $i, j = 1, 2, \dots, n$, we have

$$\begin{aligned}
(3.3) \quad & K_{(n)}(x, y, s, t) \\
&= K\left(\frac{i}{n}, \frac{j}{n}, s, t\right)[n^2xy - n(j-1)x - n(i-1)y + (i-1)(j-1)] \\
&+ K\left(\frac{i-1}{n}, \frac{j}{n}, s, t\right)[-n^2xy + n(j-1)x + niy - i(j-1)] \\
&+ K\left(\frac{i}{n}, \frac{j-1}{n}, s, t\right)[-n^2xy + njx + n(i-1)y - j(i-1)] \\
&+ K\left(\frac{i-1}{n}, \frac{j-1}{n}, s, t\right)[n^2xy - njx - niy + ij] .
\end{aligned}$$

We also see that $K_{(n)}(x, y, s, t) \rightrightarrows K(x, y, s, t)$ on I^4 as $n \rightarrow \infty$, provided that $K(x, y, s, t)$ is continuous on I^4 . Here “ \rightrightarrows ” indicates uniform convergence.

For each $\varepsilon > 0$ let

$$\begin{aligned}
(3.4) \quad & \Omega_\varepsilon(s) = \operatorname{sgn} s - s/\varepsilon \quad \text{if } |s| \leq \varepsilon \\
& = 0 \quad \text{if } |s| > \varepsilon .
\end{aligned}$$

Let $K(x, y, s, t)$ and $\psi(y, s, t)$ be as in Theorem I. Then the function defined by

$$(3.5) \quad L_\varepsilon(x, y, s, t) \equiv K(x, y, s, t) + 2^{-1}[\Omega_\varepsilon(s-x) - \Omega_\varepsilon(s^+)]\psi(y, s, t)$$

is continuous in x, s . Now define

$$(3.6) \quad J(x, s, t) \equiv L_\varepsilon(x, t^+, s, t) - L_\varepsilon(x, t^-, s, t) ,$$

and

$$\begin{aligned}
(3.7) \quad & K_\varepsilon(x, y, s, t) \equiv L_\varepsilon(x, y, s, t) + 2^{-1}[\Omega_\varepsilon(t-y) - \Omega_\varepsilon(t^+)]J(x, s, t) \\
&= K(x, y, s, t) + 2^{-1}[\Omega_\varepsilon(s-x)\Omega_\varepsilon(s^+)]\psi(y, s, t) \\
&+ 2^{-1}[\Omega_\varepsilon(t-y) - \Omega_\varepsilon(t^+)]\phi(x, s, t) \\
&+ 4^{-1}[\Omega_\varepsilon(s-x) - \Omega_\varepsilon(s^+)][\Omega_\varepsilon(t-y) - \Omega_\varepsilon(t^+)](K_1 + K_2 \\
&- K_3 - K_4)(s, t, s, t) .
\end{aligned}$$

Then $K_\varepsilon(x, y, s, t)$ is continuous on I^4 , and $K_\varepsilon(0, y, s, t) = K_\varepsilon(x, 0, s, t) \equiv 0$. Furthermore $K_\varepsilon(x, y, s, t)$ is uniformly bounded in ε, x, y, s, t , and $\lim_{\varepsilon \rightarrow 0^+} K_\varepsilon(x, y, s, t) = K(x, y, s, t)$. Now, define

$$\begin{aligned}
(3.8) \quad & C_\varepsilon(x, s) = \varepsilon/2 \quad \text{if } -\varepsilon < s-x \leq \varepsilon \\
& = 0 \quad \text{otherwise} .
\end{aligned}$$

Then from (3.7), (3.4), (3.8), and (2.3) it follows

$$\begin{aligned}
 (3.9) \quad H_\varepsilon(x, y, s, t) &\equiv \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) \\
 &= H(x, y, s, t) + C_\varepsilon(x, s)B(y, s, t) + C_\varepsilon(y, t)A(x, s, t) \\
 &\quad + C_\varepsilon(x, s) \cdot C_\varepsilon(y, t) \cdot (K_1 + K_2 - K_3 - K_4)(s, t, s, t),
 \end{aligned}$$

with the understanding that whenever the partial derivatives are not defined at a point, then the value at the point to be the left-hand limit with respect to x and y (for the uniqueness sake) as in (2.3). Thus by using the mean-value property and the dominated convergence, we see that for almost all (x, y) in I^2

$$\begin{aligned}
 (3.10) \quad &\lim_{\varepsilon \rightarrow 0+} \int_{I^2} H_\varepsilon(x, y, s, t) f(s, t) ds dt \\
 &= \int_{I^2} H(x, y, s, t) f(s, t) ds dt + \int_0^1 B(y, x, t) f(x, t) dt \\
 &\quad + \int_0^1 A(x, s, y) f(s, y) ds + (K_1 + K_2 - K_3 - K_4)(x, y, x, y) \cdot f(x, y).
 \end{aligned}$$

But the right hand side of (3.10) is exactly equal to

$$(\partial^2 / \partial y \partial x) \int_{I^2} K(x, y, s, t) f(s, t) ds dt.$$

Therefore for a.a. (x, y) in I^2

$$\begin{aligned}
 (3.11) \quad &\lim_{\varepsilon \rightarrow 0+} \int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt \\
 &= \frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds dt.
 \end{aligned}$$

4. Some Preliminary Lemmas.

LEMMA 1. *Let $K_{(n)}(x, y, s, t)$ be the n -th polyhedric function in (x, y) with the understanding that $K_{(n)}(x, y, s, t) = 0$ for all $(x, y, s, t) \notin I^4$. For $i, j, p, q = 1, 2, \dots, n$ let*

$$(4.1) \quad A_{i,j,p,q}^{(n)} \equiv \int_{(q-1)/n}^{(q+1)/n} \int_{(p-1)/n}^{(p+1)/n} K_{(n)}\left(\frac{i}{n}, \frac{j}{n}, s, t\right) (1 - |ns - p|)(1 - |nt - q|) ds dt.$$

Then for any $f \in C_Y$ we obtain:

$$(i) \quad \int_{I^2} K_{(n)}\left(\frac{i}{n}, \frac{j}{n}, s, t\right) f_{(n)}(s, t) ds dt = \sum_{p,q=1}^n A_{i,j,p,q}^{(n)} f\left(\frac{p}{n}, \frac{q}{n}\right),$$

$$(ii) \quad \int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(n)}(x, y, s, t) f_{(n)}(s, t) ds dt = n^2 \sum_{p,q=1}^n B_{i,j,p,q}^{(n)} f\left(\frac{p}{n}, \frac{q}{n}\right)$$

for $(x, y) \in \left(\frac{i-1}{n}, \frac{i}{n}\right) \times \left(\frac{j-1}{n}, \frac{j}{n}\right)$, where

$$B_{ijpq}^{(n)} \equiv A_{ijpq}^{(n)} - A_{i-1,jpq}^{(n)} - A_{i,j-1,pq}^{(n)} + A_{i-1,j-1,pq}^{(n)},$$

$$\begin{aligned} \text{(iii)} \quad & \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(n)}(x, y, s, t) f_{(n)}(s, t) ds dt \right]^2 dx dy \\ &= n^2 \sum_{i,j=1}^n \left[\sum_{p,q=1}^n B_{ijpq}^{(n)} f\left(\frac{p}{n}, \frac{q}{n}\right) \right]^2, \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad & \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(n)}(x, y, s, t) f_{(n)}(s, t) ds dt \right] d f_{(n)}(x, y) \\ &= n^2 \sum_{i,j=1}^n \left[\sum_{p,q=1}^n B_{ijpq}^{(n)} f\left(\frac{p}{n}, \frac{q}{n}\right) \right] \cdot \Delta_{ij} f_{(n)}, \end{aligned}$$

where

$$\begin{aligned} \Delta_{ij} f_{(n)} &\equiv f\left(\frac{i}{n}, \frac{j}{n}\right) - f\left(\frac{i-1}{n}, \frac{j}{n}\right) - f\left(\frac{i}{n}, \frac{j-1}{n}\right) \\ &\quad + f\left(\frac{i-1}{n}, \frac{j-1}{n}\right). \end{aligned}$$

The proof of this lemma is similar to that of corresponding results in [1]. Next, we consider a transformation of C_Y to C_Y :

$$T: g(x, y) = f_{(n)}(x, y) + \int_{I^2} K_{(n)}(x, y, s, t) f_{(n)}(s, t) ds dt.$$

Then by (i) of Lemma 1 and the fact that $f_{(n)}(i/n, j/n) = f(i/n, j/n)$ at each i and j , we have

$$(4.2) \quad T: g\left(\frac{i}{n}, \frac{j}{n}\right) = f\left(\frac{i}{n}, \frac{j}{n}\right) + \sum_{p,q=1}^n A_{ijpq}^{(n)} f\left(\frac{p}{n}, \frac{q}{n}\right); \quad i, j = 1, 2, \dots, n.$$

The determinant $\Delta(K_{(n)})$ of this transformation is given by

$$(4.3) \quad \Delta(K_{(n)}) \equiv \det (A_{IP}^* + \delta_{IP})_{I,P} = 1, 2, \dots, n^2$$

where $A_{IP}^* = A_{ijpq}^{(n)}$ with $I \equiv (i-1)n + j$, $P \equiv (p-1)n + q$, $1 \leq i, j, p, q \leq n$.

LEMMA 2. *Let $F(f)$ be a Yeh-Wiener measurable functional which depends only on the n^2 values of $f(x, y)$ at $(x, y) = (i/n, j/n)$; $i, j = 1, 2, \dots, n$. Let the n^{th} quasi-polyhedric function in x, y , $K_{(n)}(x, y, s, t)$, satisfy that $K_{(n)}(x, y, s, t) = 0$ if $x = 0$ or $y = 0$, and that $\Delta(K_{(n)}) \neq 0$. Then*

$$\begin{aligned} \int_{c_Y} F(g) d_Y g &\stackrel{*}{=} |\Delta(K_{(n)})| \left| \int_{c_Y} F[f_{(n)} + \int_{I^2} K_{(n)}(\cdot, \cdot, s, t) f_{(n)}(s, t) ds dt \right. \\ &\quad \exp \left\{ - \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(n)}(x, y, s, t) f_{(n)}(s, t) ds dt \right]^2 dx dy \right. \\ &\quad \left. \left. - 2 \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(n)}(x, y, s, t) f_{(n)}(s, t) ds dt \right] df_{(n)}(x, y) \right\} d_Y f. \end{aligned}$$

In view of Lemma 1 and a Yeh-Wiener integral formula (see Theorem I, [11]) the proof is word for word identical to that of the corresponding lemma in [1].

LEMMA 3. If $K(x, y, s, t)$ is continuous on I^4 , and if the Fredholm determinant $D(K) \neq 0$, then

$$\lim_{n \rightarrow \infty} \Delta(K_{(n)}) = D(K).$$

Proof. Using (4.3), we may expand $\Delta(K_{(n)})$ as:

$$\begin{aligned} \Delta(K_{(n)}) &= \det (A_{IP}^* + \delta_{IP})_{I, P=1, 2, \dots, n^2} \\ &= 1 + \sum_{P=1}^{n^2} A_{PP}^* + \frac{1}{2!} \sum_{P_1, P_2=1}^{n^2} \det (A_{P_i P_j}^*)_{i, j=1, 2} \\ &\quad + \dots + \frac{1}{(n^2)!} \sum_{P_1, \dots, P_{n^2}=1}^{n^2} \det (A_{P_i}^*)_{i, j=1, \dots, n^2} \\ (4.4) \quad &= 1 + \sum_{p, q=1}^n A_{pqpq}^{(n)} + \frac{1}{2!} \sum_{p_1, q_1, p_2, q_2=1}^n \det (A_{p_i q_i p_j q_j}^{(n)})_{i, j=1, 2} \\ &\quad + \dots + \frac{1}{(n^2)!} \sum_{p_1, q_1, p_2, q_2, \dots, p_{n^2}, q_{n^2}=1}^n \det (A_{p_i q_i p_j q_j}^{(n)})_{i, j=1, 2, \dots, n^2}. \end{aligned}$$

Let M be the bound for K on I^4 . Then by (4.1) we have $|n^2 A_{ijpq}^{(n)}| \leq M$ uniformly in n, i, j, p, q . Thus by Hadamard's inequality it follows:

$$(4.5) \quad |\det (A_{p_i q_i p_j q_j}^{(n)})_{i, j=1, \dots, N}| \leq (M/n^2)^N N^{N/2}.$$

Upon using (4.5) in (4.4), we conclude that $|\Delta(K_{(n)})|$ is uniformly bounded by the convergent series $1 + \sum_{N=1}^{\infty} M^N N^{N/2}/N!$, and for each N

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sum_{p_1, q_1, \dots, p_N, q_N=1}^n \det (A_{p_i q_i p_j q_j}^{(n)})_{i, j=1, \dots, N} \\ &= \int_{I^{2N}} \begin{vmatrix} K(s_1, t_1, s_1, t_1) & \dots & K(s_1, t_1, s_N, t_N) \\ \dots & \dots & \dots \\ K(s_N, t_N, s_1, t_1) & \dots & K(s_N, t_N, s_N, t_N) \end{vmatrix} ds_1 dt_1 \dots ds_N dt_N. \end{aligned}$$

Hence the conclusion follows.

Similarly it follows:

LEMMA 4. Let $K(x, y, s, t)$ be a bounded integrable function with $D(K) \neq 0$ and let $\{K_\lambda(x, y, s, t)\}$ be a set of Borel measurable functions which are uniformly bounded in λ, x, y, s, t , and let $\lim_{\lambda \rightarrow 0^+} K_\lambda(x, y, s, t) = K(x, y, s, t)$. Then $\lim_{\lambda \rightarrow 0^+} D(K_\lambda) = D(K)$.

LEMMA 5. For each $(s, t) \in I^2$ let $H(x, y, s, t)$ satisfy that

$$\sup_{(x, y) \in I^2} |H(x, y, s, t)|, \quad \text{var}_{(x, y) \in I^2} H(x, y, s, t), \quad \text{var}_{x \in I} H(x, 1, s, t), \quad \text{var}_{y \in I} H(1, y, s, t)$$

are all dominated by an integrable function $M(s, t)$. Then

$$(4.6) \quad \begin{aligned} & \int_{I^2} \left[\int_{I^2} H(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\ &= \int_{I^2} f(s, t) \left[\int_{I^2} H(x, y, s, t) df(x, y) ds dt \right]. \end{aligned}$$

Proof. Let " $\|\cdot\|$ " denote the supremum of the absolute value. Then from the fact that

$$(4.7) \quad \left. \begin{aligned} & \text{var}_{(x, y) \in I^2} \int_{I^2} H(x, y, s, t) f(s, t) ds dt \\ & \text{var}_{y \in I} \int_{I^2} H(x, 1, s, t) f(s, t) ds dt \\ & \text{var}_{y \in I} \int_{I^2} H(1, y, s, t) f(s, t) ds dt \end{aligned} \right\} \leq \|f\| \int_{I^2} M(s, t) ds dt < \infty,$$

the left member of (4.6) exists as an ordinary R - S integral. Hence for the net: $x_i = i/n, y_j = j/n, i, j = 1, 2, \dots, n$, and $x_{i-1} \leq x_i^* \leq x_i, y_{j-1} \leq y_j^* \leq y_j$, we have with $\Delta_{ij}f(x, y) = f(x_i, y_j) - f(x_{i-1}, y_j) - f(x_i, y_{j-1}) + f(x_{i-1}, y_{j-1})$,

$$(4.8) \quad \begin{aligned} & \int_{I^2} \left[\int_{I^2} H(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\ &= \lim_{n \rightarrow \infty} \sum_{i, j=1}^n \left[\int_I H(x_i^*, y_j^*, s, t) f(s, t) ds dt \right] \Delta_{ij}f(x, y) \\ &= \lim_{n \rightarrow \infty} \int_I f(s, t) \left[\sum_{i, j=1}^n H(x_i^*, y_j^*, s, t) \Delta_{ij}f(x, y) \right] ds dt. \end{aligned}$$

But

$$(4.9) \quad \begin{aligned} & \left| \sum_{i, j=1}^n H(x_i^*, y_j^*, s, t) \Delta_{ij}f(x, y) \right| \\ &\leq \|f\| \left[\text{var}_{(x, y) \in I^2} H(x, y, s, t) + \text{var}_{x \in I} H(x, 1, s, t) + \text{var}_{y \in I} H(1, y, s, t) \right. \\ &\quad \left. + |H(1, 1, s, t)| \right] \\ &\leq 4\|f\| M(s, t), \end{aligned}$$

and since $M(s, t)$ is finite a.e. we see that for a.a. $(s, t) \in I^2$

$$\lim_{n \rightarrow \infty} \sum_{i,j=1}^n H(x_i^*, y_j^*, s, t) \Delta_{ij} f(x, y) = \int_{I^2} H(x, y, s, t) df(x, y) .$$

Thus (4.6) follows from this, (4.8) and (4.9) by dominated convergence.

LEMMA 6. Let $H(x, y, s, t)$ be as in Lemma 5, and let

$$(4.10) \quad H^n(x, y, s, t) \equiv n^2 \int_{(q-1)/n}^{q/n} \int_{(p-1)/n}^{p/n} H(u, v, s, t) du dv$$

for $(x, y) \in \left(\frac{p-1}{n}, \frac{p}{n}\right] \times \left(\frac{q-1}{n}, \frac{q}{n}\right], p, q = 1, 2, \dots, n .$

Then for every $f(x, y) \in C_Y$

$$(4.11) \quad \lim_{n \rightarrow \infty} \int_{I^2} \left[\int_{I^2} H(x, y, s, t) f_{(n)}(s, t) ds dt \right]^2 dx dy$$

$$= \int_{I^2} \left[\int_{I^2} H(x, y, s, t) f(s, t) ds dt \right]^2 dx dy ,$$

$$(4.12) \quad \left| \int_{I^2} \left[\int_{I^2} H^n(x, y, s, t) f_{(n)}(s, t) ds dt \right]^2 dx dy \right| \leq \left(\|f\| \int_{I^2} M(s, t) ds dt \right)^2 ,$$

$$(4.13) \quad \lim_{n \rightarrow \infty} \int_{I^2} \left[\int_{I^2} H^n(x, y, s, t) f_{(n)}(s, t) ds dt \right] df_{(n)}(x, y)$$

$$= \int_{I^2} \left[\int_{I^2} H(x, y, s, t) f(s, t) ds dt \right] df(x, y) .$$

Proof. (4.11) and (4.12) are immediate. By (4.7) the function $\int_{I^2} H(x, y, s, t) f(s, t) ds dt$ is of B.V. Now

$$\int_{I^2} \left[\int_{I^2} H^n(x, y, s, t) f(s, t) ds dt \right] df(x, y)$$

$$= \sum_{p,q=1}^n \int_{I^2} \left[n^2 \int_{(p-1)/n}^{p/n} \int_{(q-1)/n}^{q/n} H(u, v, s, t) du, dv \right] f(s, t) ds dt \cdot \Delta_{p,q} f ,$$

$$\text{where } \Delta_{p,q} f = f\left(\frac{p}{n}, \frac{q}{n}\right) - f\left(\frac{p-1}{n}, \frac{q}{n}\right) - f\left(\frac{p}{n}, \frac{q-1}{n}\right) + f\left(\frac{p-1}{n}, \frac{q-1}{n}\right) .$$

Since

$$\int_{I^2} \left[n^2 \int_{(q-1)/n}^{q/n} \int_{(p-1)/n}^{p/n} H(u, v, s, t) du dv \right] f(s, t) ds dt$$

is the average value of $\int_{I^2} H(x, y, s, t) f(s, t) ds dt$ in the square $((q-1)/n, q/n] \times ((p-1)/n, p/n]$, the existence of the R - S integral implies

$$\begin{aligned}
 (4.14) \quad & \lim_{n \rightarrow \infty} \int_{I^2} \left[\int_{I^2} H^n(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\
 & = \int_{I^2} \left[\int_{I^2} H(x, y, s, t) f(s, t) ds dt \right] df(x, y) .
 \end{aligned}$$

Now,

$$\begin{aligned}
 (4.15) \quad & \text{var}_{(x, y) \in I^2} H^n(x, y, s, t) \leq \text{var}_{(x, y) \in I^2} H(x, y, s, t) \leq M(s, t) \\
 & \text{var}_{x \in I} H^n(x, 1, s, t) \leq \text{var}_{x \in I} H(x, 1, s, t) \leq M(s, t) \\
 & \text{var}_{y \in I} H^n(1, y, s, t) \leq \text{var}_{y \in I} H(1, y, s, t) \leq M(s, t) \\
 & |H^n(1, 1, s, t)| \leq \sup_{(x, y) \in I^2} |H(x, y, s, t)| \leq M(s, t) .
 \end{aligned}$$

Seeing the fact that $H^n(x, y, s, t)$ is constant in x, y in each square $\left(\frac{p-1}{n}, \frac{p}{n}\right] \left(\frac{q-1}{n}, \frac{q}{n}\right]$ and that $f_{(n)}(x, y)$ and $f(x, y)$ agree on the vertices of this square, we can write

$$\begin{aligned}
 & \int_{I^2} \left[\int_{I^2} H^n(x, y, s, t) f_{(n)}(s, t) ds dt \right] df_{(n)}(x, y) \\
 & = \sum_{p, q=1}^n \int_{I^2} H^n\left(\frac{p}{n}, \frac{q}{n}, s, t\right) f_{(n)}(s, t) ds dt \cdot \Delta_{pq} f \\
 & = \int_{I^2} f_{(n)}(s, t) \left[\sum_{p, q=1}^n H^n\left(\frac{p}{n}, \frac{q}{n}, s, t\right) \Delta_{pq} f \right] ds dt .
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & \left| \int_{I^2} \left[\int_{I^2} H^n(x, y, s, t) f(s, t) ds dt \right] df(x, y) \right. \\
 & \quad \left. - \int_{I^2} \left[\int_{I^2} H^n(x, y, s, t) f_{(n)}(s, t) ds dt \right] df_{(n)}(x, y) \right| \\
 & = \left| \int_{I^2} [f(s, t) - f_{(n)}(s, t)] \left[\sum_{p, q=1}^n H^n\left(\frac{p}{n}, \frac{q}{n}, s, t\right) \cdot \Delta_{pq} f \right] ds dt \right| \\
 & \leq 4 \|f - f_n\| \|f\| \int_{I^2} M(s, t) ds dt ,
 \end{aligned}$$

thus obtaining

$$\begin{aligned}
 (4.16) \quad & \lim_{n \rightarrow \infty} \int_{I^2} \left[\int_{I^2} H^n(x, y, s, t) f_{(n)}(s, t) ds dt \right] df_{(n)}(x, y) \\
 & = \lim_{n \rightarrow \infty} \int_{I^2} \left[\int_{I^2} H^n(x, y, s, t) f(s, t) ds dt \right] df(x, y) .
 \end{aligned}$$

Hence (4.13) follows from (4.14) and (4.16).

COROLLARY. *Let $K(x, y, s, t)$ be as in Theorem I, $K_\epsilon(x, y, s, t)$ the*

corresponding function defined by (3.7), and $K_{\varepsilon, (n)}(x, y, s, t)$ the n^{th} quasi-polyhedric function of K_ε in x, y . Then

$$(4.17) \quad \lim_{n \rightarrow \infty} \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{\varepsilon, (n)}(x, y, s, t) f_{(n)}(s, t) ds dt \right]^2 dx dy \\ = \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt \right]^2 dx dy ,$$

$$(4.18) \quad \lim_{n \rightarrow \infty} \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{\varepsilon, (n)}(x, y, s, t) f_{(n)}(s, t) ds dt \right] df_{(n)}(x, y) \\ = \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt \right] df(x, y) .$$

Proof. In view of (3.9), (3.8), and (2.4) we obtain

$$(4.19) \quad \sup_{(x, y) \in I^2} |H_\varepsilon(x, y, s, t)| \leq M(s, t) + \frac{\beta}{\varepsilon} + \frac{1}{4\varepsilon^2} |(K_1 + K_2 - K_3 \\ - K_4)(s, t, s, t)| , \\ \text{var}_{(x, y) \in I^2} H_\varepsilon(x, y, s, t) \leq M(s, t) + \frac{2}{\varepsilon} M(s, t) + \frac{1}{\varepsilon^2} |(K_1 + K_2 - K_3 \\ - K_4)(s, t, s, t)| , \\ \text{var}_{x \in I} H_\varepsilon(x, 1, s, t) \leq M(s, t) + \frac{1}{2\varepsilon} [2\beta + M(s, t)] + \frac{1}{2\varepsilon^2} |(K_1 + K_2 \\ - K_3 - K_4)(s, t, s, t)| , \\ \text{var}_{y \in I} H_\varepsilon(1, y, s, t) \leq M(s, t) + \frac{1}{2\varepsilon} [2\beta + M(s, t)] + \frac{1}{2\varepsilon^2} |(K_1 + K_2 \\ - K_3 - K_4)(s, t, s, t)| .$$

Let $M_\varepsilon(s, t)$ be the sum of the four right members of (4.19). From (3.3) it is obvious that

$$(4.20) \quad \frac{\partial^2}{\partial x \partial y} K_{\varepsilon, (n)}(x, y, s, t) = n^2 \int_{(q-1)/n}^{q/n} \int_{(p-1)/n}^{p/n} \frac{\partial^2}{\partial v \partial u} K_\varepsilon(u, v, s, t) du dv$$

for $(p-1)/n < x \leq p/n$, $(q-1)/n < y \leq q/n$; $p, q = 1, \dots, n$. Hence from (4.10), (4.20), and the fact that $(\partial^2/\partial v \partial u) K_\varepsilon(u, v, s, t) \equiv H_\varepsilon(u, v, s, t)$, we see that

$$(4.21) \quad \frac{\partial^2}{\partial y \partial x} H_{\varepsilon, (n)}(x, y, s, t) \equiv H_\varepsilon^n(x, y, s, t) .$$

Thus the conclusions follow directly from Lemma 6.

we may rearrange the terms in the series (5.5), and then use the uniform convergence to obtain

$$\begin{aligned}
 \mathcal{D}(x, y, s, t) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[J_n \cdot K(x, y, s, t) \right. \\
 (5.7) \quad &\quad \left. - n \int_{I^2} K(x, y, u, v) \cdot C_{n-1}(u, v, s, t) dudv \right] \\
 &= C \cdot K(x, y, s, t) - \int_{I^2} K(x, y, u, v) \cdot \mathcal{D}(u, v, s, t) dudv,
 \end{aligned}$$

where $C = \sum_{n=0}^{\infty} J_n(K)/n!$. Corresponding to each $K \in (x, y, s, t)$, define $K_{(\varepsilon)}^*(x, y, s, t)$ by

$$\begin{aligned}
 K_{(\varepsilon)}^*(x, y, s, t) &= D(K) \mathcal{D}_{(\varepsilon)}(x, y, s, t) \\
 (5.8) \quad &= D(K) \left[C \cdot K_{\varepsilon}(x, y, s, t) \right. \\
 &\quad \left. - \int_{I^2} K_{\varepsilon}(x, y, u, v) \cdot \mathcal{D}(u, v, s, t) dudv \right].
 \end{aligned}$$

Then $K_{(\varepsilon)}^*(x, y, s, t)$ is continuous on I^4 , and since $\mathcal{D}(u, v, s, t)$ is bounded, we have that $K_{(\varepsilon)}^*(x, y, s, t)$ is uniformly bounded in ε, x, y, s, t , and from (3.9), (4.19), and (5.8) it follows that

$$\begin{aligned}
 H_{(\varepsilon)}^*(x, y, s, t) &\equiv \frac{\partial^2}{\partial y \partial x} K_{(\varepsilon)}^*(x, y, s, t) \\
 (5.9) \quad &= D(K) \left[C \cdot H_{\varepsilon}(x, y, s, t) \right. \\
 &\quad \left. - \int_{I^2} H_{\varepsilon}(x, y, u, v) \cdot \mathcal{D}(u, v, s, t) dudv \right],
 \end{aligned}$$

and $\sup_{(x, y) \in I^2} |H_{(\varepsilon)}^*(x, y, s, t)|$, $\text{var}_{(x, y) \in I^2} H_{(\varepsilon)}^*(x, y, s, t)$, $\text{var}_{x \in I} H_{(\varepsilon)}^*(x, 1, s, t)$, and $\text{var}_{y \in I} H_{(\varepsilon)}^*(1, y, s, t)$ are all dominated by $|D(K)|(|C| + \|\mathcal{D}\|) \cdot M_{\varepsilon}(s, t)$. Furthermore, by Lemma 4

$$(5.10) \quad \lim_{\varepsilon \rightarrow 0+} D(K_{(\varepsilon)}^*) = D(K^*) = D^{-1}(K).$$

6. Additional lemmas. Utilizing (3.7), (3.9), (3.11), (5.8), and (5.9), we have the following

LEMMA 7. *If $K(x, y, s, t)$ satisfies the hypotheses of Theorem I, then*

$$\begin{aligned}
 &\left| \int_{I^2} [K_{\varepsilon}(x, y, s, t) - K(x, y, s, t)] f(s, t) ds dt \right| \\
 &\leq 4\varepsilon(1 + \varepsilon) \|f\| \cdot \|K\|, \\
 (6.1) \quad &\int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{\varepsilon}(x, y, s, t) f(s, t) ds dt \right]^2 dx dy
 \end{aligned}$$

$$\begin{aligned}
& \leq \|f\|^2 \left[\int_{I^2} M(s, t) ds dt + 2\beta + 4\|K\| \right]^2, \\
(6.2) \quad & \lim_{\varepsilon \rightarrow 0^+} \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt \right]^2 dx dy \\
& = \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds dt \right]^2 dx dy,
\end{aligned}$$

$$\begin{aligned}
(6.3) \quad & \left| \int_{I^2} [K_{(\varepsilon)}^*(x, y, s, t) - K^*(x, y, s, t)] f(s, t) ds dt \right| \\
& \leq 4\varepsilon(1 + \varepsilon) \|f\| \|K\| |D(K)| \cdot (|C| + \|\mathcal{D}\|),
\end{aligned}$$

$$\begin{aligned}
(6.4) \quad & \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(\varepsilon)}^*(x, y, s, t) f(s, t) ds dt \right]^2 dx dy \\
& \leq (\|f\| D(K))^2 (|C| + \|\mathcal{D}\|) \left[\int_{I^2} M(s, t) ds dt + 2\beta + 4\|K\| \right]^2,
\end{aligned}$$

$$\begin{aligned}
(6.5) \quad & \lim_{\varepsilon \rightarrow 0^+} \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(\varepsilon)}^*(x, y, s, t) f(s, t) ds dt \right]^2 dx dy \\
& = \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K_{(\varepsilon)}^*(x, y, s, t) f(s, t) ds dt \right]^2 dx dy.
\end{aligned}$$

The following two lemmas are the key results:

LEMMA 8. *Let $K(x, y, s, t)$ be as in Theorem I. Then*

$$\int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds dt \right] d^* f(x, y)$$

defined as in (1.2) converges for a.a. f in C_Y , and any two C.O.N. sets in the class lead to the same value for a.a. f in C_Y . Furthermore, we have

$$\begin{aligned}
(6.6) \quad & \text{l.i.m.}_{\varepsilon \rightarrow 0^+} \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\
& = \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds dt \right] d^* f(x, y) \text{ on } C_Y,
\end{aligned}$$

where the mean convergence is in L^2 -sense.

Proof. First we observe that

$$\int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds dt \right] d^* f(x, y)$$

$$\begin{aligned}
(6.7) \quad &= \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K(x, y, s, t) f(s, t) ds dt \right] d^* f(x, y) \\
&+ \int_{I^2} \left[\int_0^1 B(y, x, t) f(x, t) dt \right] d^* f(x, y) \\
&+ \int_{I^2} \left[\int_0^1 A(x, s, y) f(s, y) ds \right] d^* f(x, y) \\
&+ \int_{I^2} (K_1 + K_2 - K_3 - K_4)(x, y, x, y) \cdot f(x, y) d^* f(x, y),
\end{aligned}$$

and the existence and the consistency of each of last three expressions follow in the similar manner as in the generalized P.W.Z. integral

$\int_{I^2} h f d^* f$. On account of (3.9) and Lemma 5 we may write

$$\begin{aligned}
(6.8) \quad &\int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\
&= \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\
&+ \int_{I^2} f(s, t) \left[\int_{I^2} C_\varepsilon(x, s) B(y, s, t) df(x, y) \right] ds dt \\
&+ \int_{I^2} f(s, t) \left[\int_{I^2} C_\varepsilon(y, t) A(x, s, t) df(x, y) \right] ds dt \\
&+ \int_{I^2} (K_1 + K_2 - K_3 - K_4)(s, t, s, t) \left[\int_{I^2} C_\varepsilon(x, s) C_\varepsilon(y, t) df(x, y) \right] ds dt.
\end{aligned}$$

Obviously,

$$\begin{aligned}
&\int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\
&= \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K(x, y, s, t) f(s, t) ds dt \right] d^* f(x, y) \quad \text{for } \text{a.a.} f \in C_Y.
\end{aligned}$$

Thus to establish (6.6) we need only show that

$$\begin{aligned}
(6.9) \quad &\text{l.i.m.}_{\varepsilon \rightarrow 0^+} \int_{I^2} f(s, t) \left[\int_{I^2} C_\varepsilon(x, s) B(y, s, t) df(x, y) \right] ds dt \\
&= \int_{I^2} \left[\int_0^1 B(y, x, t) f(x, t) dt \right] d^* f(x, y) \quad \text{on } C_Y,
\end{aligned}$$

and so on. To see this we observe that the Yeh-Wiener integral

$$\begin{aligned}
&\int_{C_Y} \left\{ \int_{I^2} f(s, t) \left[\int_{I^2} C_\varepsilon(x, s) B(y, s, t) df(x, y) \right] ds dt \right\}^2 d_Y f \\
&= \frac{1}{4} \left\{ \int_{I^4} \left[\int_t^1 \int_s^1 C_\varepsilon(x, s') B(y, s', t') ds' dt' \right]^2 ds dt dx dy \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left[\int_{I^2} \left(\int_0^t \int_0^s C_\varepsilon(x, s) B(y, s, t) dx dy \right) ds dt \right]^2 \\
& + \int_{I^4} \left(\int_0^t \int_0^s C_\varepsilon(x, s') B(y, s', t') dx dy \right) \\
& \quad \times \left(\int_0^{t'} \int_0^{s'} C_\varepsilon(x, s) B(y, s, t) dx dy \right) ds dt ds' dt' \Big\} , \\
& \int_{C_Y} \left\{ \int_{I_1} \left[\int_0^1 B(y, x, t) f(x, t) dt \right] d^* f(x, y) \right\}^2 d_Y f \\
& = \frac{1}{4} \left\{ \int_{I^3} \left[\int_x^1 \left(\int_t^1 B(y, x', t') dt' \right)^2 dx' \right] dx dt dy \right. \\
& \quad \left. + \frac{1}{4} \left[\int_{I^2} \left(\int_y^1 B(y, x, t) dt \right) dx dy \right]^2 \right\} ,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{C_Y} \left\{ \int_{I^2} f(s, t) \left[\int_{I^2} C_\varepsilon(x, s) B(y, s, t) df(x, y) \right] ds dt \right\} \\
& \quad \cdot \left\{ \int_{I^2} \left[\int_0^1 B(y, x, t) f(x, t) dt \right] d^* f(x, y) \right\} d_Y f \\
& = \frac{1}{4} \left\{ \int_{I^3} \left[\int_{I^2} B(y, x, t') \min(s, x) C_\varepsilon(x, s) B(y, s, t) dx dy \right] \min(t, t') ds dt dt' \right. \\
& \quad + \int_{I^3} \left(\int_0^t \int_0^s B(y', x', t_i) \left[\int_0^{t'} \int_0^{x'} C_\varepsilon(x, s) B(y, s, t) dx dy \right] dx' dy' \right) ds dt dt' \\
& \quad \left. + \frac{1}{2} \int_{I^4} \left(\int_0^{t'} B(y', x', t') \left[\int_0^t \int_0^s C_\varepsilon(s, x) B(y, s, t) dx dy \right] dy' \right) dx' ds dt dt' \right\} .
\end{aligned}$$

The techniques leading to the evaluation of above integrals can be found in [3] and [4]. Using these and then taking the limit as $\varepsilon \rightarrow 0^+$, we get

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \int_{C_Y} \left\{ \int_{I^2} f(s, t) \left[\int_{I^2} C_\varepsilon(x, s) B(y, s, t) df(x, y) \right] ds dt \right. \\
& \quad \left. - \int_{I^2} \left[\int_0^1 B(y, x, t) f(x, t) dt \right] d^* f(x, y) \right\}^2 d_Y f = 0 ,
\end{aligned}$$

which is exactly the same as (6.9).

LEMMA 8'. *Let $K(x, y, s, t)$ satisfy the hypotheses of Theorem I. Then for its resolvent kernel K^* we have*

$$(6.10) \quad \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K^*(x, y, s, t) f(x, t) ds dt \right] d^* f(x, y)$$

converges for almost all f in C_Y and is essentially independent of the particular choice of the C.O.N. set, and

$$\begin{aligned}
(6.11) \quad & \text{l.i.m.}_{\varepsilon \rightarrow 0^+} \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(\varepsilon)}^*(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\
& = \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K^*(x, y, s, t) f(s, t) ds dt \right] d^* f(x, y) \quad \text{on } C_Y.
\end{aligned}$$

Proof. The claim on (6.10) can be treated as that in (6.7). As for (6.11) we may use (5.9) with $H_\varepsilon(x, y, s, t) \equiv \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t)$ to see that

$$\begin{aligned}
& \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(\varepsilon)}^*(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\
& = D(K) \left\{ C \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt \right] df(x, y) \right. \\
& \quad \left. - \int_{I^2} \left[\int_{I^2} \left(\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, u, v) \mathcal{D}(u, v, s, t) du dv \right) f(s, t) ds dt \right] df(x, y) \right\},
\end{aligned}$$

and the first term in the braces converges in the mean to

$$C \int_{I^2} \left[(\partial^2 / \partial y \partial x) \int_{I^2} K(x, y, s, t) f(s, t) ds dt \right] d^* f(x, y)$$

on C_Y by the preceding lemma. Therefore, in view of (5.3) and (5.7), it remains to show that

$$\begin{aligned}
& \text{l.i.m.}_{\varepsilon \rightarrow 0^+} \int_{I^2} \left[\int_{I^2} \left(\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, u, v) \mathcal{D}(u, v, s, t) du dv \right) f(s, t) ds dt \right] df(x, y) \\
& = \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} \left(\int_{I^2} K(x, y, u, v) \mathcal{D}(u, v, s, t) du dv \right) f(s, t) ds dt \right] d^* f(x, y)
\end{aligned}$$

on C_Y whose proof is essentially on the same lines as that of Lemma 8.

LEMMA 9. *Let $F(f)$ be a nonnegative Yeh-Wiener measurable functional on C_Y , and let $K(x, y, s, t)$ satisfy the hypotheses of Theorem I. Then under the Fredholm transformation T in (1.1), we have*

$$(6.12) \quad \int_{C_Y} F(g) d_1 g \geq \left| D(K) \right| \int_{C_Y} F(Tf) \cdot \exp \{ -\Phi(f) \} d_1 f,$$

where $\Phi(f)$ is given by (2.8).

Proof. First we assume that $F(f)$ is a bounded nonnegative functional continuous on C_Y with respect to the uniform topology, and is dependent only on the n^2 values of f at $(x, y) = (i/n, j/n)$; $i, j = 1, 2, \dots, n$. Since $\lim_{\varepsilon \rightarrow 0^+} D(K_\varepsilon) = D(K) \neq 0$ and $\lim_{n \rightarrow \infty} D(K_{\varepsilon(n)}) = D(K_\varepsilon)$

by Lemma 4, there exist ε' and $N(\varepsilon)$ such that $0 < \varepsilon < \varepsilon'$ and $n \geq N(\varepsilon)$ imply $D(K_\varepsilon) \neq 0$ and $D(K_{\varepsilon, (n)}) \neq 0$. For such ε and n we may apply Lemma 2 to obtain

$$(6.13) \quad \begin{aligned} \int_{C_Y} F(g_n) d_Y g &= |D(K_{\varepsilon, (n)})| \int_{C_Y} F \left[f_{(n)} + \int_{I^2} K_{\varepsilon, (n)}(\cdot, \cdot, s, t) f_{(n)}(s, t) ds dt \right] \\ &\cdot \exp \left\{ - \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{\varepsilon, (n)}(x, y, s, t) f_{(n)}(s, t) ds dt \right]^2 dx dy \right. \\ &\left. - 2 \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{\varepsilon, (n)}(x, y, s, t) f_{(n)}(s, t) ds dt \right] df_{(n)}(x, y) \right\} d_Y f. \end{aligned}$$

As $n \rightarrow \infty$ $f_{(n)}(x, y) \rightrightarrows f(x, y)$ and $K_{\varepsilon, (n)}(x, y, s, t) \rightrightarrows K_\varepsilon(x, y, s, t)$. Hence $g_n(x, y) \equiv f_{(n)}(x, y) + \int_{I^2} K_{\varepsilon, (n)}(x, y, s, t) f_{(n)}(s, t) ds dt \rightrightarrows g(x, y) = f(x, y) + \int_{I^2} K_\varepsilon(x, y, s, t) f(s, t) ds dt$. Therefore $F(g_n) \rightarrow F(g)$ boundedly. Thus $\lim_n \int_{C_Y} F(g_n) d_Y g = \int_{C_Y} F(g) d_Y g$. Now, by Fatou's lemma it follows from (6.13) with the help of Lemma 3, (4.11), and (4.13) that

$$(6.14) \quad \begin{aligned} \int_{C_Y} F(g) d_Y g &\geq |D(K_\varepsilon)| \int_{C_Y} F \left[f + \int_{I^2} K_\varepsilon(\cdot, \cdot, s, t) f(s, t) ds dt \right] \\ &\cdot \exp \left\{ - \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt \right]^2 dx dy \right. \\ &\left. - 2 \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt \right] df(x, y) \right\} d_Y f. \end{aligned}$$

By (6.1) $\int_{I^2} K_\varepsilon(x, y, s, t) f(s, t) ds dt \rightrightarrows \int_{I^2} K(x, y, s, t) f(s, t) ds dt$ as $\varepsilon \rightarrow 0^+$, and hence

$$F \left[f + \int_{I^2} K_\varepsilon(\cdot, \cdot, s, t) f(s, t) ds dt \right] \rightarrow F \left[f + \int_{I^2} K(\cdot, \cdot, s, t) f(s, t) ds dt \right]$$

boundedly. Since "mean convergence" implies the existence of an almost everywhere convergent subsequence, it follows from (6.6) that there exists a monotone sequence $\varepsilon_n \downarrow 0$ such that

$$(6.15) \quad \begin{aligned} \lim_{n \rightarrow \infty} \int_{I^2} \left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{\varepsilon_n}(x, y, s, t) f(s, t) ds dt \right] df(x, y) \\ = \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds dt \right] d^* f(x, y) \text{ for a.a. } f \in C_Y. \end{aligned}$$

Now we replace ε_n for ε in (6.14), and then use Fatou's Lemma together with (6.2) and (6.15) to arrive at (6.12). This completes the proof for the case when $F(f)$ is nonnegative, bounded, and continuous in uniform topology. To obtain the result for arbitrary nonnegative measurable

functionals, we go through, as usual, the following steps: after proving it for the preceding case, we go to characteristic functionals of intervals, then 0-sets, then 0_δ -sets, and then null-sets. (In the last case we get equality rather than inequality, both sides being zero.) Then to characteristic functionals of measurable sets, nonnegative simple functionals, nonnegative bounded Yeh-Wiener measurable functionals, and then finally nonnegative Yeh-Wiener measurable functionals. For the details of these steps see [2: pp. 391-392].

On account of (5.9), (5.10), (6.3), (6.5), and (6.11) we can establish the following on the same lines as above:

LEMMA 9'. *Let $F(g)$ be a nonnegative Yeh-Wiener measurable functional on C_Y , and let $K(x, y, s, t)$ be as in Theorem I and $K^*(x, y, s, t)$ its resolvent kernel. Then under the transformation*

$$T^{-1}: g(x, y) \rightarrow f(x, y) = g(x, y) + \int_{I^2} K^*(x, y, s, t)g(s, t)dsdt ,$$

we have

$$\int_{C_Y} F(f)d_Y f \geq |D^{-1}(K)| \int_{C_Y} F(T^{-1}g) \cdot \exp\{-\Phi^*(g)\}d_Y g ,$$

where

$$(6.16) \quad \begin{aligned} \Phi^*(g) = & \int_{I^2} \left[\frac{\partial^2}{\partial x \partial y} \int_{I^2} K^*(x, y, s, t)g(s, t)dsdt \right]^2 dx dy \\ & + 2 \int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K^*(x, y, s, t)g(s, t)dsdt \right] d^*g(x, y) . \end{aligned}$$

7. Proof of theorems. We prove Theorem I for nonnegative Yeh-Wiener measurable functionals first. In this case by Lemma 9 we have under the transformation T in (1.1),

$$(7.1) \quad \int_{C_Y} F(g)d_Y g \geq |D(K)| \int_{C_Y} F[Tf] \exp\{-\Phi(f)\}d_Y f ,$$

and upon applying Lemma 9' on the right-hand side of (7.1), we obtain under the transformation

$$(7.2) \quad \begin{aligned} T^{-1}: g(x, y) \rightarrow f(x, y) = & g(x, y) + \int_{I^2} K^*(x, y, s, t)g(s, t)dsdt \\ \int_{C_Y} F[Tf] \exp\{-\Phi(f)\}d_Y f & \\ \geq |D^{-1}(K)| \int_{C_Y} F(T \circ T^{-1}g) \exp\{-\Phi(T^{-1}g)\} \cdot \exp\{-\Phi^*(g)\}d_Y g . & \end{aligned}$$

Hence if we show that

$$(7.3) \quad \Phi(T^{-1}g) + \Phi^*(g) = 0 \quad \text{for a.a. } g \text{ in } C_Y,$$

then it will follow from (7.1) and (7.2) that

$$\begin{aligned} \int_{C_Y} F(g) d_Y g &\geq |D(K)| \int_{C_Y} F[Tf] \exp\{-\Phi(f)\} d_Y f \\ &\geq \int_{C_Y} F(g) d_Y g, \end{aligned}$$

and hence the theorem will follow for the case. Observe now that if $T(f) = g$ and so $T^{-1}(g) = f$, then

$$(7.4) \quad \begin{aligned} \int_{I^2} K(x, y, s, t) f(s, t) ds dt &= g(x, y) - f(x, y), \\ \int_{I^2} K^*(x, y, s, t) g(s, t) ds dt &= f(x, y) - g(x, y), \end{aligned}$$

and from (3.10) and (3.11), we see that the left-hand side of the first equation in (7.4) is absolutely continuous, and

$$\frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds dt$$

is continuous. Therefore $(\partial^2/\partial y \partial x)[g(x, y) - f(x, y)]$ is also continuous. Therefore by the corollary to Theorem 4 in [3], we see that

$$(7.5) \quad \begin{aligned} \int_{I^2} \frac{\partial^2}{\partial y \partial x} [g(x, y) - f(x, y)] d^*[g(x, y) - f(x, y)] \\ \equiv \int_{I^2} \left\{ \frac{\partial^2}{\partial y \partial x} [g(x, y) - f(x, y)] \right\}^2 dx dy. \end{aligned}$$

Hence from (2.8), (7.4), and (7.5) it follows that

$$(7.6) \quad \begin{aligned} \Phi(T^{-1}g) &= \Phi(f) \\ &= \int_{I^2} \left\{ \frac{\partial^2}{\partial y \partial x} [g(x, y) - f(x, y)] \right\}^2 dx dy \\ &\quad + 2 \int_{I^2} \frac{\partial^2}{\partial y \partial x} [g(x, y) - f(x, y)] d^* f(x, y) \\ &= \int_{I^2} \left\{ \frac{\partial^2}{\partial y \partial x} [g(x, y) - f(x, y)] \right\}^2 dx dy \\ &\quad + 2 \int_{I^2} \frac{\partial^2}{\partial y \partial x} [g(x, y) - f(x, y)] d^* f(x, y) \\ &\quad - 2 \int_{I^2} \left\{ \frac{\partial^2}{\partial y \partial x} [g(x, y) - f(x, y)] \right\}^2 dx dy, \end{aligned}$$

and from (6.16) and (7.4)

$$(7.7) \quad \begin{aligned} \Phi^*(g) &= \int_{I^2} \left\{ \frac{\partial^2}{\partial y \partial x} [f(x, y) - g(x, y)] \right\}^2 dx dy \\ &+ 2 \int_{I^2} \frac{\partial^2}{\partial y \partial x} [f(x, y) - g(x, y)] d^* g(x, y) . \end{aligned}$$

Therefore (7.3) follows from (7.6) and (7.7). Hence the theorem holds for any nonnegative Yeh-Wiener measurable functionals. For arbitrary real Yeh-Wiener measurable functionals the theorem will also hold by considering the positive part and the negative part separately. For complex functionals we get the same result once we consider real part and imaginary part separately. This completes the proof of Theorem I.

To prove Theorem II we consider the following two transformations

$$\begin{aligned} I_1: f(x, y) &\rightarrow g(x, y) = f(x, y) + f_0(x, y) , \\ I_2: f(x, y) &\rightarrow h(x, y) = f(x, y) + \int_{I^2} K(x, y, s, t) f(s, t) ds dt . \end{aligned}$$

The theorem now follows by the use of Cameron-Martin translation theorem (see [11] or more precisely Theorem 1.4 in [4] on L_1 and then our Theorem I for L_2).

REMARK. As mentioned in the introduction our theorems in one-dimensional version have slightly different forms from the ones given in [1], the difference being in the expressions of $\Phi(f)$ and $\Psi(f)$. The $\Phi(f)$ in [1] is given by

$$(7.8) \quad \begin{aligned} \Phi(f) &= \int_0^1 \left[\frac{d}{dx} \int_0^1 K(x, s) f(s) ds \right]^2 dx \\ &+ 2 \int_0^1 \left[\int_0^1 \frac{\partial}{\partial x} K(x, s) f(s) ds \right] df(x) + \int_0^1 J(x) d[f^2(x)] , \end{aligned}$$

and ours is in the form

$$(7.9) \quad \Phi(f) = \int_0^1 \left[\frac{d}{dx} \int_0^1 K(x, s) f(s) ds \right]^2 dx + 2 \int_0^1 \left[\frac{d}{dx} \int_0^1 K(x, s) f(s) ds \right] d^* f(x) .$$

However by the assumptions given on $K(x, s)$ in [1], we have that

$$\begin{aligned} \frac{d}{dx} \int_0^1 K(x, s) f(s) ds &= \frac{d}{dx} \int_0^x K_2(x, s) f(s) ds + \int_x^1 K_1(x, s) f(s) ds \\ &= [K_2(x, x) - K_1(x, x)] f(x) + \int_0^1 \frac{\partial}{\partial x} K(x, s) f(s) ds \end{aligned}$$

$$= J(x)f(x) + \int_0^1 \frac{\partial}{\partial x} K(x, s)f(s)ds .$$

But assuming absolute continuity of $J(x)$ with $J'(x) \in L^2$, it follows from Theorem 5 in [3] that

$$\int_0^1 J(x)f(x)f(x)d^*f(x) = \frac{1}{2} \int_0^1 J(x)d[f^2(x)] \quad \text{for a.a. } f \in C_w .$$

Also as mentioned earlier

$$\int_0^1 \left[\int_0^1 \frac{\partial}{\partial x} K(x, s)f(s)ds \right] d^*f(x) = \int_0^1 \left[\int_0^1 \frac{\partial}{\partial x} K(x, s)f(s)ds \right] df(x)$$

for almost all f in C_w . Thus (7.8) and (7.9) represent essentially the same thing.

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