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NORM CONVERGENCE OF MARTINGALES OF RADON-NIKODYM DERIVATIVES GIVEN A σ -LATTICE

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R. B. DARST AND G. A. DEBOTH

Suppose that $\{\mathscr{M}_k\}$ is an increasing sequence of sub σ -lattices of a σ -algebra $\mathscr A$ of subsets of a non-empty set Ω . Let $\mathscr M$ be the sub σ -lattice generated by $\bigcup_k \mathscr M_k$. Suppose that $L^{\mathfrak p}$ is an associated Orlicz space of $\mathscr A$ -measurable functions, where Φ satisfies the Δ_2 -condition, and let $h \in L^{\mathfrak p}$. It is verified that the Radon-Nikodym derivative, f_k , of h given $\mathscr M_k$ is in $L^{\mathfrak p}$ and shown that the sequence $\{f_k\}$ converges to f in $L^{\mathfrak p}$, where f is the Radon-Nikodym derivative of h given $\mathscr M$.

- 1. Introduction. H. D. Brunk defined conditional expectation given a σ -lattice and established several of its properties in [1]. Subsequently S. Johansen [5] described a Radon-Nikodym derivative given a σ -lattice and showed that the Radon-Nikodym derivative was the conditional expectation in the appropriate case. Then H. D. Brunk and S. Johansen [2] proved an almost everywhere martingale convergence theorem for the Radon-Nikodym derivatives given an increasing sequence of σ -lattices. We shall establish norm convergence of these derivatives in L_1 and in the Orlicz spaces L^{ϕ} , where Φ satisfies the Δ_2 -condition. The theory of these Orlicz spaces can be found in [6], so we shall assume and build upon the results therein. Thereby, we can place fewer restrictions on Φ and obtain L_1 -convergence as a byproduct.
- 2. Notation. Let \mathscr{A} be a σ -algebra of subsets of a (non-empty) set Ω , and let μ be a non-negative (bounded) σ -additive function defined on \mathscr{A} .

For our purposes the following information about Φ will suffice: Φ is an even, convex function defined on the real numbers, R, with $\Phi(0)=0$ and $\Phi(x)\neq 0$ for some x. Moreover, there exists K>0 with $\Phi(2x)\leq K\Phi(x)$ for all $x\in R$. This latter property is called the \varDelta_2 -condition; it implies

$$(1) \quad \varPhi(x+y) = \varPhi\Big(\,2\Big(\frac{x+y}{2}\Big)\Big) \leqq \mathit{K}\varPhi\Big(\frac{x+y}{2}\Big) \leqq \Big(\frac{\mathit{K}}{2}\Big) \left[\varPhi(x) + \varPhi(y)\right] \;.$$

Then L^{σ} denotes the collection of (real valued) \mathscr{A} -measurable functions h defined on Ω with $\int_{\Omega} \Phi(h) d\mu < \infty$. Since Φ is convex and not

identically zero, $L^{\phi} \subset L_1$; L^{ϕ} is usually a proper subset of L_1 if $\lim_{x \to \infty} \Phi(x)/x = \infty$. This latter property and $\lim_{x \to 0} \Phi(x)/x = 0$ are required of an Orlicz space; but, these two properties are not necessary for our estimates to be valid. Examples are $\Phi(x) = |x|^p$, $1 \le p < \infty$.

Let $h \in L^{\varphi}$ and $\lambda(E) = \int_{E} h d\mu$, where $E \in \mathscr{A}$. Let \mathscr{M} be a sub σ -lattice of \mathscr{A} and let f be the Radon-Nikodym derivative of λ with respect to μ . Thus, f is the \mathscr{M} -measurable function defined on Ω (ϕ : the empty set, Ω , and [f > a] belong to \mathscr{M} , for all $a \in R$) satisfying

(2)
$$\lambda(A\cap [f\leqq b])\leqq b\mu(A\cap [f\leqq b])\;,\quad \text{where }A\in M\text{ and }b\in R\;,$$
 and

(3)
$$\lambda([f>a]\cap B^c) \geq a\mu([f>a]\cap B^c) \;,$$
 where $B^c=\varOmega-B$, $B\in\mathscr{M}$, and $a\in R$.

Our first result is a preliminary step to an L^{σ} martingale convergence theorem.

3. The derivative of an L^{ρ} -function is an L^{ρ} -function. We shall verify this assertion by establishing a sequence of estimates, the first of which is

(4)
$$\int_{[f>a]} \Phi(f) d\mu \leq \int_{[f>a]} \Phi(h) d\mu , \quad \text{for all } a \geq 0 .$$

To verify (4), choose $\delta > 0$ and $a = a_0 < a_1 < a_2 < \cdots$ with $\Phi(a_k) = \Phi(a_{k-1}) + \delta$. Let $A_k = [a_k \ge f > a_{k-1}]$ and notice that (3) implies

$$|\lambda|(\Omega) \ge \lambda([f > a_k]) \ge a_k \mu([f > a_k])$$
.

Thus, $\mu([f > a_k]) \rightarrow 0$ and

$$\int_{[f>a]} \Phi\left(\cdot\right) d\mu = \sum_{k=1}^{n} \int_{A_{k}} \Phi\left(\cdot\right) d\mu + \int_{[f>a_{n}]} \Phi\left(\cdot\right) d\mu = \sum_{k=1}^{\infty} \int_{A_{k}} \Phi\left(\cdot\right) d\mu.$$

Applying (3) again,
$$\int_{A_k} h d\mu = \lambda(A_k) \ge a_{k-1}\mu(A_k)$$
, so $a_{k-1} \le \frac{1}{\alpha_k} \int_{A_k} h d\mu$, where $\alpha_k = \mu(A_k) > 0$.

Then, applying Jensen's inequality,

$$\Phi(a_{k-1}) \leq \Phi\left(\frac{1}{\alpha_k} \int_{A_k} h d\mu\right) \leq \frac{1}{\alpha_k} \int_{A_k} \Phi(h) d\mu$$

Next, notice that

$$\int_{A_k} \Phi(f) d\mu \leq \Phi(a_k) \mu(A_k) = \left(\Phi(a_{k-1}) + \delta \right) \mu(A_k) \leq \int_{A_k} \Phi(h) d\mu + \delta \mu(A_k).$$

Thus $\int_{[f>a]} \Phi(f) d\mu \leq \int_{[f>a]} \Phi(h) d\mu + \delta \mu(\Omega)$, for all $\delta > 0$, which implies (4).

By a similar argument, one obtains

(5)
$$\int_{\{f \leq a\}} \Phi(f) d\mu \leq \int_{\{f \leq a\}} \Phi(h) d\mu, \quad \text{for all } a \leq 0.$$

Hence, splitting Ω into two pieces, [f>0] and $[f\leq 0]$, and applying (4) and (5), yields

(6)
$$\int_{\mathcal{Q}} \Phi(f) d\mu \le \int_{\mathcal{Q}} \Phi(h) d\mu;$$

thus verifying Theorem 1.

Theorem 1. The Radon-Nikodym derivative of an L° -function is an L° -function.

4. A Martingale convergence theorem. Suppose that $\{\mathscr{M}_k\}_{k=1}^\infty$ is an increasing sequence of σ -lattices of subsets of Ω , and \mathscr{M} is the σ -lattice generated by the lattice $\mathscr{M}_\infty = \bigcup_k \mathscr{M}_k$. Denote by \mathscr{M}_k the σ -algebra that is generated by \mathscr{M}_k and by λ_k and μ_k the restrictions of λ and μ to \mathscr{M}_k . Let h_k be an \mathscr{M}_k -measurable function satisfying $\lambda(E) = \int_E h_k d\mu$, where $E \in \mathscr{M}_k$, and denote by f_k the Radon-Nikodym derivative of λ_k with respect to μ_k on \mathscr{M}_k .

Theorem 2. The sequence $\{f_k\}$ converges to f in L^{σ} -norm:

$$\lim_{k\to\infty}\int_{a}\Phi(f-f_{k})d\mu=0.$$

Proof. To begin, notice that applying (4) and (5) to f_k yields

(8)
$$\int_{[f_k>a]} \Phi(h_k) d\mu \ge \int_{[f_k>a]} \Phi(f_k) d\mu , \quad \text{for all } a \ge 0 ,$$

and

(9)
$$\int_{[f_k \le a]} \Phi(h_k) d\mu \ge \int_{[f_k \le a]} \Phi(f_k) d\mu , \quad \text{for all } a \le 0 .$$

Since λ_k is the restriction of λ to \mathscr{A}_k , a variation on the theme which established (4) verifies

(10)
$$\int_E \varPhi(h) d\mu \ge \int_E \varPhi(h_k) d\mu , \qquad \text{for all } E \in \mathscr{L}_k.$$

To substantiate this latter assertion, suppose $a \geq 0$, $\delta > 0$, b > a, $\varPhi(b) = \varPhi(a) + \delta$, $E \in \mathscr{A}_k$, $F = E \cap [b \geq h_k > a]$, and $\mu(F) > 0$. Then $\int_F h_k d\mu = \int_F h d\mu$, since $F \in \mathscr{A}_k$. Moreover,

and

$$egin{aligned} arPhi(a) & \leq arPhi\left(rac{1}{\mu(F)}\int_{\mathbb{F}}h_kd\mu
ight) & = arPhi\Big(rac{1}{\mu(F)}\int_{\mathbb{F}}hd\mu\Big) \ & \leq rac{1}{\mu(F)}\!\int_{\mathbb{F}}\!arPhi(h)d\mu \;. \end{aligned}$$

Thus,

$$\int_{F} \Phi(h_{k}) d\mu \leq \int_{F} \Phi(h) d\mu + \delta \mu(F) .$$

Hence, appealing to the proof of (4) and to the sentence containing (5), we claim (10). Consequently,

(11)
$$\int_{[f_k>a]} \Phi(h) d\mu \ge \int_{[f_k>a]} \Phi(f_k) d\mu ,$$
 where $a \ge 0$ and $k = 1, 2, \cdots$,

and

(12)
$$\int_{[f_k \le a]} \Phi(h) d\mu \ge \int_{[f_k \le a]} \Phi(f_k) d\mu ,$$
 where $a \le 0$ and $k = 1, 2, \cdots$.

Moreover, $a\mu([|f_k|>a]) \le |\lambda|([|f_k|>a]) \le |\lambda|(\Omega)$, where $a \ge 0$; thus,

(13)
$$\lim_{n\to\infty}\sup_{k}\int_{[(f_k)>n]}\Phi(f_k)d\mu=0.$$

So we can truncate the functions and still approximate them uniformly as follows. Whenever n is a positive integer and u is a (real valued) function defined on Ω , let $u^n(x) = u(x)$, where $|u(x)| \le n$, and $u^n(x) = nu(x)/|u(x)|$ otherwise. Then, using (1) and setting $M = \max\{(K/2), (K^2/4)\}$,

$$\int_{\Omega} \Phi(f - f_k) d\mu = \int_{\Omega} \Phi(\{f - f^n\} + \{f^n - (f_k)^n\} + \{(f_k)^n - f_k\}) d\mu$$

$$\leq M(A_n + B_n + C_n),$$

where

$$A_n = \int_{[|f|>n]} \Phi(f) d\mu$$
 ,

$$B_n = \int_{\Omega} \varPhi(f_n - (f_k)^n) d\mu$$
,

and

$$C_n = \int_{[|f_k| > n]} \varPhi(f_k) d\mu$$
 .

From (4), (5) and (13), we obtain $A_n \to 0$ and $C_n \to 0$. Moreover, for each $\delta > 0$,

$$B_n \leq \Phi(2n)\mu([|f^n - (f_k)^n| > \delta]) + \Phi(\delta)\mu(\Omega)$$

$$\leq \Phi(2n)\mu([|f - f_k| > \delta]) + \Phi(\delta)\mu(\Omega).$$

But, Brunk and Johansen have shown that $\lim_k \mu([|f - f_k| > \delta]) = 0$, where $\delta > 0$, so Theorem 2 is established.

Because of the approximation properties which are verified in [4], the results of this paper extend immediately to analogous results for the derivatives of additive set functions defined on algebras of subsets of Ω given a sub-lattice (cf. [3]). Results for vector valued functions with respect to lattices which are related to the results: [7], [8], [9], of J. J. Uhl, Jr. for vector valued functions with respect to algebras should appear subsequently.

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