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VECTOR SPACE DECOMPOSITIONS AND THE ABSTRACT IMITATION PROBLEM

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VECTOR SPACE DECOMPOSITIONS AND THE ABSTRACT IMITATION PROBLEM

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Let \mathscr{S} be a Hilbert space, \mathscr{P} a closed subspace of \mathscr{S} , L an orthogonal projection operator on \mathscr{S} . The "imitation problem" consists of finding the solutions $p \in \mathscr{P}$ of the equation

$$p - s = L(p - s)$$

for given $s \in \mathscr{S}$. If \overline{W} is a compact bordered Riemann surface, A a boundary neighborhood, s a "singularity differential" defined on \overline{A} , p will be a harmonic exact differential which imitates s on \overline{A} in a sense precised by L (hence the name "imitation problem"). Existence and uniqueness theorems are given for the solution. Some concrete applications are described. The paper ends with a constructive method of solution in the case of L^2 -normal operators.

0. Introduction. The "imitation problem" has originally been formulated by L. Sario (see for instance [1]). It is fundamental in the construction of harmonic functions on a Riemann surface with given singularities and given boundary behavior. It can be formulated as follows: given a "singularity function" s defined in a boundary neighborhood, and a "normal operator L", construct a harmonic function pdefined on the whole Riemann surface and satisfying in the given boundary neighborhood the equation

$$p-s = L(p-s)$$
.

Sario's original solution uses the sup norm. For problems involving harmonic differentials, the L^2 norm is introduced somewhat more naturally and progress has been made in various directions. (see [5]). In §1 we study the abstract "imitation problem" for an arbitrary Hilbert space and give an existence and uniqueness theorem for the solution. In §2 we consider some decompositions of the vector space $\mathfrak{S}(\overline{A})$ of harmonic exact differentials defined on a boundary neighborhood A of a compact bordered Riemann surface \overline{R} and continuous in \overline{A} , and study some corresponding "imitation problems". In §3 we return to the L^2 case and give a constructive method of solution when the operator L is L^2 -normal. The method may be applied to the case of harmonic differentials on a Riemannian manifold of dimension > 2, and also to open manifolds.

1. The abstract imitation problem in a Hilbert space. Let

 \mathscr{S} be a Hilbert space, $\mathscr{B}(\mathscr{S})$ the algebra of bounded linear operators on \mathscr{S} . We are given a closed subspace $\mathscr{P} \subset \mathscr{S}$ corresponding to the orthogonal projection F. Given the orthogonal projection L on \mathscr{S} we want to solve the equation

$$(*) p - s = L(p - s)$$

for $p \in \mathscr{P}$ given the "singularity" $s \in \mathscr{S}$. We assume moreover that p = Ts where $T \in \mathscr{B}(\mathscr{S})$.

We are going to prove the theorem:

THEOREM. Let \mathscr{P} be a closed subspace of the Hilbert space \mathscr{S} and let F denote orthogonal projection on \mathscr{P} . Let L be an arbitrary orthogonal projection operator on a subspace of \mathscr{S} . Then the imitation problem

$$p - s = L(p - s)$$

admits a unique solution in \mathcal{P} of the form

p = Ts

(where T is a bounded linear operator on \mathcal{S}) if and only if

 $\operatorname{Im} L \dotplus \operatorname{Im} F = \mathscr{S}.$

Proof. Observe that (*) may be written as:

$$(I-L)(I-T)s=0$$

which is true for each $s \in \mathcal{S}$.

It follows that I - T belongs to the right annihilator of I - L. Now $\mathscr{B}(\mathscr{S})$ being a Baer ring [4] it follows that there exists $X \in \mathscr{B}(\mathscr{S})$ such that

(**) I - T = LX.

Moreover, since $p \in \operatorname{Im} F$ we have $Ts \in \operatorname{Im} F$ hence

$$(I-F)Ts=0.$$

We conclude that T belongs to the right annihilator of I - F hence there exists $Y \in \mathscr{B}(\mathscr{S})$ such that

$$(***) T = FY.$$

Adding up (**) and (***) we get the equation

$$(\dagger) I = LX + FY$$

where, we recall L, F are given orthogonal projection operators and X,

Y are unknown elements of $\mathscr{B}(\mathscr{S})$. Clearly, if $\operatorname{Im} L + \operatorname{Im} F \subseteq \mathscr{S}$, the last equation has no solution. We show conversely that if $\operatorname{Im} L + \operatorname{Im} F = \mathscr{S}$ the problem has always a solution. We need the:

LEMMA. Let A, B be closed subspaces of a Hilbert space S such that A + B = S (vector sum). Then, there exist closed subspaces $A_m \subset A, B_m \subset B$ such that

$$A_m \dotplus B_m = \mathscr{S}$$
 (direct sum).

Proof. Let $\{e_{\alpha}\}$ be a basis for A, $\{e_{\beta}\}$ a basis for B. Then, $\{e_{\alpha}, e_{\beta}\}$ is a set of generators for \mathscr{S} . It contains a basis $\{e_{\alpha_i}, e_{\beta_j}\}$ where $\{e_{\alpha_i}\} \subset \{e_{\alpha}\}$ and $\{e_{\beta_j}\} \subset \{e_{\beta}\}$. Let then A_m be the closed span of $\{e_{\alpha_i}\}$, B_m be the closed span of $|e_{\alpha_i}|$. Then $A_m + B_m = \mathscr{S}$ and $A_m \cap B_m = \{0\}$.

We apply the lemma to $A = \operatorname{Im} L, B = \operatorname{Im} F$. There exist subspaces $A_m \subset \operatorname{Im} L, B_m \subset \operatorname{Im} F$ such that

$$A_m \dotplus B_m = \mathscr{S}.$$

Let X_0 and Y_0 be orthogonal projections on A_m and B_m respectively. Then

$$I = LX_{\scriptscriptstyle 0} + FY_{\scriptscriptstyle 0}$$

and (X_0, Y_0) is a solution of (\dagger) . To study uniqueness, let (X, Y) be another solution of (\dagger) . One must have:

$$L(X - X_0) = F(Y_0 - Y)$$
.

So if $\operatorname{Im} F \cap \operatorname{Im} L = \{0\}$ then necessarily $X = X_0$ $Y = Y_0$. If $\operatorname{Im} F \cap \operatorname{Im} L \neq \{0\}$ then, the operators of the form $L(X - X_0) = F(Y_0 - Y)$ are the elements of the right annihilator of the set $\{I - L, I - F\}$ hence of the form $G\mathscr{B}(\mathscr{S})$ for some orthogonal projection G. The T's we are looking for are of the form $FY = FY_0 - F(Y_0 - Y) = FY_0 - G\mathscr{B}(\mathscr{S})$. $G\mathscr{B}(\mathscr{S})$ is non void: if $\operatorname{Im} F \cap \operatorname{Im} L = \operatorname{Im} M$ where M is a projection, M satisfies LM = FM. In that case uniqueness is lost and we have proved the theorem.

Notes. (1) there is actually no restriction when dealing with operators L which are projections: if L denotes any element of $\mathscr{B}(\mathscr{S})$, (*) becomes (I - L)(I - T) = 0. So I - T belongs to the right annihilator of I - L and therefore $I - F = \Lambda U$ where Λ is the orthogonal projection generating the right annihilator of I - L.

(2) The preceding proof can be applied to the Baer ring of linear endomorphisms of a vector space. Orthogonal projections should be replaced by projection operators. As an example we apply the previous theory to the construction of harmonic differentials on a Riemann surface which "imitate" some singularity differential in the neighborhood of the ideal boundary (whence the name "imitation problem").

2. Vector space decompositions and the corresponding "imitation problems". Let \overline{R} be any compact bordered Riemann surface. We consider the space $\mathfrak{S}(\overline{R})$ consisting of harmonic exact differentials on Int (\overline{R}) , which are continuous on \overline{R} . Let γ be a cycle on \overline{R} , $[\gamma]$ the corresponding homology class. We introduce the space

$$H_{[\tau]}(\bar{R}) = \left\{ \omega \in \mathfrak{F}(\bar{R}) \colon \int_{\tau} *\omega = 0 \right\}$$

(see [1]).

Let now \overline{W} be a compact bordered Riemann surface, \overline{A} the complement of a regularly embedded domain Ω . We use the standard notation

$$\begin{aligned} \alpha &= Bd\Omega\\ \beta &= Bd\,\bar{W} \end{aligned}$$

In the vector space $\mathfrak{H}(\overline{A})$ we consider the subspaces

$$egin{aligned} H_{_{0eta}}(ar{A}) &= \{\omega \in \mathfrak{S}(A), \, \omega = df, \, df \mid_{eta} = 0\} \ H_{_{0lpha}}(ar{A}) &= \{\omega \in \mathfrak{S}(ar{A}), \, \omega = df, \, df \mid_{lpha} = 0\} \ H_{_{0eta}}^{lpha}(ar{A}) &= \left\{\omega \in \mathfrak{S}(ar{A}); \, \omega = df, \, *df \mid_{eta} = 0, \, \int_{lpha_i} *df \ &= 0, \, ext{for each component } lpha_i \, ext{of } lpha
ight\} \ H_{_{0lpha}}^{lpha}(ar{A}) &= \left\{\omega \in \mathfrak{S}(ar{A}); \, \omega = df, \, *df \mid_{lpha} = 0, \, \int_{eta_i} *df \ &= 0, \, ext{for each component } lpha_i \, ext{of } lpha
ight\} \ H_{_{0lpha}}^{lpha}(ar{A}) &= \left\{\omega \in \mathfrak{S}(ar{A}); \, \omega = df, \, *df \mid_{lpha} = 0, \, \int_{eta_i} *df \ &= 0, \, ext{for each component } eta_i \, ext{of } \beta
ight\} \ H_{_{0eta}}'(ar{A}) &= H_{_{0eta}}(ar{A}) \cap H_{_{[eta]}}(ar{A}) \ H_{_{0eta}}'(ar{A}) &= H_{_{0eta}}(ar{A}) \cap H_{_{[eta]}}(ar{A}) \ . \end{aligned}$$

Observe that:

$$H^{lpha}_{\scriptscriptstyle 0eta}(ar{A}) = H^{lpha}_{\scriptscriptstyle 0eta}(ar{A}) \cap H_{\scriptscriptstyle [eta]}(ar{A}) = H^{lpha}_{\scriptscriptstyle 0eta}(ar{A})$$
 .

Another important subspace will be

$$H_{ext}(\bar{A}) = \{ \omega \in \mathfrak{F}(\bar{A}) \colon \omega = \hat{\omega}|_{\bar{A}} \text{ where } \hat{\omega} \in \mathfrak{F}(\bar{W}) \}$$

Clearly $H_{ext}(\bar{A}) \subset H_{[\beta]}(\bar{A})$.

Let now $\Gamma(\overline{A})$ be the space of square integrable harmonic differentials on \overline{A} . We denote

$$egin{aligned} h_{\left[eta
ight]}(A) &= ext{closure in } arGamma(ar{A}) ext{ of } H_{\left[eta
ight]}(ar{A}) \ h_{0_{7}}(ar{A}) &= ext{closure in } arGamma(ar{A}) ext{ of } H_{0_{7}}'(ar{A}) \ h_{0_{7}}^{\star}(ar{A}) &= ext{closure in } arGamma(ar{A}) ext{ of } H_{0_{7}}^{\star}(ar{A}) \end{aligned}$$

where γ stands for α or β . We have the following vector space decompositions:

PROPOSITION.

$$egin{aligned} h_{1eta]}(ar{A}) &= h_{0lpha}'(ar{A}) \bigoplus h_{0eta}^\star(ar{A}) \ h_{1eta]}(ar{A}) &= h_{0lpha}^\star(ar{A}) \bigoplus h_{0eta}'(ar{A}) \ . \end{aligned}$$

Proof. We prove the first equality. The second is obtained by symmetry. First, we show

$$h_{\scriptscriptstyle [eta]}(ar{A}) \,=\, H_{\scriptscriptstyle 0lpha}'(ar{A}) igoplus H_{\scriptscriptstyle 0eta}^{st}(ar{A})$$
 .

Observe that $H'_{0\alpha}(\bar{A})$ is orthogonal to $h^{\star}_{0\beta}(\bar{A})$: let $df \in H'_{0\alpha}(\bar{A})$, $dg \in H^{\star}_{0\beta}(\bar{A})$. The innner product on the Hilbert space $\Gamma(\bar{A})$ induces an inner product on $H_{[\beta]}(\bar{A})$. So,

$$(df, dg)_{\overline{A}} = \int_{\beta-\alpha} f * \overline{dg} = \int_{\beta} f * \overline{dg} = 0$$
.

Let now dk be an element of $H_{[\rho]}(\bar{A})$. We want to find $df \in H'_{0\alpha}$ and $dg \in H^{\star}_{0,0}(\bar{A})$ such that

$$dk = df + dg$$
.

We must have $dg|_{\alpha} = dk|_{\alpha}$, $*dg|_{\beta} = 0$, $\int_{\alpha_i} *dg = 0$ for each component α_i of α . Also $df|_{\alpha} = 0$, $*df|_{\beta} = *dk|_{\beta}$ and $\int_{\alpha_i} *df = \int_{\alpha_i} *dh$ for each component α_i of α . Such a problem has a unique solution.

We now take closures in $\Gamma(\bar{A})$. Observe that $h'_{0\alpha}(\bar{A})$ and $h^{\star}_{0\beta}(\bar{A})$ are orthogonal since $H'_{0\alpha}(\bar{A})$ and $H^{\star}_{0\beta}(\bar{A})$ are dense and orthogonal. It follows that

$$h_{[\beta]}(\bar{A}) = h'_{\scriptscriptstyle 0lpha}(\bar{A}) + h^{\star}_{\scriptscriptstyle 0eta}(\bar{A})$$
.

We now consider some orthogonal projections in the space $h_{[\beta]}(\overline{A})$, which may be used as operators L of §1.

(1) Let Λ_0 be orthogonal projection on $h_{0\beta}^{\star}(\bar{A})$. We have

$$\ker \Lambda_{\scriptscriptstyle 0} = h_{\scriptscriptstyle 0lpha}'(ar A)$$
 .

In particular $*\Lambda_0 df \in h_{0\beta}(\overline{A})$ and hence $\Lambda_0 df$ has "vanishing normal derivative" on β . Moreover $(I - \Lambda_0) df \in h'_{0\alpha}$. So $\Lambda_0 df|_{\alpha} = df|_{\alpha}$ and Λ_0 has the property of Sario's " L_0 operator".

(2) Let Λ_1 be orthogonal projection on $h'_{0\beta}(\overline{A})$. We have

$$\ker arLambda_{\scriptscriptstyle 1} = h_{\scriptscriptstyle 0lpha}(ar{A})$$
 .

So $\Lambda_1 df|_{\beta} \in h'_{0\beta}$ and " $\Lambda_1 df$ vanishes on β " However $*(I - \Lambda_1) df|_{\alpha} \in h'_{0\alpha}$ hence

$$* \Lambda_1 df \mid_{lpha} = * df \mid_{lpha}$$

and Λ_1 differs from Sario's " L_1 operator" by its behavior on α . Some other vector space decompositions will be of interest:

PROPOSITION. $H_{[\beta]}(\overline{A}) = H_{ext}(\overline{A}) \bigoplus H'_{0\beta}(\overline{A}).$

Proof. Observe that $H_{ext}(\bar{A}) \cap H_{o\beta}^*(A) = \{0\}$. This is a consequence of the fact that on \bar{W} , $\Gamma_{hse}^* \cap \Gamma_{he}$ is orthogonal to $\Gamma_{he} \cap \Gamma_{ho}$.

Now consider any $df \in H_{[\beta]}(\overline{A})$. Let df be the unique harmonic exact differential on \overline{W} which has same boundary values as df. Now:

$$df = d\widehat{f}|_{\overline{A}} + (df - d\widehat{f})|_{\overline{A}}$$

and

$$\widehat{df}|_{\overline{A}} \in H_{ext}(\overline{A})$$
, $(df - \widehat{df})|_{\overline{A}} \in H_{0\beta}'$.

which proves the validity of the direct sum decomposition.

We shall denote by K_1 the corresponding projection on $H_{ext}(\bar{A})$ and by L_1 the corresponding projection on $H'_{0\beta}(\bar{A})$.

Proposition. $H_{[eta]}(ar{A}) = H_{ext}(ar{A}) \oplus H_{_{0eta}}^{\star}(ar{A})$.

Proof. $H_{ext}(\bar{A}) \cap H_{o\beta}^{\star}(\bar{A}) = \{0\}$. Thus assume $\omega = df \in H_{ext}(\bar{A})$ and $*df \mid \beta = 0$. By the uniqueness of the solution to the Neumann problem df = 0. Consider now any $df \in H_{[\beta]}(\bar{A})$. Let df be the harmonic exact differential on \bar{W} such that $*(\hat{df})\mid_{\beta} = *df\mid_{\beta}$ and $\int_{\alpha_i} *\hat{df} = \int_{\alpha_i} *df$ for each component α_i of α . We can write

$$df=\hat{df}ert_{ar{A}}+(df-(\hat{df}))ert_{ar{A}}$$

where

$$\hat{df}|_{ar{A}}\in H_{ext}(ar{A})$$
 , $(df-(\hat{df})|_{ar{A}}\in H_{_{0ar{eta}}}(A)$,

which proves the validity of the direct sum decomposition. We denote by K_0 the projection on $H_{ext}(\bar{A})$ and by L_0 the projection on $H_{0\beta}^{\star}(\bar{A})$.

Application. Solution to the "imitation problem" for harmonic differentials in $H_{[\beta]}(\bar{A})$. (cf. §1. note 2).

Assume that we have a decomposition:

$$H_{[\beta]}(\bar{A}) = H_{ext}(\bar{A}) \bigoplus \tilde{H}(\bar{A})$$
 . (direct sum) .

We denote by L the corresponding projection onto $H(\bar{A})$ and by K the corresponding projection onto $H_{ext}(\bar{A})$.

The "imitation problem" consists in studying the solutions $\omega \in H_{ext}(\bar{A})$ of the equation:

(*)
$$\omega - s = L(\omega - s)$$
, $s \in H_{[\beta]}(\overline{A})$.

One can apply the existence and uniqueness theorem of §1, or check directly.

Uniqueness of the solution: let ω_1, ω_2 be two solutions of (*) then

$$\omega_1 - \omega_2 = L(\omega_1 - \omega_2)$$

or

$$(I-L)(\omega_1-\omega_2)=0$$

i.e. $\omega_1-\omega_2\in \operatorname{Ker}\left(I-L
ight)=\operatorname{Im} L$.

Now $\omega_1 - \omega_2 \in \text{Im } K$ and $\text{Im } K \cap \text{Im } L = \{0\}$. It follows that $\omega_1 - \omega_2 = 0$ and the solution is unique.

Existence of the solution. To solve $(I - L)(\omega - s) = 0$ set $\omega = Ks$. We then get:

$$(I-L)(I-K)s=0$$

which is verified for all $s \in H_{[\beta]}(\overline{A})$ since

$$\operatorname{Im}\left(I-K\right)=\operatorname{Ker}L;$$

from the direct sum decomposition.

EXAMPLES. (1) L_1 and K_1 . The unique solution to

$$\omega - s = L_1(\omega - s)$$

is given by $\omega = K_1 s$. Such a ω has the same boundary behavior as s.

(2) L_0 and K_0 . The unique solution to

$$\omega - s = L_0(\omega - s)$$

is given by $\omega = K_0 s$ and $*\omega$ and *s have same boundary behavior.

3. $-L^2$ -normal operators and the "imitation problem". We

now return to the L^2 theory and show a constructive method of solution. We consider the Hilbert space \mathfrak{G}_1 defined as the closure in the L^2 -norm on \overline{A} of the space of harmonic exact differential on \overline{A} . We are considering operators

 $L: \mathfrak{H}_1 \longrightarrow \mathfrak{H}_1$

such that

(i) L is an orthogonal projection operator. (in particular $L^2 = L$ and ||L|| = 1)

(ii) Im $(I - L) \cap H_{ext}(\overline{A}) = \{0\}$.

Such operators will be called L^2 -normal.

We consider in particular the operator

$$K: \mathfrak{H}_1 \longrightarrow \mathfrak{H}_1$$

where K denotes orthogonal projection onto the subspace \Re of exact harmonic differentials in \mathfrak{G}_1 which admit a harmonic extension to all of W. The next generalized q-lemma shows that \Re is closed.

GENERALIZED q-LEMMA. There exist numbers $q(\bar{A})$ and $q'(\bar{A})$ lying between 0 and 1 such that for each $\omega \in \Gamma_{he}(\bar{W})$. $q'(\bar{A}) ||\omega||_{\overline{W}} \leq ||\omega||_{\overline{A}} \leq q(\bar{A}) ||\omega||_{\overline{W}}$.

Proof. We know that $\Gamma_{he}(\bar{W})$ has the Montel property. Consider the subset $S \subset \Gamma_{he}(\bar{W})$ defined as

$$\mathrm{S}=\{\omega\in arGamma_{{}^{he}}(ar W){:}\,||\,\omega\,||_{ar w}=1\}$$
 .

We first want to show that then exists $q(\bar{A}), 0 < q(\bar{A}) < 1$ such that

$$||\omega||_{\bar{A}} \leq q(\bar{A})$$

for each $\omega \in S$.

If this is not the case, there is a sequence (ω_n) from S such that $||\omega_n||_{\overline{A}} \rightarrow 1$.

By the Montel property, (ω_n) has a convergent subsequence (ω_{n_i}) and $\omega_{n_i} \to \hat{\omega} \in S$. (since S is closed). Now $||\omega_{n_i}||_{\bar{A}} \to 1$ and hence $||\hat{\omega}||_{\bar{A}} = 1$ and so supp $\hat{\omega} \subseteq \bar{A}$. But no element of $\Gamma_{he}(\bar{W})$ has support contained in \bar{A} a proper subset of Int \bar{W} . ([3] p. 186). Hence there exists $q(\bar{A}), 0 < q(\bar{A}) < 1$ such that

 $||\omega||_{\bar{A}} \leq q(\bar{A}) ||\omega||_{\bar{w}}$.

To get the second inequality, consider $\bar{\varOmega}$:

 $||\omega||_{\bar{a}} \leq q(\bar{\Omega}) ||\omega||_{\bar{w}}$

hence

 $||\omega||_w - ||\omega||_{\bar{A}} \leq q(\bar{\Omega}) ||\omega||_{\overline{w}}$

or

$$(1-q(ar{ar{\Omega}})) \, \| \, oldsymbol{\omega} \, \|_{ar{w}} \leq \| \, oldsymbol{\omega} \, \|_{ar{A}}$$

and we have $q'(\bar{A}) = 1 - q(\bar{\Omega})$. Which proves the lemma.

NOTE. We have
$$1 - q(\overline{\Omega}) \leq q(\overline{A})$$
. So $q(\overline{A}) + q(\overline{W} - \overline{A}) \geq 1$.

COROLLARY. \Re is a closed subspace of \mathfrak{H}_1 .

Proof. We show \Re contains all the limits of its Cauchy sequences. Let (ω_n) be Cauchy in \Re . Let $(\hat{\omega}_n)$ be the corresponding sequence in $\Gamma_{he}(\overline{W})$ (such that $\hat{\omega}_n|_{\overline{A}} = \omega_n$). Now $(\hat{\omega}_n) \to \hat{\omega} \in \Gamma_{he}(\overline{W})$ in the L^2 norm on $\Gamma_{he}(\overline{W})$. Since the L^2 -norms on $\Gamma_{he}(\overline{W})$ and \Re are equivalent. It follows that

 $(\omega_n) \longrightarrow \hat{\omega}|_A$

in the L^2 norm on \Re and hence \Re is closed. We now prove:

THEOREM. Let L be a L²-normal operator on \mathfrak{F}_1 . Then the equation $\omega - s = L(\omega - s)$ admits a solution $\omega \in \mathfrak{R}$. The solution is unique provided $\mathfrak{R} \cap \operatorname{Im} L = (0)$.

Proof. Assume there exists $p \in \mathfrak{H}_1$ such that

(+)
$$-Kp - s = L(p - s)$$
.

We then have

$$L(-Kp - s) = L^{2}(p - s) = L(p - s) = -Kp - s$$
.

Setting $\omega = -Kp$ we obtain an element of \Re such that

$$\omega - s = L(\omega - s)$$
.

It then suffices to solve (+). We rewrite it as:

$$(^{++}) \qquad [I - (I - (K + L))]p = -(I - L)s.$$

The latter admits a solution $p \in \mathfrak{H}_1$ (which can be written as a Neumann series) if

||I - (K + L)|| < 1

or, what is the same, if the aperture

 $\theta(\operatorname{Im}(I-K), \operatorname{Im}(K)) < 1$.

(For the definition and properties of the aperture see [2] p. 69.) Now

$$\begin{split} \theta(\operatorname{Im}\,(I-L),\,\operatorname{Im}(K)) \\ &= \max \left\{ \operatorname{dist}\left[S(\operatorname{Im}\,(I-L)),\,\operatorname{Im}\,K \right],\,\operatorname{dist}\left[S(\operatorname{Im}\,K),\,\operatorname{Im}\,(I-L) \right] \right\} \end{split}$$

(where S(V) denotes the unit sphere in the subspace V).

Now the unit spheres in Im (I - L), im K are closed and bounded hence compact since \mathfrak{H}_1 has the Montel property.

Assume that the max is given by the first term; let $x \in S(\text{Im }(I - L))$. The projection of x on Im K lies in the unit ball of Im K which is compact. Hence we can consider in the computation of θ the distance from S(Im (I - L)) to the unit ball of Im K and the distance is thus attained.

Let

$$df \in S(\operatorname{Im}(I-L)), \qquad dg \in \operatorname{Im} K$$

be corresponding points. One has

$$heta=rac{|\,(df,\,dg)\,|}{||\,df\,||\,||\,dg||}$$
 .

If now $\theta = 1$, then |(df, dg)| = ||df|| ||dg|| and hence $df = \lambda dg$ where λ is a constant, and also df = (I - L)dh.

Now dg is extendable and $df \in \text{Im}(I - L)$. It follows that df = 0, a contradiction.

(A similar reasoning is valid in case the max in the definition of θ is given by the second term.)

It follows that $\theta < 1$ and $(^{++})$ has the solution.

$$\omega = -Kp = K \sum_{n=0}^{\infty} [I - (K + L)]^n (I - L)s$$
.

NOTE. Instead of \mathfrak{H}_1 one could work in a closed subset of \mathfrak{H}_1 e.g. $h_{[\beta]}(\overline{A})$.

The uniqueness is discussed as before: we get uniqueness provided

$$\operatorname{Im} K \cap \operatorname{Im} L = \{0\}$$
.

i.e. no differential in the image of L is extendable to \overline{W} .

If ω_1 and ω_2 are solutions, then

$$(1-L)(\omega_1-\omega_2)=0$$
.

Now

$$\omega_i=-Kp_i$$
 $i=1,2$.

 \mathbf{So}

$$(I-L)K(p_1-p_2)=0$$
.

Hence if

$$\operatorname{Im} K \cap \operatorname{Im} L = \{0\}$$
, $p_1 = p_2$ and $\omega_1 = \omega_2$.

Conversely, if there is a differential $\tau \in \mathfrak{H}_1$ such that

$$\tau = L\mu = K\nu$$

then if ω is a solution in \Re of

$$\omega - s = L(\omega - s)$$

we have

$$\omega + \tau - s = \tau + L(\omega - s) = L(\mu + \omega - s) = L(\tau + \omega - s)$$

and uniqueness is lost.

As examples we could take:

(i)
$$L = \Lambda_0$$
 orthogonal projection on $h_{0\beta}^{\pm}(A)$.

Then

$$\operatorname{Im}\left(I-L\right)=h_{0\alpha}'(\bar{A})$$

and

$$\mathrm{Im} \ (I-L) \cap H_{ext}(\bar{A}) = \{0\} \quad ext{and} \quad \mathrm{Im} \ L \cap H_{ext}(\bar{A}) = \{0\} \ ;$$

(ii) $L = \Lambda_1$ orthogonal projection on $h'_{0\beta}(\overline{A})$. Similar results are valid.

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