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## **ARITHMETICAL PROPERTIES OF GENERALIZED RAMANUJAN SUMS**

TOM M. (MIKE) APOSTOL

# ARITHMETICAL PROPERTIES OF GENERALIZED RAMANUJAN SUMS

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The sums studied in this paper are defined as follows.  
For any two arithmetical functions  $f$  and  $g$ , let

$$(1) \quad S_{f,g}(m, k) = \sum_{d \mid (m, k)} f(d)g(k/d),$$

where the sum extends over the divisors of the greatest common divisor  $(m, k)$  of the positive integers  $m$  and  $k$ . It should be noted that  $m$  and  $k$  do not enter symmetrically in (1) unless  $g$  is constant.

The sums  $S_{f,g}(m, k)$  generalize the Dirichlet convolution

$$(2) \quad (f * g)(k) = \sum_{d \mid k} f(d)g(k/d),$$

to which they reduce when  $(m, k) = k$ . Multiplicative properties and finite Fourier expansions were obtained in [1]. A famous special case is Ramanujan's sum  $c_k(m)$ , the sum of the  $m$ th powers of the primitive  $k$ th roots of unity, for which we have

$$(3) \quad c_k(m) = \sum_{\substack{h \bmod k \\ (h, k) = 1}} \exp(2\pi i m h/k) = \sum_{d \mid (m, k)} d \mu(k/d),$$

where  $\mu$  is the Möbius function. The second sum in (3) is an example of (1) with  $f(n) = n$  and  $g(n) = \mu(n)$  for all  $n$ . When  $(m, k) = 1$  we have  $c_k(m) = \mu(k)$ , and when  $(m, k) = k$  we have  $c_k(m) = \varphi(m)$ , Euler's totient.

In a study on cyclotomic polynomials, Hölder [4] showed that Ramanujan's sum can also be expressed in closed form as follows:

$$(4) \quad c_k(m) = \frac{\varphi(m)}{\varphi(m/(k, m))} \mu(m/(k, m)).$$

The number on the right is called the Von Sterneck function and is denoted by  $\Phi(m, k)$ . Thus, (4) states that

$$c_k(m) = \Phi(m, k).$$

The function  $\Phi(m, k)$  was encountered by Von Sterneck in 1902 [11] in a study of restricted partitions with summands reduced to their least residues module  $m$ . Its properties were also studied by Nicol and Vandiver [7].

We derive further properties of the sums  $S_{f,g}(m, k)$ . Some of them

generalize known properties of Ramanujan's sum or Von Sterneck's function. Others, when specialized, give new properties of these numbers. The methods of this paper are simpler than those used by earlier writers. In particular, roots of unity and restricted partitions mod  $m$  play no role.

**2. Properties of Dirichlet convolution.** The Dirichlet convolution (2) provides a natural setting for the results of this paper. It is well known that the operation  $*$  is commutative and associative and has the identity element  $I$ , where

$$I(n) = \left[ \frac{1}{n} \right] = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

If  $g(1) \neq 0$  we let  $g^{-1}$  denote the Dirichlet inverse of  $g$ , defined by the equation  $g * g^{-1} = I$ . In particular,  $\mu^{-1}(n) = 1$  for all  $n$ .

The set of all functions  $g$  with  $g(1) \neq 0$  forms an abelian group under the operation  $*$ . The subset of multiplicative functions is a subgroup. [A function  $g$  is called multiplicative if  $g(1) = 1$  and if  $g(mn) = g(m)g(n)$  whenever  $(m, n) = 1$ .]

If  $a$  is any arithmetical function and if  $g(1) \neq 0$ , the equation

$$S(n) = \sum_{d|n} a(d)g(n/d)$$

holds if, and only if,

$$a(n) = \sum_{d|n} S(d)g^{-1}(n/d).$$

The special case with  $g = \mu$  is the usual Möbius inversion formula.

By introducing the function

$$(5) \quad \alpha_{n,d} = \begin{cases} 1 & \text{if } d|n \\ 0 & \text{if } d \nmid n, \end{cases}$$

we can rewrite the sum in (1) in two alternate forms:

$$(6) \quad S_{f,g}(m, k) = \sum_{d|m} \alpha_{k,d} f(d)g(k/d),$$

and

$$(7) \quad S_{f,g}(m, k) = \sum_{d|k} \alpha_{m,d} f(d)g(k/d).$$

For fixed  $k$ , Equation (6) expresses  $S_{f,g}(m, k)$  as a Dirichlet convolution,

$$(8) \quad S_{f,g}(m, k) = (h_k * \mu^{-1})(m),$$

where

$$h_k(d) = \alpha_{k,d} f(d) g(k/d) .$$

For fixed  $m$ , Equation (7) expresses  $S_{f,g}(m, k)$  as another Dirichlet convolution,

$$(9) \quad S_{f,g}(m, k) = (w_m * g)(k) ,$$

where

$$w_m(d) = \alpha_{m,d} f(d) .$$

**3. Dirichlet convolutions involving  $S_{f,g}(m, k)$ .** This section derives arithmetical identities in the form of divisor sums whose terms contain the numbers  $S_{f,g}(m, k)$ .

**NOTATION.** The function  $\alpha$  is defined by (5) and, unless otherwise stated, the functions  $f$  and  $g$  are arbitrary. For any arithmetical function  $a$ , we denote the Dirichlet convolution  $a * \mu^{-1}$  by  $a^*$ . Thus,

$$a^*(n) = \sum_{d|n} a(d) .$$

For example, we have  $\mu^* = \mu * \mu^{-1} = I$ , and  $(\mu^{-1})^* = \sigma$ , where  $\sigma(n)$  is the sum of the divisors of  $n$ .

**THEOREM 1.** *If  $n \geq 1, k \geq 1$  we have*

$$(10) \quad \sum_{d|n} S_{f,g}(d, k) = \sum_{d|(n,k)} f(d) g(k/d) \sigma(n/d) .$$

*Proof.* For fixed  $k$ , let  $S(m) = S_{f,g}(m, k)$ . Then by (8) we have  $S(m) = (h_k * \mu^{-1})(m)$ , so

$$\begin{aligned} \sum_{d|n} S_{f,g}(d, k) &= \sum_{d|n} S(d) = (S * \mu^{-1})(n) \\ &= (h_k * \mu^{-1} * \mu^{-1})(n) = (h_k * \sigma)(n) \\ &= \sum_{d|n} \alpha_{k,d} f(d) g(k/d) \sigma(n/d) = \sum_{d|n, d|k} f(d) g(k/d) \sigma(n/d) , \end{aligned}$$

which proves Theorem 1.

**EXAMPLES.** Theorem 1 has a number of interesting corollaries. If  $(n, k) = n$  we obtain

$$\sum_{d|n} S_{f,g}(d, k) = \sum_{d|n} f(d) g(k/d) \sigma(n/d) .$$

If  $k = n$  this gives

$$\sum_{d|n} S_{f,g}(d, n) = \sum_{d|n} f(d) g(n/d) \sigma(n/d) .$$

When  $g = \mu$  we can write this as

$$(11) \quad \sum_{d|n} S_{f,\mu}(d, n) = \sum_{d|n} \mu(d)\sigma(d)f(n/d) .$$

If  $f$  is completely multiplicative, that is, if  $f(ab) = f(a)f(b)$  for all  $a, b$ , and if  $f(n) \neq 0$ , then  $f(n/d) = f(n)/f(d)$  and (11) gives us

$$\begin{aligned} \sum_{d|n} S_{f,\mu}(d, n) &= f(n) \sum_{d|n} \mu(d)\sigma(d)/f(d) \\ &= f(n) \prod_{p|n} \left(1 - \frac{\sigma(p)}{f(p)}\right) = f(n) \prod_{p|n} \left(1 - \frac{2}{f(p)}\right), \end{aligned}$$

where the product is taken over all prime divisors of  $n$ . The special case  $f(n) = n$  gives a formula of Nicol and Vandiver ([7], Theorem VIII),

$$(12) \quad \sum_{d|n} \phi(d, n) = n \prod_{p|n} \left(1 - \frac{2}{p}\right) .$$

We also have the following more general result.

**THEOREM 2.** *Let  $a^*(n) = \sum_{d|n} a(d)$ , where  $a(n)$  is any arithmetical function. Then for  $n \geq 1, k \geq 1$  we have*

$$(13) \quad \sum_{d|n} S_{f,g}(d, k)a(n/d) = \sum_{d|(n,k)} f(d)g(k/d)a^*(n/d) .$$

*Proof.* With the notation used in the proof of Theorem 1 we have

$$\begin{aligned} \sum_{d|n} S_{f,g}(d, k)a(n/d) &= (S^*a)(n) = (h_k^* \mu^{-1} * a)(n) = (h_k^* a^*)(n) \\ &= \sum_{d|k} \alpha_{k,d} f(d)g(k/d)a^*(n/d) = \sum_{d|n, d|k} f(d)g(k/d)a^*(n/d), \end{aligned}$$

which proves Theorem 2.

**EXAMPLES.** When  $a = \mu^{-1}$  then  $a^* = \sigma$  and Theorem 2 reduces to Theorem 1. When  $a = \mu$  then  $a^* = I$  and the second sum in (13) is

$$\sum_{d|(n,k)} f(d)g(k/d)a^*(n/d) = \sum_{d|(n,k)} f(d)g(k/d)[d/n] = \alpha_{k,n} f(n)g(k/n) .$$

Hence (13) becomes

$$\sum_{d|n} S_{f,g}(d, k)\mu(n/d) = \begin{cases} f(n)g(k/n) & \text{if } n|k, \\ 0 & \text{if } n \nmid k. \end{cases}$$

This can also be deduced by Möbius inversion of (6) for fixed  $k$ .

When  $k = n$ , Equation (13) can be written as

$$\sum_{d|n} S_{f,g}(d, n)a(n/d) = \sum_{d|n} g(d)a^*(d)f(n/d) .$$

If we take  $g = \mu$  and assume that  $a$  is multiplicative and that  $f$  is

completely multiplicative, with  $f(n) \neq 0$ , we obtain the following product formula, a further generalization of (12):

$$\sum_{d|n} S_{f,\mu}(d, n) a(n/d) = f(n) \prod_{p|n} \left(1 - \frac{a^*(p)}{f(p)}\right).$$

**THEOREM 3.** *If  $n \geq 1$ ,  $m \geq 1$  we have*

$$(14) \quad \sum_{d|n} S_{f,g}(m, d) = \sum_{d|(m,n)} f(d) g^*(n/d),$$

where  $g^*(n) = \sum_{d|n} g(d)$ .

*Proof.* For fixed  $m$ , let  $S(k) = S_{f,g}(m, k)$ . Then by (9) we have  $S(k) = (w_m * g)(k)$ , so

$$\begin{aligned} \sum_{d|n} S_{f,g}(m, d) &= \sum_{d|n} S(d) = (S * \mu^{-1})(n) \\ &= (w_m * g * \mu^{-1})(n) = (w_m * g^*)(n) \\ &= \sum_{d|n} \alpha_{m,d} f(d) g^*(n/d) = \sum_{d|n, d|m} f(d) g^*(n/d), \end{aligned}$$

which proves Theorem 3.

**EXAMPLES.** For the special case  $g = \mu$  we have  $g^* = I$  so (14) becomes

$$(15) \quad \sum_{d|n} S_{f,\mu}(m, d) = \sum_{d|(m,n)} f(d) [d/n] = \alpha_{m,n} f(n) = \begin{cases} f(n) & \text{if } n|m, \\ 0 & \text{if } n \nmid m. \end{cases}$$

When  $f(n) = n$  this gives a formula of Von Sterneck ([7], p. 825),

$$\sum_{d|n} \Phi(m, d) = \begin{cases} n & \text{if } n|m, \\ 0 & \text{if } n \nmid m. \end{cases}$$

The type of argument used to prove Theorem 3 also gives the following more general result.

**THEOREM 4.** *For any arithmetical function  $a$ , let  $B = g * a$ . Then for  $m \geq 1$ ,  $n \geq 1$  we have*

$$(16) \quad \sum_{d|n} S_{f,g}(m, d) a(n/d) = \sum_{d|(m,n)} f(d) B(n/d).$$

If  $g(1) \neq 0$  and  $a = g^{-1}$ , then  $B = I$  and (16) reduces to

$$\sum_{d|n} S_{f,g}(m, d) g^{-1}(n/d) = \alpha_{m,n} f(n) = \begin{cases} f(n) & \text{if } n|m, \\ 0 & \text{if } n \nmid m. \end{cases}$$

When  $g = \mu$  this is the same as formula (15).

EXAMPLES. When  $g = \mu$  and  $a(n) = n$ , then  $B = \mu * a = \varphi$  and (16) implies

$$\sum_{d|n} dS_{f,\mu}\left(m, \frac{n}{d}\right) = \sum_{d|(m,n)} f(d)\varphi(n/d) .$$

When  $f(n) = n$  this gives the following identity for Von Sterneck's function,

$$\sum_{d|n} d\Phi(m, n/d) = \sum_{d|(m,n)} d\varphi(n/d) .$$

4. **Partial sums involving the functions  $S_{f,g}(m, k)$ .** The theorems of this section deal with sums whose terms include the numbers  $S_{f,g}(m, k)$  where one of  $m$  or  $k$  is fixed and the other ranges over consecutive integers. First we introduce some notation.

As in the previous section we denote by  $a^*$  the divisor sum

$$a^*(n) = \sum_{d|n} a(d) ,$$

where  $a$  is any arithmetical function. For real  $x \geq 1$  we also write

$$a^\wedge(x) = \sum_{n \leq x} a(n) ,$$

and we put  $a^\wedge(x) = 0$  if  $x < 1$ . We shall make use of the following lemma.

LEMMA. *For any two arithmetical functions  $a$  and  $b$  we have*

$$\sum_{n \leq x} (a*b)(n) = \sum_{n \leq x} a(n)b^\wedge(x/n) .$$

*Proof.* We have

$$\sum_{n \leq x} (a*b)(n) = \sum_{n \leq x} \sum_{d|n} a(d)b(n/d) .$$

If  $d|n$  we can write  $n = qd$  and we obtain

$$\sum_{n \leq x} \sum_{d|n} a(d)b(n/d) = \sum_{d \leq x} a(d) \sum_{q \leq x/d} b(q) = \sum_{d \leq x} a(d)b^\wedge(x/d) .$$

THEOREM 5. *For  $k \geq 1$ ,  $n \geq 1$ , and any arithmetical function  $a$  we have*

$$(17) \quad \sum_{m=1}^n S_{f,g}(m, k)a^\wedge(n/m) = \sum_{d|k, d \leq n} f(d)g(k/d)(a^*)^\wedge(n/m) .$$

*Proof.* We use the lemma, then Equation (8), and then the lemma once more to get

$$\begin{aligned} \sum_{m=1}^n S_{f,g}(m, k) a^\wedge(n/m) &= \sum_{m=1}^n (h_k * \mu^{-1} * a)(m) = \sum_{m=1}^n (h_k * a^*)(m) \\ &= \sum_{m=1}^n h_k(m) (a^*)^\wedge(n/m) = \sum_{m=1}^n \alpha_{k,m} f(m) g(k/m) (a^*)^\wedge(n/m), \end{aligned}$$

which proves (17).

EXAMPLES. If we take  $a = I$ , then  $a^\wedge(x) = 1$  for all  $x \geq 1$  and  $a^*(n) = 1$  for all  $n$ , so  $(a^*)^\wedge(x) = [x]$ , and Theorem 5 becomes

$$(18) \quad \sum_{m=1}^n S_{f,g}(m, k) = \sum_{d|k, d \leq n} f(d) g(k/d) [n/d].$$

If  $k|n$ , say  $n = qk$ , then every divisor  $d$  of  $k$  is  $\leq n$  and (18) takes the form

$$\sum_{m=1}^{qk} S_{f,g}(m, k) = qk \sum_{d|k} \frac{f(d)}{d} g\left(\frac{k}{d}\right).$$

For the special case  $f(n) = n$  this gives us

$$\sum_{m=1}^{qk} S_{f,g}(m, k) = qk g^*(k).$$

In particular, when  $g = \mu$  we obtain the following formula of Von Sterneck ([7], p. 825):

$$\sum_{m=1}^{qk} \Phi(m, k) = qk \sum_{d|k} \mu(d) = \begin{cases} q & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$

THEOREM 6. For  $m \geq 1$ ,  $n \geq 1$  and any arithmetical function  $a$  we have

$$(19) \quad \sum_{m=1}^n S_{f,g}(m, k) a^\wedge(n/k) = \sum_{d|m, d \leq n} f(d) (g*a)^\wedge(n/d).$$

*Proof.* This time we use the lemma in conjunction with (9) to obtain

$$\begin{aligned} \sum_{k=1}^n S_{f,g}(m, k) a^\wedge(n/k) &= \sum_{k=1}^n (w_m * g*a)(k) = \sum_{k=1}^n w_m(k) (g*a)^\wedge(n/k) \\ &= \sum_{k=1}^n \alpha_{m,k} f(k) (g*a)^\wedge(n/k) = \sum_{k|m, k \leq n} f(k) (g*a)^\wedge(n/k). \end{aligned}$$

This proves Theorem 6.

EXAMPLES. If  $a = g^{-1}$  we have  $g*a = I$  and  $I^\wedge(x) = 1$  for all  $x \geq 1$ , so Theorem 6 gives us the formula



$$(20) \quad \sum_{k=1}^n S_{f,g}(m, k)(g^{-1})^{\wedge}(n/k) = \sum_{d|m, d \leq n} f(d) .$$

In particular, if  $g = \mu$  then  $(g^{-1})^{\wedge}(x) = [x]$  and (20) becomes

$$\sum_{k=1}^n \left[ \frac{n}{k} \right] S_{f,\mu}(m, k) = \sum_{d|m, d \leq n} f(d) .$$

When  $f(n) = n$  this gives a theorem of Nicol ([6], p. 965),

$$\sum_{k=1}^n \left[ \frac{n}{k} \right] \Phi(m, k) = \sum_{d|m, d \leq n} d = \sigma(m, n) .$$

Here  $\sigma(m, n)$  is the sum of the divisors of  $m$  which are  $\leq n$ .

If we take  $a = I$ , then  $g*a = g$  and Theorem 6 becomes

$$(21) \quad \sum_{k=1}^n S_{f,g}(m, k) = \sum_{d|m, d \leq n} f(d)g^{\wedge}(n/d) .$$

In particular, if  $f(n) = n$  and  $g = \mu$  we obtain the following formula for the sum of the  $m$ th powers of all the primitive  $k$ th roots of unity for  $k = 1, 2, \dots, n$ :

$$\sum_{k=1}^n c_k(m) = \sum_{d|m, d \leq n} d\mu^{\wedge}(n/d) .$$

When  $n = m$  this becomes

$$(22) \quad \sum_{k=1}^m c_k(m) = \sum_{d|m} d\mu^{\wedge}(m/d) .$$

The right-hand member of (22) has the form  $(f*\mu^{\wedge})(m)$  where  $f(n) = n$ . Since  $f^{-1}(n) = n\mu(n)$ , inversion of (22) gives us the following formula for the partial sums of the Möbius function:

$$(23) \quad \sum_{n=1}^m \mu(n) = m \sum_{d|m} \frac{\mu(m/d)}{d} \sum_{k=1}^d c_k(d) .$$

**THEOREM 7.** For  $k \geq 1$ ,  $n \geq 1$  and any completely multiplicative function  $a$  we have

$$\sum_{m=1}^n a(m)S_{f,g}(m, k) = \sum_{d|k, d \leq n} a(d)f(d)g(k/d)a^{\wedge}(n/d) .$$

*Proof.* We use formula (8) to obtain

$$\begin{aligned} \sum_{m=1}^n a(m)S_{f,g}(m, k) &= \sum_{m=1}^n a(m) \sum_{d|m} h_k(d) = \sum_{d=1}^n a(d)h_k(d) \sum_{q \leq n/d} a(q) \\ &= \sum_{d=1}^n a(d)h_k(d)a^{\wedge}(n/d) = \sum_{d|k, d \leq n} a(d)f(d)g(k/d)a^{\wedge}(n/d) . \end{aligned}$$

This proves Theorem 7.

EXAMPLES. If  $a(n) = n$  then  $\alpha^\wedge(x) = [x]([x] + 1)/2$  and Theorem 7 gives us

$$(24) \quad \sum_{m=1}^n m S_{f,g}(m, k) = \frac{1}{2} \sum_{d|k, d \leq n} d f(d) g(k/d) [n/d] ([n/d] + 1).$$

If  $k|n$ , say  $n = qk$ , then  $[n/d] = qk/d$  if  $d|k$  and (24) becomes

$$\sum_{m=1}^{qk} m S_{f,g}(m, k) = \frac{1}{2} q^2 k^2 \sum_{d|k} \frac{f(d)}{d} g(k/d) + \frac{1}{2} qk \sum_{d|k} f(d) g(k/d).$$

When  $g = \mu$  and  $f(n) = n$  we obtain the formula

$$\sum_{m=1}^{qk} m \Phi(m, k) = \frac{1}{2} q^2 k^2 I(k) + \frac{1}{2} qk \varphi(k) = \begin{cases} \frac{1}{2} q(q+1) & \text{if } k = 1, \\ \frac{1}{2} qk \varphi(k) & \text{if } k > 1. \end{cases}$$

When  $q = 1$  this gives Theorem III of Nicol and Vandiner ([7], p. 830).

In Theorem 5 of [1] it was shown that for  $R(s) > 1$  we have

$$(25) \quad \sum_{m=1}^{\infty} \frac{S_{f,g}(m, k)}{m^s} = \zeta(s) \sum_{d|k} f(d) g(k/d) d^{-s}$$

where  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ . Theorem 7 leads to another proof of this formula and also gives an estimate for the growth of the partial sums of the series on the left. Taking  $a(m) = m^{-s}$ , and putting  $\sigma = R(s)$ , we have (see [2], p. 618)

$$(26) \quad \alpha^\wedge(x) = \sum_{m \leq x} \frac{1}{m^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-\sigma}).$$

This formula is valid for all complex  $s \neq 1$  with  $\sigma > 0$ . Using this in Theorem 7 and taking  $n = qk$ , we obtain the formula

$$(27) \quad \sum_{m=1}^{qk} \frac{S_{f,g}(m, k)}{m^s} = \frac{q^{1-s} k^{1-s}}{1-s} \sum_{d|k} f(d) g(k/d) d^{-1} + \zeta(s) \sum_{d|k} f(d) g(k/d) d^{-s} + O(q^{-\sigma} k^{-\sigma} \sum_{d|k} |f(d) g(k/d)|).$$

If  $\sigma > 1$ , the terms on the right which contain  $q$  approach 0 as  $q \rightarrow \infty$  and we obtain (25).

For the special case  $f(d) = d$ ,  $g = \mu$ , we have

$$\sum_{d|k} f(d) g(k/d) d^{-1} = \sum_{d|k} \mu(k/d) = 0 \quad \text{if } k > 1.$$

In this case the first term on the right of (27) vanishes if  $k > 1$  and the third term approaches 0 as  $q \rightarrow \infty$  when  $\sigma > 0$ . This gives a celebrated formula of Ramanujan ([9], p. 199)

$$\sum_{m=1}^{\infty} \frac{c_k(m)}{m^s} = \zeta(s) \sum_{d|k} d^{1-s} \mu(k/d) ,$$

valid for  $k > 1$  and all complex  $s \neq 1$  with  $\sigma > 0$ .

For  $s = 1$  we have, instead of (26), the estimate

$$\alpha^{\wedge}(x) = \sum_{m \leq x} \frac{1}{m} = \log x + C + O\left(\frac{1}{x}\right) ,$$

where  $C$  is Euler's constant. Using this in Theorem 7 with  $n = qk$  we obtain the formula

$$\begin{aligned} \sum_{m=1}^{qk} \frac{S_{f,g}(m, k)}{m} &= (C + \log k + \log q) \sum_{d|k} f(d) d^{-1} g(k/d) \\ &\quad - \sum_{d|k} f(d) d^{-1} g(k/d) \log d + O(q^{-1} k^{-1} \sum_{d|k} |f(d) g(k/d)|) . \end{aligned}$$

This shows that the series  $\sum_{m=1}^{\infty} S_{f,g}(m, k)/m$  converges if and only if the coefficient of  $\log q$  vanishes, that is, if and only if

$$(28) \quad \sum_{d|k} f(d) d^{-1} g(k/d) = 0 ,$$

in which case we obtain

$$\sum_{m=1}^{\infty} \frac{S_{f,g}(m, k)}{m} = - \sum_{d|k} f(d) d^{-1} g(k/d) \log d .$$

When  $f(d) = d$  and  $g = \mu$ , Equation (28) is satisfied for  $k > 1$  and we obtain another formula of Ramanujan ([9], p. 199),

$$\sum_{m=1}^{\infty} \frac{c_k(m)}{m} = - \sum_{d|k} \mu(k/d) \log d = - \Lambda(k) ,$$

where  $k > 1$  and  $\Lambda(k)$  is Mangoldt's function.

**THEOREM 8.** *For  $m \geq 1$ ,  $n \geq 1$  and any completely multiplicative function  $a$  we have*

$$\sum_{k=1}^n a(k) S_{f,g}(m, k) = \sum_{d|m, d \leq n} a(d) f(d) (ag)^{\wedge}(n/d) ,$$

where  $(ag)^{\wedge}(r) = \sum_{d \leq r} a(d) g(d)$ .

*Proof.* We use formula (9) to obtain

$$\begin{aligned} \sum_{k=1}^n a(k) S_{f,g}(m, k) &= \sum_{k=1}^n a(k) \sum_{d|k} w_m(d) g(k/d) = \sum_{d=1}^n a(d) w_m(d) \sum_{q \leq n/d} a(q) g(q) \\ &= \sum_{d=1}^n a(d) \alpha_{m,d} f(d) (ag)^{\wedge}(n/d) = \sum_{d|m, d \leq n} a(d) f(d) (ag)^{\wedge}(n/d) . \end{aligned}$$

EXAMPLES. In Theorem 5 of [1] it was shown that

$$(29) \quad \sum_{k=1}^{\infty} \frac{S_{f,g}(m, k)}{k^s} = G(s) \sum_{d|m} f(d) d^{-s}$$

for each  $s$  for which the Dirichlet series  $G(s) = \sum_{r=1}^{\infty} g(r) r^{-s}$  is convergent. We can derive this also from Theorem 8 and obtain a formula for the partial sums of the series on the left.

Taking  $a(n) = n^{-s}$  we have

$$(ag)^{\wedge}(x) = \sum_{r \leq x} g(r) r^{-s} = G(s) - \sum_{n > x} g(r) r^{-s}$$

if the series for  $G(s)$  converges. Using this in Theorem 8 and taking  $n = qm$  we find

$$\sum_{k=1}^{qm} \frac{S_{f,g}(m, k)}{k^s} = G(s) \sum_{d|m} f(d) d^{-s} - \sum_{d|m} f(d) d^{-s} \sum_{r > qm/d} g(r) r^{-s} .$$

Letting  $q \rightarrow \infty$  we obtain (29).

In the special case with  $f(d) = d$ ,  $g = \mu$ , we have

$$G(s) = \sum_{r=1}^{\infty} \frac{\mu(r)}{r^s} = \frac{1}{\zeta(s)} \quad \text{for } R(s) > 1 .$$

This series also converges for  $s = 1$  and  $G(1) = 0$ . Also,

$$\sum_{d|m} f(d) d^{-s} = \sum_{d|m} d^{1-s} = \sigma_{1-s}(m) .$$

In this case (29) gives another formula of Ramanujan ([9], p. 185)

$$\sum_{k=1}^{\infty} \frac{c_k(m)}{k^s} = \frac{\sigma_{1-s}(m)}{\zeta(s)}$$

valid for  $R(s) > 1$  and also for  $s = 1$ .

5. An extension of Smith's determinant. For any arithmetical function  $f$ , let

$$f^*(r) = \sum_{d|r} f(d) ,$$

and let  $A = [f^*(m, k)]$  be the  $n \times n$  matrix whose  $m, k$  entry is the value of  $f^*$  at the greatest common divisor  $(m, k)$  of  $m$  and  $k$ . H. J. S. Smith [10] proved the determinant formula

$$(30) \quad \det A = f(1)f(2) \cdots f(n) .$$

Dickson ([3], pp. 122–129) reports on several papers by Catalan, Cesaro, Gegenbauer, Mansion, and others, devoted to proofs and extensions of (30). D. H. Lehmer [5] also generalized (30) to higher-dimensional determinants.

A simple proof of (30), suggested by Pólya and Szegő (see [8], p. 330), is based on the observation that  $A = BC^t$ , where  $B$  and  $C$  are lower triangular  $n \times n$  matrices with  $\det B = f(1)f(2) \cdots f(n)$  and  $\det C = 1$ . This section extends this proof to provide the following new generalization of (30).

**THEOREM 9.** *Let  $A$  be the  $n \times n$  matrix whose  $m, k$  entry is  $S_{f,g}(m, k)$ . Then we have*

$$(31) \quad \det A = f(1)f(2) \cdots f(n)g(1)^n .$$

*Proof.* We express the  $n \times n$  matrix  $A$  as a product,  $A = B(f)C(g)^t$ , where  $B(f)$  and  $C(g)$  are lower triangular  $n \times n$  matrices given by

$$B(f) = [\alpha_{m,k}f(m)], \quad C(g) = [\alpha_{m,k}g(m/k)] .$$

Then  $m, k$  entry of  $B(f)C(g)^t$  is equal to

$$\sum_{r=1}^n \alpha_{m,r}f(r)\alpha_{k,r}g(k/r) = \sum_{r|m, r|k} f(r)g(k/r) = S_{f,g}(m, k) ,$$

so  $A = B(f)C(g)^t$ , as asserted. Since  $\det B(f) = f(1) \cdots f(n)$  and  $\det C(g) = g(1)^n$ , we obtain (31).

**EXAMPLE.** When  $f(n) = n$  and  $g = \mu$ , we obtain the following formula for the determinant of the  $n \times n$  matrix whose  $m, k$  entry is the Ramanujan sum  $c_k(m)$ :

$$\det [c_k(m)] = n! .$$

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