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**A GENERALIZATION OF INJECTIVITY**

JOHN ALLEN BEACHY

# A GENERALIZATION OF INJECTIVITY

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**In a category of modules the notions of  $\rho$ -injectivity (with respect to a torsion radical  $\rho$ ) and quasi-injectivity can be generalized to a notion of injectivity with respect to two preradicals simultaneously. Using this general definition an analog of Baer's condition for injectivity is obtained, as well as other generalizations of results for injective and quasi-injective modules. An alternate approach (not requiring the existence of injective envelopes) is given for abelian categories, with the results stated in dual form for projectivity.**

In the first section of the paper we give some preliminary definitions and results, including a definition of density with respect to a preradical which is weaker than the standard one, and the definitions of preradicals  $\text{rad}^M$  and  $\text{Rad}^M$  associated with a module  $M$  (the smallest preradical and smallest torsion preradical, respectively, for which  $M$  is torsion). In the second section we define and study the notion of  $(\rho, \sigma)$ -injectivity, for preradicals  $\rho$  and  $\sigma$ . A module  $Q$  is called  $(\rho, \sigma)$ -injective if every homomorphism  $f: N_0 \rightarrow Q$ , where  $N_0$  is a  $\rho$ -dense submodule of  $N$  and  $\ker(f)$  is  $\sigma$ -dense in  $N$ , can be extended to  $N$ . This definition is motivated by a theorem of L. Fuchs [3, Lemma 1] giving a characterization of quasi-injectivity. Many of the results are motivated by those of G. Azumaya in his paper on  $M$ -projective and  $M$ -injective modules [1]. We prove that a module is  $M$ -injective if and only if it is  $(\rho, \sigma)$ -injective, where  $\rho$  is the identity functor and  $\sigma$  is either  $\text{rad}^M$  or  $\text{Rad}^M$ . This approach depends heavily on the existence of injective envelopes in categories of modules. In the third section of the paper we drop this assumption and obtain slightly weaker results valid in any abelian category. These results are stated in their dual form, for projectivity, and we show that our definition specializes, for modules with a projective cover, to that of  $M$ -projectivity.

1. Preliminary definitions and results. We will use the terminology of J.-M. Maranda [6]. A subfunctor  $\rho$  of the identity functor on an abelian category  $\underline{A}$  is called a preradical of  $\underline{A}$ . Thus a preradical  $\rho$  of  $\underline{A}$  assigns to each object  $A$  of  $\underline{A}$  a subobject  $\rho(A)$  and to each morphism  $f: A \rightarrow B$  in  $\underline{A}$  its restriction  $\rho(f): \rho(A) \rightarrow \rho(B)$ . It is said to be idempotent if  $\rho^2 = \rho$  and is called a torsion preradical

if  $\rho$  is left exact. If  $\rho(A/\rho(A)) = 0$  for all  $A$  in  $\underline{A}$ , then  $\rho$  is called a radical. An object  $A$  is  $\rho$ -torsion if  $\rho(A) = A$  and  $\rho$ -torsionfree if  $\rho(A) = 0$ .

We are primarily interested in the category  ${}_R\mathbf{M}$  of unital left  $R$ -modules over an associative ring  $R$  with identity. Any preradical  $\rho$  of  ${}_R\mathbf{M}$  defines a closure operation on submodules in the following way: for a submodule  $N_0$  of  ${}_R N$  let the closure  $C_\rho(N_0; N)$  of  $N_0$  in  $N$  be the inverse image of  $\rho(N/N_0)$  under the projection  $p: N \rightarrow N/N_0$ . The  $\rho$ -closure of  $N$  in its injective envelope  $E(N)$  will be written simply as  $C_\rho(N)$ . A submodule  $N_0$  of  $N$  is said to be  $\rho$ -closed in  $N$  if  $C_\rho(N_0; N) = N_0$ . Note that this is the case if and only if  $N/N_0$  is  $\rho$ -torsionfree. This closure operation has the property that if  $M, N \in {}_R\mathbf{M}$  with submodules  $M_0, N_0$  respectively and  $f \in \text{Hom}_R(M, N)$ , then  $f(M_0) \subseteq N_0$  implies  $f(C_\rho(M_0; M)) \subseteq C_\rho(N_0; N)$ . This can be shown by considering the homomorphism from  $M/M_0$  to  $N/N_0$  induced by  $f$ . Associated with the closure operation is a notion of density.

**DEFINITION 1.1.** Let  $\rho$  be a preradical of  ${}_R\mathbf{M}$ . A submodule  $N_0$  of a module  ${}_R N$  is said to be  $\rho$ -dense in  $N$  if there exists an extension  ${}_R M$  of  $N$  such that  $N \subseteq C_\rho(N_0; M)$ .

**LEMMA 1.2.** A submodule  $N_0 \subseteq {}_R N$  is  $\rho$ -dense in

$$N \iff N \subseteq C_\rho(N_0; E(N)).$$

*Proof.* If  $N_0$  is  $\rho$ -dense in  $N$  then there exists an extension  $M$  of  $N$  with  $N \subseteq C_\rho(N_0; M)$ . The inclusion  $N \rightarrow E(N)$  extends to a homomorphism  $f: M \rightarrow E(N)$ , since  $E(N)$  is injective, and then  $f(N_0) \subseteq N_0$  implies that  $N = f(N) \subseteq f(C_\rho(N_0; M)) \subseteq C_\rho(N_0; E(N))$ . The converse follows immediately from the definition.

**PROPOSITION 1.3.** Let  $\rho$  be a preradical of  ${}_R\mathbf{M}$ , and  $N_1, N_2$  be submodules of the module  ${}_R N$ , with  $N_1 \subseteq N_2 \subseteq N$ .

- (a) If  $N_1$  is  $\rho$ -dense in  $N$ , then  $N_2$  is  $\rho$ -dense in  $N$ .
- (b) If  $N_2$  is  $\rho$ -dense in  $N$ , then  $N_2/N_1$  is  $\rho$ -dense in  $N/N_1$ .

*Proof.* (a) If  $N_1$  is  $\rho$ -dense in  $N$ , then by Lemma 1.2 we have  $N \subseteq C_\rho(N_1; E(N))$ . Since  $N_1 \subseteq N_2$  implies  $C_\rho(N_1; E(N)) \subseteq C_\rho(N_2; E(N))$ , it follows that  $N \subseteq C_\rho(N_2; E(N))$ , and  $N_2$  is  $\rho$ -dense in  $N$ .

(b) If  $N_2$  is  $\rho$ -dense in  $N$ , then since  $N \subseteq C_\rho(N_2; E(N))$  it follows that  $N/N_1 \subseteq C_\rho(N_2/N_1; E(N)/N_1)$ .

The usual definition of  $\rho$ -density (see [5] and [6]) states that a submodule  $N_0$  is  $\rho$ -dense in  $N$  if  $\rho(N/N_0) = N/N_0$ . Note that this

occurs if and only if  $N \subseteq C_\rho(N_0; M)$  for every extension  $M$  of  $N$ . Our definition has some of the usual properties, as shown in the above proposition, and in addition guarantees that  $N$  is always  $\rho$ -dense in  $C_\rho(N)$ . The next proposition shows that the two definitions are equivalent when  $\rho$  is a torsion preradical.

**PROPOSITION 1.4.** *The following conditions are equivalent for a preradical  $\rho$  of  ${}_R\mathbf{M}$ .*

- (a)  $\rho$  is a torsion preradical.
- (b) For all  $N \in {}_R\mathbf{M}$ , a submodule  $N_0$  is  $\rho$ -dense in

$$N \iff \rho(N/N_0) = N/N_0.$$

*Proof.* Recall that  $\rho$  is a torsion preradical if and only if for all modules  $N$  and submodules  $N_0$ ,  $N_0 \subseteq \rho(N)$  implies  $\rho(N_0) = N_0$ .

(a)  $\implies$  (b). If  $N_0$  is  $\rho$ -dense in  $N$ , then there exists an extension  $M$  of  $N$  with  $N \subseteq C_\rho(N_0; M)$ . This implies that  $N/N_0$  is isomorphic to a submodule of  $\rho(M/N_0)$ , and since  $\rho$  is assumed to be a torsion preradical, we must have  $\rho(N/N_0) = N/N_0$ . On the other hand, it is always true that  $\rho(N/N_0) = N/N_0$  implies that  $N_0$  is  $\rho$ -dense in  $N$ .

(b)  $\implies$  (a). If  $N_0$  is a submodule of  ${}_R N$  such that  $N_0 \subseteq \rho(N)$ , then the zero submodule is  $\rho$ -dense in  $N_0$ , since  $C_\rho(0; N) = \rho(N)$ . By assumption we must have  $\rho(N_0) = N_0$ , and this shows that  $\rho$  is a torsion preradical.

If  $\rho$  and  $\sigma$  are preradicals such that  $\rho(M) \subseteq \sigma(M)$ , for all  $M \in {}_R\mathbf{M}$ , we write  $\rho \leq \sigma$ . The smallest preradical of  ${}_R\mathbf{M}$ , the zero functor, is denoted by  $0$  and the largest preradical, the identity functor, is denoted by  $\infty$ . If  $M \in {}_R\mathbf{M}$  and  $\rho$  is a preradical with  $\rho(M) = M$ , then since  $\rho$  is a preradical we must have  $f(M) \subseteq \rho(N)$ , for any  $N \in {}_R\mathbf{M}$  and  $f \in \text{Hom}_R(M, N)$ . Letting  $\text{rad}^M(N) = \sum f(M)$ , where  $f$  runs through all elements of  $\text{Hom}_R(M, N)$ , we have  $\text{rad}^M \leq \rho$ . It can be verified that  $\text{rad}^M$  is an idempotent preradical, and is the smallest preradical for which  $M$  is torsion. Furthermore,  $\text{rad}^M(N) = N$  if and only if  $M$  generates  $N$  in the categorical sense. The module  $M$  is called cofaithful if it generates every injective module in  ${}_R\mathbf{M}$ . This occurs if and only if  ${}_R R$  can be embedded in a finite direct sum of copies of  $M$  (see [2]). There is also a smallest torsion preradical for which  $M$  is torsion. This can be shown by considering for each module  ${}_R N$  the intersection of all submodules which are the torsion submodule of  $N$  for some torsion preradical for which  $M$  is torsion. The formal definition and some properties follow. Using the notation of Fuchs [3] we denote by  $\bar{\Omega}(M)$  the set of all left ideals of  $R$  which

contain a finite intersection of left ideals of the form  $\text{Ann}(m) = \{r \in R: rm = 0\}$ , for some  $m \in M$ .

**DEFINITION 1.5.** Let  ${}_R M \in {}_R \underline{M}$ . The smallest preradical and smallest torsion preradical of  ${}_R \underline{M}$  for which  $M$  is torsion will be denoted by  $\text{rad}^M$  and  $\text{Rad}^M$ , respectively.

**PROPOSITION 1.6.** Let  ${}_R M \in {}_R \underline{M}$ .

- (a) For all  $N \in {}_R \underline{M}$ ,  $\text{Rad}^M(N) = N \cap \text{rad}^M(E(N))$ .
- (b) The left ideal  $A$  of  $R$  is  $\text{Rad}^M$ -dense in  $R \Leftrightarrow A \in \bar{\mathcal{Q}}(M)$ .
- (c) If  $N$  is a submodule or factor module of  $M$  then  $\text{Rad}^N \leq \text{Rad}^M$ .
- (d)  $\text{Rad}^M = \infty \Leftrightarrow M$  is cofaithful.

*Proof.* (a) Since  $\text{Rad}^M$  is a torsion preradical and  $\text{rad}^M \leq \text{Rad}^M$ , it follows that  $\text{Rad}^M(N) = N \cap \text{Rad}^M(E(N)) \supseteq N \cap \text{rad}^M(E(N))$ , for all  $N \in {}_R \underline{M}$ . On the other hand, setting  $\rho(N) = N \cap \text{rad}^M(E(N))$  for all  $N \in {}_R \underline{M}$  defines a torsion preradical of  ${}_R \underline{M}$ . That  $\rho$  is a preradical follows from the fact that if  $f \in \text{Hom}_R(N, Q)$  then  $f$  extends to  $g: E(N) \rightarrow E(Q)$ , with  $g(\text{rad}^M(E(N))) \subseteq \text{rad}^M(E(Q))$ , and consequently  $f(N \cap \text{rad}^M(E(N))) \subseteq Q \cap \text{rad}^M(E(Q))$ . Furthermore,  $\rho$  is a torsion preradical, since if  $N_0$  is a submodule of  $N$ , then  $E(N_0)$  is a direct summand of  $E(N)$ , and so  $\text{rad}^M(E(N_0)) = E(N_0) \cap \text{rad}^M(E(N))$ . Hence  $\rho(N_0) = N_0 \cap \text{rad}^M(E(N_0)) = N_0 \cap \text{rad}^M(E(N)) = N_0 \cap \rho(N)$ . Thus  $\rho$  is a torsion preradical with  $\rho(M) = M$  and  $\rho \leq \text{Rad}^M$ , and so we must have  $\rho = \text{Rad}^M$ .

(b) We will show that for any injective module  ${}_R Q$ ,  $x \in \text{rad}^M(Q)$  if and only if  $\text{Ann}(x) \in \bar{\mathcal{Q}}(M)$ . The result then follows from part (a). If  ${}_R Q$  is injective and  $x \in \text{rad}^M(Q)$ , then by the remarks preceding the proposition,  $x = \sum_{i=1}^n f_i(m_i)$ , for  $m_i \in M$  and  $f_i \in \text{Hom}_R(M, Q)$ . It is clear that  $\bigcap_{i=1}^n \text{Ann}(m_i) \subseteq \text{Ann}(x)$ , and  $\text{Ann}(x) \in \bar{\mathcal{Q}}(M)$ . On the other hand, if  $x \in Q$  and  $\text{Ann}(x) \supseteq A = \bigcap_{i=1}^n \text{Ann}(m_i)$ , for elements  $m_i \in M$ , then the homomorphism  $g: R/A \rightarrow R x$  with  $g(1) = x$  is well-defined. Since  $R/A$  can be embedded in the direct sum  $M^n$  of  $n$  copies of  $M$ , the homomorphism  $g$  can be extended by the injectivity of  $Q$  to  $f: M^n \rightarrow Q$ , with  $f(m_1, m_2, \dots, m_n) = x$ . For the components  $f_i$  of  $f$  we have  $x = \sum_{i=1}^n f_i(m_i)$ , and so  $x \in \text{rad}^M(Q)$ .

(c) If  $N$  is a submodule or factor module of  $M$ , then since  $\text{Rad}^M(M) = M$  and  $\text{Rad}^M$  is a torsion preradical we must have  $\text{Rad}^M(N) = N$ . It follows from the definition of  $\text{Rad}^N$  that  $\text{Rad}^N \leq \text{Rad}^M$ .

(d) If  $\text{Rad}^M = \infty$ , then for any injective module  $Q$ ,  $\text{rad}^M(Q) = \text{Rad}^M(Q) = Q$  and  $M$  generates  $Q$ . Conversely, if  $M$  is cofaithful,

then for any module  ${}_R N$ ,  $\text{Rad}^M(N) = N \cap \text{rad}^M(E(N)) = N$ .

The definitions in 1.5 can be dualized, and we give here only the constructions of preradicals  $\text{rad}_M$  and  $\text{Rad}_M$  associated with a module  ${}_R M$ . It can be shown that the preradical  $\text{rad}_M$  defined below is the largest preradical for which  $M$  is torsionfree, and is in fact a radical. We have  $\text{Rad}_M \leq \text{rad}_M$ , although Azumaya has shown [1, Prop. 7] that the preradicals coincide for any projective module.

DEFINITION 1.7. Let  ${}_R M \in {}_R \underline{M}$ . Let  $\text{rad}_M$  and  $\text{Rad}_M$  be the preradicals defined by  $\text{rad}_M(N) = \bigcap_{f \in \text{Hom}_R(N, M)} \ker(f)$  and  $\text{Rad}_M(N) = \text{Ann}(M) \cdot N$ , for all modules  $N \in {}_R \underline{M}$ .

The following lemma and its dual will be used in both of the following sections. Recall that a monomorphism  $f: A \rightarrow B$  in an abelian category  $\underline{A}$  is essential if for all  $C \in \underline{A}$  and  $g: B \rightarrow C$  in  $\underline{A}$ ,  $gf$  is a monomorphism implies that  $g$  is a monomorphism. Equivalently, the monomorphism  $f: A \rightarrow B$  is essential if and only if  $g \neq 0$  implies  $h \neq 0$  in every pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & C \\ \downarrow h & & \downarrow g \\ A & \xrightarrow{f} & B \end{array} .$$

LEMMA 1.8. *If*

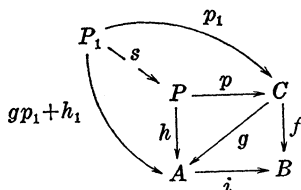
$$\begin{array}{ccc} P & \xrightarrow{p} & C \\ \downarrow h & \swarrow g & \downarrow f \\ A & \xrightarrow{i} & B \end{array}$$

*is a pullback diagram in an abelian category, with  $i: A \rightarrow B$  an essential monomorphism, then  $gp = h$  implies  $ig = f$ .*

*Proof.* Consider the pullback diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{p_1} & C \\ \downarrow h_1 & & \downarrow f \circ i \\ A & \xrightarrow{i} & B \end{array} .$$

Then  $(f-ig)p_1 = ih_1$  implies  $fp_1 = i(gp_1 + h_1)$ . Since  $P$  is a pullback, this induces a factorization  $s: P_1 \rightarrow P$ .



But then  $i(gp_1 + h_1) = ihs = igps = igp_1$ . This shows that  $ih_1 = 0$ , and then  $h_1 = 0$  since  $i$  is a monomorphism. But  $i: A \rightarrow B$  is an essential monomorphism and  $P_1$  is a pullback diagram, so  $h_1 = 0$  implies  $f - ig = 0$ , and thus  $f = ig$ .

**2.  $(\rho, \sigma)$ -injective modules.** A module  ${}_R Q$  is injective if (i) for each module  ${}_R N$ , (ii) each submodule  $N_0$  of  $N$ , and (iii) each homomorphism  $f: N_0 \rightarrow Q$ , there exists an extension of  $f$  to  $N$ . (By an extension of  $f$  to  $N$  we mean a homomorphism  $g: N \rightarrow Q$  such that  $gi = f$ , for the inclusion  $i: N_0 \rightarrow N$ .) This definition can be generalized by suitably restricting the class of modules in (i), the class of submodules in (ii), or the class of homomorphisms in (iii). Azumaya has studied the first of these in a recent paper [1], in which a module  ${}_R Q$  is called  $M$ -injective if each homomorphism  $f: M_0 \rightarrow Q$  from a submodule  $M_0$  of the fixed module  $M$  can be extended to  $M$ . The second of these possible generalizations has been studied extensively in connection with rings of quotients ([5] and [6]), where the class of submodules is restricted to those submodules which are dense with respect to a fixed torsion radical. Finally, a condition placing restrictions on the class of allowable homomorphisms in (iii) has been used by Fuchs [3, Lemma 2] to characterize quasi-injective modules. The following definition combines these approaches.

**DEFINITION 2.1.** Let  $\rho$  and  $\sigma$  be preradicals of  ${}_R \underline{M}$ . A module  ${}_R Q$  will be called  $(\rho, \sigma)$ -injective if each homomorphism  $f: N_0 \rightarrow Q$  such that (i)  $N_0$  is a  $\rho$ -dense submodule of  ${}_R N$  and (ii)  $\ker(f)$  is a  $\sigma$ -dense submodule of  $N$  can be extended to  $N$ .

It is immediate from the definition that for preradicals  $\rho_1 \geq \rho_2$  and  $\sigma_1 \geq \sigma_2$ , any  $(\rho_1, \sigma_1)$ -injective module is also  $(\rho_2, \sigma_2)$ -injective. The standard argument can be used to show that a direct product of modules is  $(\rho, \sigma)$ -injective if and only if each factor is  $(\rho, \sigma)$ -injective.

**DEFINITION 2.2.** Let  $\rho$  be a preradical of  ${}_R \underline{M}$ , and  $M \in {}_R \underline{M}$ .

A module  ${}_R Q$  will be called  $(\rho, M)$ -injective if each homomorphism  $f: M_0 \rightarrow Q$  such that  $M_0$  is a  $\rho$ -dense submodule of  $M$  can be extended to  $M$ .

**THEOREM 2.3.** *Let  $\rho$  and  $\sigma$  be preradicals of  ${}_R \underline{M}$ . A module  ${}_R Q$  is  $(\rho, \sigma)$ -injective  $\Leftrightarrow Q$  is  $(\rho, M)$ -injective for all  $M \in {}_R \underline{M}$  such that  $0$  is  $\sigma$ -dense in  $M$ .*

*Proof.*  $\Rightarrow$ ). If the zero submodule of  $M$  is  $\sigma$ -dense in  $M$ , then by Proposition 1.3 every submodule of  $M$  is  $\sigma$ -dense in  $M$ .

$\Leftarrow$ ). Assume that  $Q$  is  $(\rho, M)$ -injective for all  $M$  such that  $0$  is  $\sigma$ -dense in  $M$ , and that  $f: N_0 \rightarrow Q$  is a homomorphism with  $N_0$  a  $\rho$ -dense submodule of  $N$  and  $\ker(f)$  a  $\sigma$ -dense submodule of  $N$ . Then  $f$  induces a homomorphism  $f_1: N_0/\ker(f) \rightarrow Q$ . Since  $N_0$  is  $\rho$ -dense in  $N$  it follows from Proposition 1.3 that  $N_0/\ker(f)$  is  $\rho$ -dense in  $N/\ker(f)$ . Furthermore, since  $\ker(f)$  is  $\sigma$ -dense in  $N$  it follows that  $0$  is  $\sigma$ -dense in  $N/\ker(f)$ . Then, by assumption,  $f_1$  can be extended to  $g_1: N/\ker(f) \rightarrow Q$ , and it is clear that  $g_1 p$  gives the desired extension of  $f$ , where  $p$  is the projection  $p: N \rightarrow N/\ker(f)$ .

The next theorem extends Baer's condition for injective modules to  $(\rho, \sigma)$ -injective modules.

**THEOREM 2.4.** *Let  $\rho$  and  $\sigma$  be torsion preradicals of  ${}_R \underline{M}$ . A module  ${}_R Q$  is  $(\rho, \sigma)$ -injective  $\Leftrightarrow$  each homomorphism  $f: A \rightarrow Q$  such that  $A$  is a  $\rho$ -dense left ideal of  $R$  and  $\ker(f)$  is  $\sigma$ -dense in  $R$  can be extended to  $R$ .*

*Proof.* Our proof follows very closely the standard proof for injectivity. Given a  $\rho$ -dense submodule  $N_0$  of  ${}_R N$  and a homomorphism  $f: N_0 \rightarrow Q$  with  $\ker(f)$   $\sigma$ -dense in  $N$ , there exists a maximal extension  $f_1: N_1 \rightarrow Q$ , with  $N_0 \subseteq N_1 \subseteq N$ . If there exists  $x \in N$ ,  $x \notin N_1$ , then let  $A = \{r \in R: rx \in N_1\}$  and define  $g: A \rightarrow Q$  by  $g(r) = f_1(rx)$ . An extension of  $g$  to  $R$  gives rise to an extension of  $f_1$  to  $N_1 + Rx$ , a contradiction, which then shows that  $N_1 = N$ . Thus to complete the proof we must show that the homomorphism  $g: A \rightarrow Q$  defined above can be extended to  $R$ . If we assume that  $Q$  satisfies the condition of the theorem, then it is sufficient to show that  $A$  is  $\rho$ -dense in  $R$  and  $\ker(g)$  is  $\sigma$ -dense in  $R$ . By assumption  $N_0$  is  $\rho$ -dense in  $N$ , so  $N_1$  is also  $\rho$ -dense in  $N$ , and since  $\rho$  is a torsion preradical,  $N/N_1$  is  $\rho$ -torsion. From the definition of  $A$  it follows that  $R/A$  is isomorphic to a submodule of  $N/N_1$ , and again since  $\rho$  is a torsion preradical, it follows that  $R/A$  is  $\rho$ -torsion and  $A$  is  $\rho$ -dense in  $R$ .



From the definition of  $g$  it follows that  $\ker(g) = \{r \in R: rx \in \ker(f_1)\}$ , and so  $R/\ker(g)$  is isomorphic to a submodule of  $N/\ker(f_1)$ . But  $\ker(f_1) \supseteq \ker(f)$ , and by assumption  $\ker(f)$  is  $\sigma$ -dense in  $N$ . It follows as before that  $\ker(g)$  is  $\sigma$ -dense in  $R$ , using the assumption that  $\sigma$  is a torsion preradical.

The following theorem, with  $\sigma = \infty$ , extends the known results for  $\rho$ -injective modules to the case in which  $\rho$  is only a preradical, rather than a torsion radical.

**THEOREM 2.5.** *Let  $\rho$  and  $\sigma$  be preradicals of  ${}_R M$  and  $Q \in {}_R M$ . The following conditions (a) – (c) are equivalent and imply (d). If  $\rho$  is a torsion preradical, then all four conditions are equivalent.*

(a)  *$Q$  is a direct summand in each extension  ${}_R M \supseteq Q$  such that  $Q$  is a  $\rho$ -dense submodule of  $M$  and  $Q + M \cap \sigma(E(M)) = M$ .*

(b)  *$Q \supseteq C_\rho(Q) \cap \sigma(E(Q))$ .*

(c) *Each homomorphism  $f: N_0 \rightarrow Q$  such that  $N_0$  is a  $\rho$ -dense submodule of  ${}_R N$  and  $N_0 + N \cap \sigma(E(N)) = N$  can be extended to  $N$ .*

(d)  *$Q$  is  $(\rho, \sigma)$ -injective.*

*Proof.* (a)  $\Rightarrow$  (b). Let  $M = Q + C_\rho(Q) \cap \sigma(E(Q))$ . Then since  $M \subseteq C_\rho(Q)$  it follows that  $Q$  is a  $\rho$ -dense submodule of  $M$ . Furthermore, since  $Q \subseteq M \subseteq E(Q)$ , it follows that  $E(M) = E(Q)$ , and consequently  $M = Q + M \cap \sigma(E(M))$ . If  $Q$  satisfies condition (a) then it must be a direct summand of  $M$ , and since  $Q$  is an essential submodule of  $M$  this implies that  $Q = M$ , or equivalently, that  $Q \supseteq C_\rho(Q) \cap \sigma(E(Q))$ .

(b)  $\Rightarrow$  (c). Let  $f: N_0 \rightarrow Q$  be a homomorphism which satisfies the conditions of (c). Then  $f$  can be extended to  $g: E(N) \rightarrow E(Q)$ , and since  $N \subseteq N_0 + C_\rho(N_0: E(N)) \cap \sigma(E(N))$  it follows that

$$\begin{aligned} g(N) &\subseteq g(N_0) + g(C_\rho(N_0: E(N))) \cap g(\sigma(E(N))) \\ &\subseteq Q + C_\rho(Q) \cap \sigma(E(Q)) \subseteq Q \end{aligned}$$

and so  $g$  restricted to  $N$  gives the desired extension of  $f$  to  $N$ .

(c)  $\Rightarrow$  (a). This is immediate.

(c)  $\Rightarrow$  (d). This follows from Theorem 2.3, since if  $0$  is  $\sigma$ -dense in  $N$ , then  $N \cap \sigma(E(N)) = N$ , and  $N$  satisfies the conditions of (c) for every  $\rho$ -dense submodule.

(d)  $\Rightarrow$  (b). It is necessary to assume that  $\rho$  is a torsion preradical. In this case, since  $Q$  is  $\rho$ -dense in  $C_\rho(Q)$ , it follows that  $Q \cap \sigma(E(Q))$  is  $\rho$ -dense in  $C_\rho(Q) \cap \sigma(E(Q))$ . If  $Q$  is  $(\rho, \sigma)$ -injective, then the inclusion  $i: Q \cap \sigma(E(Q)) \rightarrow Q$  extends to  $f: C_\rho(Q) \cap \sigma(E(Q)) \rightarrow Q$ , since  $\ker(i) = 0$  is  $\sigma$ -dense in  $C_\rho(Q) \cap \sigma(E(Q))$ . In fact  $f$  extends to an

endomorphism  $g$  of  $E(Q)$ , and then  $g(\sigma(E(Q))) \subseteq \sigma(E(Q))$  implies that  $f(C_\rho(Q) \cap \sigma(E(Q))) \subseteq Q \cap \sigma(E(Q))$ . Thus  $f_i$  is the identity on  $Q \cap \sigma(E(Q))$ , and since  $Q \cap \sigma(E(Q))$  is essential in  $C_\rho(Q) \cap \sigma(E(Q))$ , this implies that  $Q \cap \sigma(E(Q)) = C_\rho(Q) \cap \sigma(E(Q))$ . Therefore  $Q \subseteq C_\rho(Q) \cap \sigma(E(Q))$ , and the proof is complete.

**COROLLARY 2.6.** *Let  $\rho$  and  $\sigma$  be preradicals of  ${}_R\mathbf{M}$  and let  $M, N \in {}_R\mathbf{M}$ , with  $N$  a submodule of  $M$ . If  $M$  is  $(\rho, \sigma)$ -injective and  $N \subseteq C_\rho(M) \cap \sigma(E(M))$ , then  $N$  is  $(\rho, \sigma)$ -injective. The converse is true if  $\rho$  is a torsion radical and  $N$  is an essential  $\rho$ -dense submodule of  $M$ .*

*Proof.* If  $N$  is a submodule of  $M$  then  $C_\rho(M) \subseteq C_\rho(N)$  and  $\sigma(E(M)) \subseteq \sigma(E(N))$ . Hence if

$$N \subseteq C_\rho(M) \cap \sigma(E(M)) \subseteq C_\rho(N) \cap \sigma(E(N)),$$

then  $N$  is  $(\rho, \sigma)$ -injective by Theorem 2.5. Conversely, if  $N$  is essential in  $M$ , then  $E(N) = E(M)$  and  $\sigma(E(N)) = \sigma(E(M))$ . If  $\rho$  is a radical, then since  $N$  is  $\rho$ -dense in  $M$  it follows that  $C_\rho(N) = C_\rho(M)$ , and then the result follows from Theorem 2.5.

These results are simplified considerably if  $\sigma$  is a torsion preradical. In this event, for any module  ${}_R N$  we have  $N \cap \sigma(E(N)) = \sigma(N)$  and  $C_\rho(N) \cap \sigma(E(N)) = \sigma(C_\rho(N))$ . The next corollary can be used to show the existence of a “ $(\rho, \sigma)$ -injective envelope” when  $\rho$  is a torsion radical.

**COROLLARY 2.7.** *Let  $\rho$  be a torsion radical of  ${}_R\mathbf{M}$  and let  $\sigma$  be a preradical of  ${}_R\mathbf{M}$ . Then for any module  ${}_R N$ ,  $N + C_\rho(N) \cap \sigma(E(N))$  is  $(\rho, \sigma)$ -injective and is contained in every  $(\rho, \sigma)$ -injective submodule of  $E(N)$  which contains  $N$ .*

*Proof.* Let  $Q = N + C_\rho(N) \cap \sigma(E(N))$ . Then  $N \subseteq Q \subseteq C_\rho(N)$ , and since  $\rho$  is a radical it follows that  $C_\rho(Q) = C_\rho(N)$ . Furthermore,  $E(Q) = E(N)$ , and so  $Q \subseteq C_\rho(Q) \cap \sigma(E(Q))$ . Theorem 2.5 implies that  $Q$  is  $(\rho, \sigma)$ -injective. If  $N \subseteq M \subseteq E(N)$ , with  $M$   $(\rho, \sigma)$ -injective, then by Theorem 2.5,  $M \subseteq C_\rho(M) \cap \sigma(E(M)) \subseteq C_\rho(N) \cap \sigma(E(N))$ , and so  $M \subseteq Q$ .

The next theorem gives a condition equivalent to  $(\rho, M)$ -injectivity. It generalizes Theorem 15 of [1] and a theorem of [8]. Its application in Corollary 2.9 shows the connection between  $(\rho, \sigma)$ -injectivity and  $M$ -injectivity.

**THEOREM 2.8.** *Let  $\rho$  be a torsion preradical of  ${}_R\mathbf{M}$ , and let  $M, Q \in {}_R\mathbf{M}$ . Then  $Q$  is  $(\rho, M)$ -injective  $\Leftrightarrow f(M) \subseteq Q$ , for all*

$$f \in \text{Hom}_R(M, C_\rho(Q)) .$$

*Proof.*  $\Rightarrow$ ). Let  $f: M \rightarrow C_\rho(Q)$  and consider the pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & M \\ \downarrow h & \nearrow g & \downarrow f \\ Q & \xrightarrow{i} & C_\rho(Q) \end{array} .$$

Since  $Q \rightarrow C_\rho(Q)$  is a monomorphism, so is  $P \rightarrow M$ . Furthermore, viewing  $P$  as a submodule of  $M$ ,  $M/P$  is isomorphic to a submodule of  $C_\rho(Q)/Q$ , and so  $P$  is  $\rho$ -dense in  $M$  since  $Q$  is  $\rho$ -dense in  $C_\rho(Q)$ . If we assume that  $Q$  is  $(\rho, M)$ -injective, then  $h: P \rightarrow Q$  extends to  $g: M \rightarrow Q$ . Lemma 1.8 implies that  $f = ig$ , and so  $f(M) \subseteq Q$ .

$\Leftarrow$ ). If  $g: M_0 \rightarrow Q$ , with  $M_0$  a  $\rho$ -dense submodule of  $M$ , then  $g$  extends to  $f: E(M) \rightarrow E(Q)$ , and  $f(M) \subseteq f(C_\rho(M_0: E(M))) \subseteq C_\rho(Q)$ . By assumption,  $f(M) \subseteq Q$ , and so  $f$  yields the required extension of  $g$  to  $M$ .

**COROLLARY 2.9.** *The following conditions are equivalent for any modules  ${}_RQ$  and  ${}_RM$ .*

- (a)  $Q$  is  $M$ -injective.
- (b)  $Q$  is  $(\infty, \text{rad}^M)$ -injective.
- (c)  $Q$  is  $(\infty, \text{Rad}^M)$ -injective.

*Proof.* Theorem 2.8 implies that  $Q$  is  $M$ -injective (equivalently,  $Q$  is  $(\infty, M)$ -injective) if and only if  $Q \supseteq \text{rad}^M(E(Q)) = \text{Rad}^M(E(Q))$ . The three conditions are then equivalent as a consequence of Theorem 2.5.

A module  ${}_RQ$  is quasi-injective if and only if it is  $Q$ -injective, so Theorem 2.3 and 2.4 imply Lemma 1 and 2 of Fuchs [3]. (A module  $N$  is  $\text{Rad}^2$ -torsion if and only if  $\bar{\mathcal{Q}}(N) \subseteq \bar{\mathcal{Q}}(Q)$ .) Theorem 2.5 is closely related to Theorem 11 of [1], which can be seen by taking  $\rho = \infty$  and  $\sigma = \text{rad}^M$ . Corollary 2.6 shows the existence of an “ $M$ -injective envelope” ([9]). If  $N$  is a submodule or factor module of  $M$ , then  $\text{Rad}^N \leq \text{Rad}^M$ , and every  $M$ -injective module is  $N$ -injective ([1, Proposition 10]). If  $M$  is cofaithful, then  $\text{Rad}^M = \infty$ , and every  $M$ -injective module is injective ([1, Theorem 14]). If  $A = \text{Ann}(M)$ , and  $A \in \bar{\mathcal{Q}}(M)$ , then  $M$  is a cofaithful  $R/A$ -module, and hence any  $R/A$ -module

$Q$  is  $M$ -injective if and only if it is injective as an  $R/A$ -module (compare [3, Theorem 1]).

3.  $(\underline{E}, \sigma)$ -projective objects. The results in §2 depend on the existence of injective envelopes. The dual of this condition does not hold in many categories of modules, so the earlier results cannot simply be dualized. We can, however, give results dual to those obtained when  $\rho$  is a torsion preradical and  $\sigma$  is an idempotent preradical.

Let  $\underline{A}$  be an abelian category and let  $\sigma$  be a radical of  $\underline{A}$ . These will remain fixed throughout, along with a nonempty class  $\underline{E}$  of epimorphisms of  $\underline{A}$  which satisfies the following two conditions: (i) if  $A \rightarrow B \in \underline{E}$  and

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & Q \end{array}$$

is a pushout diagram in  $\underline{A}$ , then  $C \rightarrow Q \in \underline{E}$  and (ii) if  $A \rightarrow B \in \underline{E}$  and

$$\begin{array}{ccc} Q & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is a pullback diagram in  $\underline{A}$ , then  $Q \rightarrow C \in \underline{E}$ . (Note that the dual of (i) and (ii) is satisfied by the class of monomorphisms  $A \rightarrow B$  such that  $A$  is  $\rho$ -dense in  $B$ , where  $\rho$  is a torsion preradical.)

DEFINITION 3.1. An object  $P \in \underline{A}$  is called  $(\underline{E}, \sigma)$ -projective if each diagram

$$\begin{array}{ccc} & P & \\ & \swarrow & \downarrow f \\ A & \longrightarrow & B \end{array}$$

such that  $A \rightarrow B \in \underline{E}$  and  $\ker(A \rightarrow B/f(P))$  is  $\sigma$ -torsionfree can be completed to a commutative diagram.

Here we use  $f(P)$  for the image of  $f$  and  $B/f(P)$  rather than cokernel ( $f$ ). Note that  $\ker(A \rightarrow B/f(P))$  is  $\sigma$ -torsionfree if

$$\ker(A \rightarrow B/f(P)) \cap \sigma(A) = 0.$$

An object  $P \in \underline{A}$  is called  $(\underline{E}, A)$ -projective, for  $A \in \underline{A}$ , if each diagram

$$\begin{array}{ccc}
 & P & \\
 \swarrow & & \downarrow \\
 A & \longrightarrow & B
 \end{array}$$

such that  $A \rightarrow B \in \underline{E}$  can be completed to a commutative diagram.

**THEOREM 3.2.** *The following conditions are equivalent for an object  $P \in \underline{A}$ .*

- (a)  $P$  is  $(\underline{E}, \sigma)$ -projective.
- (b)  $P$  is  $(\underline{E}, A)$ -projective for all  $A \in \underline{A}$  such that  $A$  is  $\sigma$ -torsion-free.
- (c) *There exists a lifting  $h: P \rightarrow A$  in all diagrams*

$$\begin{array}{ccc}
 & P & \\
 \swarrow h & & \downarrow f \\
 A & \xrightarrow{g} & B
 \end{array}$$

such that (i)  $A \rightarrow B \in \underline{E}$  and (ii)  $\ker(g) \cap \sigma(A) = 0$ .

- (d) *Each epimorphism  $p: C \rightarrow P$  in  $\underline{E}$  such that  $\ker(p) \cap \sigma(C) = 0$  has a splitting morphism  $i: P \rightarrow C$  with  $pi = 1_P$ .*

*Proof.* The implications (a)  $\Rightarrow$  (b) and (c)  $\Rightarrow$  (d) are obvious. We prove (b)  $\Rightarrow$  (c) and (d)  $\Rightarrow$  (a), to complete the argument.

(b)  $\Rightarrow$  (c). Given

$$\begin{array}{ccc}
 & P & \\
 & \downarrow f & \\
 A & \xrightarrow{g} & B
 \end{array}$$

such that  $A \rightarrow B \in \underline{E}$  and  $\ker(g) \cap \sigma(A) = 0$ , form the pushout diagram

$$\begin{array}{ccccc}
 & & P & & \\
 & & \downarrow f & & \\
 A & \xrightarrow{\quad} & B & & \\
 \downarrow & \nearrow & \downarrow p & & \\
 A/\sigma(A) & \longrightarrow & D & & .
 \end{array}$$

Then  $A \rightarrow B \in \underline{E}$  implies  $A/\sigma(A) \rightarrow D \in \underline{E}$ , and  $A/\sigma(A)$  is  $\sigma$ -torsionfree since  $\sigma$  is a radical, so  $pf$  lifts by assumption to  $P \rightarrow A/\sigma(A)$ . But  $\ker(g) \cap \sigma(A) = 0$  implies that

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/\sigma(A) & \longrightarrow & D \end{array}$$

is also a pullback diagram, and this gives the desired factorization  $P \rightarrow B = P \rightarrow A \rightarrow B$ .

(d)  $\Rightarrow$  (a). Given

$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ A & \xrightarrow{g} & B \end{array}$$

with  $A \rightarrow B \in \underline{E}$  and  $\sigma(\ker(A \rightarrow B/f(P))) = 0$ , consider the pullback

$$\begin{array}{ccc} C & \xrightarrow{p} & P \\ h \downarrow & \dashrightarrow & \downarrow f \\ A & \xrightarrow{g} & B \end{array} .$$

Then  $C \rightarrow P \in \underline{E}$ . Let  $D = \ker(p) \cap \sigma(C)$ . Now  $h(D) \subseteq h(\sigma(C)) \subseteq \sigma(A)$ , and  $gh(C) = fp(C) \subseteq f(P)$  implies  $h(C) \subseteq \ker(A \rightarrow B/f(P))$ . Hence  $h(D) \subseteq \sigma(\ker(A \rightarrow B/f(P))) = 0$ , and so because  $C$  is a pullback and  $h(D) = p(D) = 0$ , it follows that  $D = 0$ . Therefore  $p: C \rightarrow P$  satisfies the conditions of (d), and the splitting morphism  $P \rightarrow C$  induces the required lifting  $P \rightarrow C \rightarrow A$  of  $P \rightarrow B$ .

**COROLLARY 3.3.** *If  $P \in \underline{A}$  is  $(\underline{E}, \sigma)$ -projective and  $p: P \rightarrow A \in \underline{E}$ , with  $\ker(p) \subseteq \sigma(P)$ , then  $A$  is  $(\underline{E}, \sigma)$ -projective.*

*Proof.* We will use Theorem 3.2 (b). If  $B \rightarrow C \in \underline{E}$  and  $\sigma(B) = 0$ , then for any morphism  $A \rightarrow C \in \underline{A}$ ,  $P \rightarrow A \rightarrow C$  lifts to  $g: P \rightarrow B$ , since  $P$  is  $(\underline{E}, \sigma)$ -projective. But  $\sigma(B) = 0$  implies  $g(\sigma(P)) = 0$ , so  $\ker(g) \supseteq \sigma(P) \supseteq \ker(p)$ . Therefore  $g$  factors through  $A$ , and since  $p$  is an epimorphism this induces the desired lifting of  $A \rightarrow C$  to  $A \rightarrow B$ .

**THEOREM 3.4.** *Let  $p: C \rightarrow P \in \underline{E}$  and  $p$  be a coessential epimorphism. If  $P$  is  $(\underline{E}, A)$ -projective, for  $A \in \underline{A}$ , then  $\ker(f) \supseteq \ker(p)$  for all morphisms  $f: C \rightarrow A$  in  $\underline{A}$ .*

*Proof.* Consider the pushout diagram

$$\begin{array}{ccc} C & \xrightarrow{p} & P \\ f \downarrow & \nearrow h & \downarrow g \\ A & \longrightarrow & D \end{array} .$$

Then  $C \rightarrow P \in \underline{E}$  implies  $A \rightarrow D \in \underline{E}$ . Since  $P$  is  $(\underline{E}, A)$ -projective,  $g$  lifts to  $h: P \rightarrow A$ . But then the dual of Lemma 1.8 implies that  $f = hp$  and so  $\ker(f) = \ker(hp) \supseteq \ker(p)$ .

By an  $\underline{E}$ -projective cover  $p: C \rightarrow P$  of an object  $P \in \underline{A}$  we mean a coessential epimorphism  $p \in \underline{E}$  and an object  $C \in \underline{A}$  such that  $C$  is  $(\underline{E}, A)$ -projective for all  $A \in \underline{A}$ .

**THEOREM 3.5.** *If  $P \in \underline{A}$  has an  $\underline{E}$ -projective cover  $p: C \rightarrow P$ , then the following conditions are equivalent.*

- (a)  $P$  is  $(\underline{E}, \sigma)$ -projective.
- (b)  $P$  is  $(\underline{E}, C/\sigma(C))$ -projective.
- (c)  $\ker(p) \subseteq \sigma(C)$ .

*Proof.* (a)  $\Rightarrow$  (b). This follows from Theorem 3.2 (b) and the fact that  $C/\sigma(C)$  is  $\sigma$ -torsionfree.

(b)  $\Rightarrow$  (c). Apply Theorem 3.4 to the projection  $C \rightarrow C/\sigma(C)$ .

(c)  $\Rightarrow$  (a). This follows from Corollary 3.3 since  $C$  is certainly  $(\underline{E}, \sigma)$ -projective.

We now assume that  $\underline{A} = {}_R\underline{M}$  is a category of modules. Let  $\underline{E}$  be the class of all epimorphisms of  ${}_R\underline{M}$ . If  $\sigma$  is a radical of  ${}_R\underline{M}$  and  ${}_R P$  is  $(\underline{E}, \sigma)$ -projective, we will simply say that  $P$  is  $\sigma$ -projective. If  $P$  is  $(\underline{E}, M)$ -projective, for  $M \in {}_R\underline{M}$ , we say that  $P$  is  $M$ -projective. Our final corollary to these results is in essence Theorem 8 of Azumaya [1]. From this it follows immediately that if  ${}_R P$  has a projective cover and is  $M$ -projective for a faithful module  ${}_R M$ , then  $\text{Rad}_M = 0$  and  $P$  is projective ([1, Theorem 9]). In general, our results on  $M$ -projective modules are not as good as those of Azumaya, since our characterization of  $M$ -projective modules holds only for those modules with projective covers.

**COROLLARY 3.6.** *Let  $P, M \in {}_R\underline{M}$ . If  $P$  has a projective cover, then the following conditions are equivalent.*

- (a)  $P$  is  $M$ -projective.
- (b)  $P$  is  $\text{rad}_M$ -projective.
- (c)  $P$  is  $\text{Rad}_M$ -projective.

*Proof.* Let  $p: C \rightarrow P$  be the projective cover of  $P$ .

(a)  $\Rightarrow$  (b). By Theorem 3.4, if  $P$  is  $M$ -projective then  $\text{rad}_M(C) \supseteq \ker(p)$ , and therefore by Theorem 3.5 (c) it follows that  $P$  is  $\text{rad}_M$ -projective.

(b)  $\Rightarrow$  (c). This is immediate from Theorem 3.5, since  $C$  is projective and therefore  $\text{rad}_M(C) = \text{Rad}_M(C)$ .

(c)  $\Rightarrow$  (a). Because  $M$  is  $\text{Rad}_M$ -torsionfree, this follows from Theorem 3.2 (b).

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