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ON ARITHMETIC PROPERTIES OF THE TAYLOR SERIES OF RATIONAL FUNCTIONS. II

DAVID GEOFFREY CANTOR

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ON ARITHMETIC PROPERTIES OF THE TAYLOR SERIES OF RATIONAL FUNCTIONS, II

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Suppose a_n , b_n , and $c_n = a_n b_n$ are sequences of algebraic integers and that all b_n are nonzero. It is easy to verify that if both $a(z) = \sum_{n=0}^{\infty} a_n z^n$ and $b(z) = \sum_{n=0}^{\infty} b_n z^n$ are rational functions, then so is $c(z) = \sum_{n=0}^{\infty} c_n z^n$. We are interested in studying the conjecture that if b(z) and c(z) are rational functions, then so is a(z). We shall prove this in the case that b(z) has no more than three distinct singularities.

Let k be an algebraic number field; denote by M_k the set of valuations of k, normalized so as to satisfy the Artin product-formula. We assume, whenever convenient, that each valuation in M_k has been extended in some fashion to Ω , the algebraic closure of k. Let S be a finite subset of M_k containing all Archimedean valuations. We say that $\alpha \in k$ is an S-integer if $|\alpha|_v \leq 1$ for all $v \in M_k - S$ and that α is an S-unit if α and $1/\alpha$ are both S-integers. Let a_n be a sequence of Sintegers of k. Suppose there exist rational functions $b(z) = \sum_{n=0}^{\infty} b_n z^n$ and $c(z) = \sum_{n=0}^{\infty} c_n z^n$ whose coefficients lie in an extension field K (possibly transcendental) of k; suppose that none of the b_n are 0 and that $a_n = c_n/b_n$ for $n \geq 0$. In [1], I showed that if b(z) has only one singularity (possibly a pole of high multiplicity) then $a(z) = \sum_{n=0}^{\infty} a_n z^n$ is a rational function. In [6] G. Pathiaux extended this result by showing that, under the additional assumption that K is algebraic, if b(z) has at most two distinct singularities, then a(z) is rational.

Here we shall study various extensions of these results. In particular we shall show that if b(z) has at most three distinct singularities, then a(z) is rational.

We note that since b(z) and c(z) are rational functions, we may write b_n and c_n as exponential polynomials:

(1)
$$b_n = \sum_{i=1}^r \lambda_i(n) \theta_i^n$$

(2)
$$c_n = \sum_{i=1}^s \mu_i(n) \varphi_i^n$$

for all sufficiently large n. Here the $\lambda_i(n)$ and $\mu_i(n)$ are polynomials in n. By appropriately enlarging K, if necessary, we may assume that the θ_i , the φ_i , and the coefficients of the polynomials $\lambda_i(n)$ and $\mu_i(n)$ all lie in K. By omitting a finite number of terms from each of the sequences a_n , b_n , c_n we may assume that (1) and (2) hold for all $n \geq 0$. The purpose of the first lemma is to show that we may assume that K is algebraic over h.

LEMMA 1. Suppose a_n , b_n , c_n are sequences as above. There exist sequences \overline{b}_n , \overline{c}_n lying in a finite algebraic extension of k with $\overline{b}(z) = \sum_{n=0}^{\infty} \overline{b}_n z^n$ and $\overline{c}(z) = \sum_{n=0}^{\infty} c_n z^n$ rational functions such that $a_n \overline{b}_n = \overline{c}_n$ for all integral $n \ge 0$ and such that only finitely many \overline{b}_n are 0.

Proof. As above we may write $b_n = \sum_{i=1}^r \lambda_i(n)\theta_i^n$ and $c_n = \sum_{i=1}^s \mu_i(n)\varphi_i^n$.

If all the coefficients of the λ_i and the μ_i , and the θ_i and φ_i are in k, then the Lemma is true with the $\overline{b}_n = b_n$ and $\overline{c}_n = c_n$. We henceforth assume this not the case. Let R be the ring generated by adjoining the θ_i , the φ_i , the ratios θ_i/θ_i , and the coefficients of the λ_i and the μ_i to k. By the assumption above the transcendence degree t of R/k is ≥ 1 . We are going to construct a homomorphism τ of R into a finite algebraic extension of k such that τ , when restricted to k, will be the identity. If $\tau \alpha$ is abbreviated $\bar{\alpha}$ then $\bar{b}_n = \sum_{i=1}^r \bar{\lambda}_i(n) \bar{\theta}_i^n$ and thus $\sum \overline{b}_n z^n$ is rational; similarly $\sum \overline{c}_n z^n$ is rational and since $a_n b_n = c_n$ clearly $a_n \overline{b}_n = \overline{c}_n$. The remainder of this proof is devoted to constructing such a homomorphism τ for which only finitely many \bar{b}_n are zero. By the Noether normalization lemma [3], there exists a transcendence basis $x = (x_1, x_2, \dots, x_t)$ for R/k such that each element of R is integral over k[x]. Since R/k[x] is algebraic and finitely generated, its degree d is finite. Each element α in R satisfies a polynomial equation $f(\alpha) = 0$, where

$$f(Y) = \sum_{i=0}^{r} p_i(x) Y^{e-i}$$

is a polynomial with coefficients $p_i(x)$ in k[x], of degree $e \leq d$, and monic $(p_0(x) \equiv 1)$. Any homomorphism τ of k[x] into k, which is the identity on k, has the form $p(x) \rightarrow p(u)$ where $u = (u_1, u_2, \dots, u_t)$ is a t-tuple of elements of k and p(x) is in k[x]. Such a homomorphism τ can be extended to a homomorphism of R into Ω , the algebraic closure of k [3]. The image $\bar{\alpha}$ of α will satisfy the monic polynomial $\sum_{i=0}^{e} p_i(u) Y^{e-i}$ and hence have degree $\leq e$ over k. Since $e \leq d$, every element in τR will have degree $\leq d$ over k and hence τR will be contained in a finite algebraic extension of k. Moreover if $p_{e}(u) \neq 0$, then $\bar{\alpha} \neq 0$. Denote by $\Phi_k(h)$ the degree of the field generated by the primitive h^{th} roots of unity over k. It is easy to verify that $\Phi_k(h) \geq \Phi_0(h)/[k:Q]$ where Q is the field of rational numbers and $\Phi_{Q}(h)$ is, of course, Euler's phi-Since $\Phi_o(h) \to \infty$ as $h \to \infty$, so does $\Phi_k(h)$. Let h be the function. largest integer for which $\Phi_k(h) \leq d$. Let *m* be the least common multiple of all of the orders of all of the roots of unity which can be written in the form θ_i/θ_i . We can write

$$b_{mn+s} = \sum_{i=1}^{q} \eta_{is}(n) \sigma_{i}^{n}$$

where the σ_i are the distinct m^{th} powers of the θ_i , and the $\eta_{is}(n)$ are polynomials, not all 0 (for each value of s). Let α be the product of all the nonzero coefficients of the $\eta_{is}(n)$ and the elements $(\sigma_i/\sigma_j)^{k!} - 1$ for $i \neq j$ (the latter quantities are not 0 since the ratios σ_i/σ_i cannot be roots of unity). Now let $u = (u_1, u_2, \dots, u_t)$ be elements of h for which $p_{\epsilon}(u) \neq 0$. Then under the homomorphism τ , defined above, $\bar{\alpha} = \tau \alpha$ will be nonzero, and $\bar{\eta}_{is}(n)$ (the polynomial obtained by applying τ to each coefficient of the polynomial $\eta_{is}(n)$ will be the zero-polynomial if and only if $\eta_{is}(n)$ is the zero-polynomial. None of the ratios $\bar{\sigma}_i/\bar{\sigma}_i$, with $i \neq j$, are roots of unity, for since $(\bar{\sigma}_i/\bar{\sigma}_j)^{h_1} \neq 1$, if $\bar{\sigma}_i/\bar{\sigma}_j$ were a root of unity, it would have to have order >h and hence degree >dover k; but the latter is not the case. If any of the m sequences \overline{b}_{mn+s} had infinitely many zeros then either all of the polynomials $\overline{\eta}_{is}(n)$ would be zero or by a theorem of Mähler [4] and Lech [5] the zeros would be periodic, and two of the $\bar{\sigma}_i$ would have ratio a root of unity. Thus the sequence b_n has only finitely many zeros.

LEMMA 2. Suppose a_n is a sequence of S-integers of k, that $a_n = c_n/b_n$ where $b_n = \sum \lambda_i(n)\theta_i^n$ is never 0 and $c_n = \sum \mu_i(n)\varphi_i^n$; suppose the θ_i, φ_i and the coefficients of the $\lambda_i(n)$ and the $\mu_i(n)$ are integers of k. Suppose there exists a valuation $v_0 \in S$ such that $|\theta_1|_{v_0} > |\theta_i|_{v_0}$ for $i \ge 2$. Then $\sum_{n=0}^{\infty} a_n z^n$ is rational.

Proof. Elementary estimates show there exist M > 0 and R > 0 such that $|b_n|_v$ and $|c_n|_v$ are $\leq MR^n$ for all $v \in S$ and $n \geq 0$, and that $|b_n|_v \leq 1$ for all $v \notin S$ and $n \geq 0$. Since $\prod_{v \in S} |b_n|_v \geq 1$, if $w \in S$, then

$$\left|rac{1}{b_n}
ight|_w \leq \prod_{v \in S top v
eq w} |b_n|_v \leq M^{s-1} R^{(s-1)n}$$

where s is the cardinality of S. Then $|a_n|_w = |c_n/b_n|_w \leq M^s R^{sn}$. It follows that $\sum_{n=0}^{\infty} a_n z^n$ has positive radius of convergence in k_w , the completion of k under the valuation w. Let \tilde{k}_w be the algebraic closure of k_w and assume that w has been extended to \tilde{k}_w . Let R_w be the radius convergence of $a(z) = \sum_{n=0}^{\infty} a_n z^n$ in k_w . Then a(z) is analytic in \tilde{k}_w for $|z|_w < R_w$. Now

$$\lambda_1(n) heta_1^na_n=c_n-\sum\limits_{i=2}^r\lambda_i(n) heta_i^na_n$$

or

$$\sum_{n=0}^{\infty}\lambda_1(n)a_nz^n\,=\,c\Bigl(rac{oldsymbol{z}}{oldsymbol{ heta}}\Bigr)\,-\,\sum_{i=2}^r\lambda_i(n)\Bigl(rac{oldsymbol{ heta}_i}{oldsymbol{ heta}_1}\Bigr)^na_nz^n$$
 .

In the field \tilde{k}_{v_0} , the algebraic closure of k_{v_0} , the last equation expresses

$$\begin{split} &\sum_{n=0}^{\infty}\lambda_1(n)a_nz^n \text{ as a rational function plus a sum of functions each} \\ & \text{meromorphic for } |z|_{v_0} \leq \delta R_{v_0} \text{ where } \delta = \min_{i\geq 2} |\theta_1/\theta_i|_{v_0} \text{ is } > 1. \\ & \text{Thus by} \\ & \text{analytic extension } \sum_{n=0}^{\infty}\lambda_1(n)a_nz^n \text{ is meromorphic for } |z|_{v_0} < \delta R_{v_0}. \\ & \text{Repeated applications of the above transformation show that} \\ & \sum_{n=0}^{\infty}\lambda_1(n)^ja_nz_u \text{ is meromorphic for } |z|_{v_0} < \delta^j R_{v_0}. \\ & \text{Elementary estimates} \\ & \text{show that } \sum_{n=0}^{\infty}\lambda_1(n)^ja_nz^n \text{ has radius of convergence } R_v \text{ for all } v \in S. \\ & \text{Choosing } j \text{ so large that } \delta^j \prod_{v \in S} R_v \text{ is } > 1, \text{ we find, by a theorem of } \\ & \text{Dwork [2], that } \sum_{n=0}^{\infty}\lambda_1(n)^ja_nz^n \text{ is a rational function. By [1] so is } \\ & \sum_{n=0}^{\infty}a_nz^n. \end{split}$$

LEMMA 3. Suppose a_n is a sequence of S-integers of k, that $a_n = c_n/b_n$ where $b_n = \sum \lambda_i(n)\theta_i^n$ is never zero and $c_n = \sum \mu_i(n)\varphi_i^n$, suppose the θ_i , φ_i and the nonzero coefficients of the $\lambda_i(n)$ and the $\mu_i(n)$ are S-units of h. Suppose there exists a valuation v_0 of h such that $|\theta_1|_{v_0} < |\theta_i|_{v_0}$ for $i \ge 2$. Then $\sum_{n=0}^{\infty} a_n z^n$ is rational.

Extend the definition of c_n and b_n to negative n by their Poof. formulas. If infinitely many such b_n were zero, then by a theorem of Lech [4] and Mähler [5], b_n would be zero for all n in a doubly infinite arithmetic progression, contradicting the hypotheses. Extend the definition of a_n to negative n by putting $a_n = c_n/b_n$ if $b_n \neq 0$ and otherwise put $a_n = 0$. Now let v be any valuation of M_k not in S. Then v is an extension of a p-adic valuation $| |_{v}$ of Q. There exists an integer f such that if $\alpha \in k$ and $|\alpha|_v = 1$ then $|\alpha^{p^f} - 1|_v < 1$. Letting m be an integer of the form $p^{h}(p^{f}-1)$, where h is large, we find that $|\alpha^m - 1|_v$ can be made very small. In particular we can choose m so large that if $b_n \neq 0$ then $|b_{n+m}|_v = |b_n|_v$ and that $|c_{n+m} - c_n|_v < |b_n|_v$ $|b_n|_v$. We can choose m so large that $m + n \ge 0$ and then $|c_{m+n}|_v \le 1$. Thus $|a_n|_v \leq 1$. Restating all this, we have shown that there exists n_0 such that if $n \leq n_0$ then $b_n \neq 0$ and if $v \in S$ then $|a_n|_v \leq 1$. We apply Lemma 2 to the sequences $a'_n = a_{n_0-n}$, $b'_n = b_{n_0-n}$ and $c'_n = c_{n_0-n}$, to conclude that $\sum_{n=0}^{\infty} a'_n z^n$ is rational. It follows that a_n can be written in the form

$$a_n = \sum_{i=1}^t \gamma_i(n) \sigma_i^n$$

for $n \leq n_0$. Then the exponential polynomial

$$\sum_{i=1}^{s} \mu_i(n) arphi_i^n - \sum_{i=1}^{r} \lambda_i heta_i^n \sum_{i=1}^{t} \eta_i(n) \sigma_i^n$$

is 0 for $n \leq n_0$. By the theorem of Mähler [5] and Lech [4], it is identically 0. Thus $a_n = \sum_{i=1}^t \gamma_i(n) \sigma_i^n$ for $n \geq 0$ and $a(z) = \sum_{n=0}^\infty a_n z^n$ is a rational function.

We now come to the result mentioned at the beginning of this

paper.

THEOREM 4. Suppose a_n is a sequence of S-integers of k and that b_n and c_n are sequences of elements of an extension field K of k such that $\sum_{n=0}^{\infty} b_n z^n$ and $\sum_{n=0}^{\infty} c_n z^n$ are rational functions and b_n is never zero. If $a_n = c_n/b_n$ and the rational function $\sum_{n=0}^{\infty} b_n z^n$ has at most 3 distinct singularities then $\sum_{n=0}^{\infty} a_n z^n$ is rational.

Proof. By Lemma 1, we may assume K is algebraic over k and that $b_n = \sum_{i=1}^r \lambda_i(n) \theta_i^n$ and that $c_n = \sum_{i=1}^s \mu_i(n) \varphi_i^n$ where the θ_i, φ_i and all coefficients of the λ_i and μ_i are algebraic over k. By replacing k by a larger field and S by the set of extensions of the valuations in S to this new field, we may assume that the above quantities are, in fact, in k. By increasing S appropriately, we may assume that those of the above quantities which are not zero are S-units. Now if r = 1, the theorem follows immediately from [1]. If r = 2 then either θ_1/θ_2 is a root of unity, in which case the theorem follows from the case r=1 or there is a valuation v such that $|\theta_1|_v > |\theta_2|_v$, and the theorem follows from Lemma 2. If r = 3 then either $|\theta_1|_v = |\theta_2|_v = |\theta_3|_v$, for all $v \in S$ and θ_1/θ_2 and θ_1/θ_3 are roots of unity, so the theorem follows from the case r = 1, or there is a valuation $v_0 \in S$ for which not all of the three values are equal. In the latter case we may assume that $|\theta_1|_{v_0} \leq |\theta_2|_{v_0} \leq |\theta_3|_{v_0}$ and $|\theta_1|_{v_0} < |\theta_3|_{v_0}$. If $|\theta_2|_{v_0} = |\theta_3|_{v_0}$ then the theorem follows from Lemma 3, and otherwise from Lemma 2.

It is worth noting that the method of the theorem cannot be extended to the case where b(z) has 4 singularities. In fact, consider the case where k is the field Q(i) where $i = \sqrt{-1}$ and $\theta_1 = (1 + 2i) \times$ $(1 + 4i), \theta_2 = (1 + 2i)(1 - 4i), \theta_3 = (1 - 2i)(1 + 4i), \theta_4 = (1 - 2i)(1 - 4i)$. The ideals generated by (1 + 2i), (1 - 2i), (1 + 4i), (1 - 4i) are prime and give rise to 4 valuations of Q(i). At each of these valuations, two of the θ_j take one value and two another. For example at the valuation corresponding to the prime ideal generated by $1 - 2i, \theta_1$ and θ_2 both have value 1, while θ_3 and θ_4 both have the same value which is less than 1. All 4 θ_j take the same value at all other valuations. Thus the hypotheses of Lemma 2 or Lemma 3 cannot be met.

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