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The purpose of this paper is to characterize planar dynamical systems satisfying certain stability criterion. These flows are called dynamical systems of characteristic 0. Basically the set S of critical points of such a flow is shown to be in one of three categories: $S=\emptyset$; S consists of at most two Poincaré centers; or $S=R^2$.

1. Introduction. In §2 we give the basic concepts used throughout the paper. In §3 we give examples of flows of characteristic 0^+ and 0^- that are not of characteristic 0. We also give examples of flows of characteristic 0 which are not of characteristic 0^+, 0^-, or $0^\pm$. Further, by Examples 2 and 3 we show that the set S of critical points in Theorem 4.8 may actually consist of one or two local Poincaré centers. In §4 we give necessary and sufficient conditions for a flow (R^2, π) to have characteristic 0.

2. Definitions, notations, and basic theorems. We shall denote the real numbers, nonnegative real numbers, nonpositive real numbers, and Euclidean plane by R, R^+, R^-, and R^2, respectively. We shall use R^{2*} to designate the one point compactification of R^2.

A pair (X, π) consisting of a topological space X and a continuous mapping $\pi: X \times R \to X$ from the product space $X \times R$ into X is called a *dynamical system* or (continuous) *flow* whenever the following conditions are satisfied.

1. Identity axiom: $\pi(x, 0) = x$ for each $x \in X$.
2. Homomorphism axiom: $\pi(\pi(x, t), s) = \pi(x, t+s)$ for each $x \in X$ and $t, s \in R$.
3. Continuity axiom: π is continuous on $X \times R$.

In this paper X will always be Hausdorff.

We shall let $\pi(x, t) = xt$ for brevity. For each $x \in X$, $C(x) = xR$, $C^+(x) = xR^+$, and $C^-(x) = xR^-$ are called the *trajectory* (or *orbit*), *positive semi-trajectory*, and *negative semi-trajectory* through x, respectively. A point $x \in X$ is called a *critical* or *rest* point if $xR = x$. If x is not critical and $xt = x$ for some $t > 0$, then x is called *periodic*. For $M \subset X$, M is said to be *invariant* if $C(M) = M$ and *positively* (*negatively*) *invariant* if $C^+(M) = M$ $(C^-(M) = M)$.

We shall denote the boundary, interior, and closure of a set $M \subset X$ by ∂M, M^0, and $\bar{M}$, respectively. For any simple closed curve C in R^2 we shall denote the bounded and unbounded components of $R^2 - C$ by int C and ext C, respectively. We shall let $\eta(x)$ and $\eta(M)$ denote the neighborhood filters of $x \in X$ and $M \subset X$, respectively.

The sets $\overline{C(x)}$, $\overline{C^+(x)}$, and $\overline{C^-(x)}$ shall be denoted by $K(x)$, $K^+(x)$, and $K^-(x)$, respectively. The *positive* (*negative*) *limit set* of $x \in X$ is $L^+(x) = \bigcap_{t \in R} K^+(xt) (L^-(x) = \bigcap_{t \in R} K^-(xt))$. The *limit set* of $x \in X$ is $L(x) = L^+(x) \cup L^-(x)$.

A set $M \subset X$ is called *positively* (*orbitally*) *stable* if for every $U \in \eta(M)$ there exists a $V \in \eta(M)$ such that $V = C^+(V) \subset U$. *Negative* and *bilateral stability* are defined by replacing $C^+(V)$ above by $C^-(V)$ and $C(V)$, respectively. One can easily verify that a set M is bilaterally stable if and only if M is both positively and negatively stable. When we write *stable* we shall mean positively stable.

For each $x \in X$, the *positive* (*negative*) *prolongation* of x is given by

$$D^+(x) = \bigcap_{M \in \eta(x)} \overline{C^+(M)} \qquad (D^-(x) = \bigcap_{M \in \eta(x)} \overline{C^-(M)}) .$$

The *prolongation* of x is $D(x) = D^+(x) \cup D^-(x)$. The *positive* (*negative*) *prolongational limit set* of x is given by

$$J^+(x) = \bigcap_{t \in R} D^+(xt) \qquad (J^-(x) = \bigcap_{t \in R} D^-(xt)) .$$

The *prolongational limit set* is $J(x) = J^+(x) \cup J^-(x)$.

The following theorem which we shall use several times in this paper is due to Ura (see [6] and [11]).

THEOREM 2.1. *Let X be locally compact and ∂M be compact. Then M is stable* (negatively stable) *if and only if $D^+(M) = M$ $(D^-(M) = M.)$ Furthermore, M is bilaterally stable if and only if $D(M) = M$.*

A flow (X, π) is called *parallelizable* if it is isomorphic to a parallel flow; that is, if there is a flow $(Y \times R, \pi')$ such that $(y, t)s = (y, t + s)$ for each $y \in Y$ and $t, s \in R$ and a homeomorphism $f: X \to Y \times R$ such that $f(xt) = f(x)t$ for each $x \in X$ and $t \in R$. We shall use the following characterization of a parallelizable flow. For a proof see [3] and [4].

THEOREM 2.2. *Let X be a locally compact separable metric space. A flow (X, π) is parallelizable if and only if for each $x \in X$, $D^+(x) = C^+(x)$ $(D^-(x) = C^-(x))$ and there are no rest points or periodic orbits.*

A flow (X, π) is said to have *characteristic* $0^+(0^-)$ if $D^+(x) = K^+(x)(D^-(x) = K^-(x))$ for each $x \in X$. A flow having both characteristic 0^+ and 0^- is said to have *characteristic* $0^\pm$. A flow (X, π) is said to have *characteristic* 0 if $D(x) = K(x)$ for each $x \in X$. The flow (X, π) has characteristic $0^+(0^-)$ if and only if $J^+(x) = L^+(x)$ $(J^-(x) = L^-(x))$

for each $x \varepsilon X$. The corresponding statement does not hold for flows of characteristic 0 ($J(x_0) \neq L(x_0)$ in Example 2).

The basic properties of dynamical systems used in succeeding sections are contained in [5], [6], and [7].

3. Flows of characteristic 0 in R^2. The characteristic 0^+ and 0^- concepts were introduced by Ahmad in [1]. He classified such systems on the plane in terms of their critical points. In [8] necessary and sufficient conditions are given for a flow (R^2, π) to have characteristic 0^+ or 0^- in terms of the set S of critical points. Ahmad showed that a flow (R^2, π) has characteristic $0^\pm$ if and only if $S = \varnothing$ and (R^2, π) is parallelizable, $S = R^2$, or $S = \{s_0\}$ is a global Poincaré center (that is, all trajectories in $R^2 - \{s_0\}$ are periodic orbits surrounding s_0).

It seems natural to ask whether there is a connection between flows of characteristic 0 and flows of characteristic 0^+, 0^-, or $0^\pm$. Since $D^+(x) = K^+(x)$ and $D^-(x) = K^-(x)$ for each $x \in R^2$ implies $D(x) = K(x)$, any flow of characteristic $0^\pm$ is a flow of characteristic 0. A flow which has characteristic 0^+ (0^-) but not characteristic 0 is given below in Example 1. Examples 2 and 3 consist of flows of characteristic 0 that are not of characteristic 0^+, 0^-, or $0^\pm$.

EXAMPLE 1. The system of differential equations

$$\dot{x} = -x$$
$$\dot{y} = -y$$

defines a flow of characteristic 0^+ in which the origin is a proper node. Note, however, that $D((0, 0)) = R^2 \neq \{(0, 0)\} = K((0, 0))$ so that the flow does not have characteristic 0.

Similarly, the flow defined by $\dot{x} = x$ and $\dot{y} = y$ is of characteristic 0^- but not of characteristic 0.

EXAMPLE 2. Let a flow be defined by the system

$$\begin{aligned} \dot{r} &= -r^2 \sin\theta \\ \dot{\theta} &= 1 \end{aligned} \tag{1}$$

for $r \geqq 0$. Figure 1 illustrates the trajectories of the flow.

This flow is of characteristic 0 but not characteristic 0^+, 0^-, or $0^\pm$. For let x_0 be a point on the parabolic boundary of the region consisting of the pole and the periodic orbits surrounding the pole. Then $D(x_0) = D^+(x_0) = D^-(x_0) = C(x_0) = K(x_0)$ implies that $D^+(x_0) \neq K^+(x_0)$ and $D^-(x_0) \neq K^-(x_0)$.

EXAMPLE 3. The flow defined by the system of differential equations

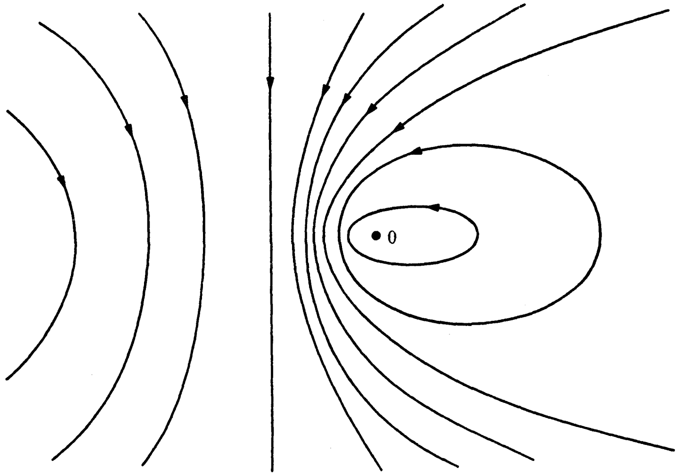

FIGURE 1

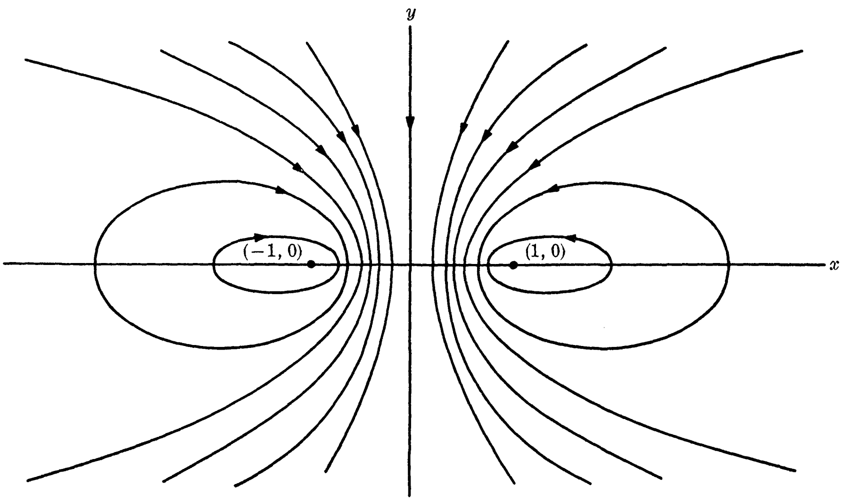

FIGURE 2

$$(2)\qquad \begin{aligned} \dot{x} &= -xy \\ \dot{y} &= \begin{cases} x - 1 - y^2 & \text{for } x \geqq 0 \\ -x - 1 - y^2 & \text{for } x < 0 \end{cases} \end{aligned}$$

is of characteristic 0. After changing system (1) to Cartesian coordinates, system (2) can be obtained by translation and reflection. The phase plane of (2) is illustrated in Figure 2.

4. Characterization of flows having characteristic 0. The purpose of this section is to give necessary and sufficient conditions for a flow (R^2, π) to have characteristic 0. Unless otherwise specified we shall let (R^2, π) be a fixed flow of characteristic 0 and S be the set of critical points. We shall first prove a few lemmas.

LEMMA 4.1. *If $L^+(x) \neq \emptyset$ ($L^-(x) \neq \emptyset$) for some $x \in R^2$, then x is either periodic or critical.*

Proof. Let $y \in L^+(x)$. Then $x \in J^-(y)$ since $y \in J^+(x)$. Hence, $x \in D(y) = K(y) \subset L^+(x)$. Seibert and Tulley have shown in [10] that a point is positively (negatively) Poisson stable if and only if it is either a critical point or a periodic point. The result for $L^-(x) \neq \emptyset$ follows similarly.

LEMMA 4.2. *If $x \in S$ or x is periodic, then $C(x)$ is bilaterally stable.*

Proof. The proof follows from Theorem 2.1 since $D(xt) = K(xt) = K(x) = C(x)$ for each t in R implies $D(C(x)) = C(x)$.

NOTATION. For any $s \in S$ we shall henceforth let

$$N_s = \{x \in R^2 \colon x = s \text{ or } x \text{ is periodic and } S \cap \operatorname{int} C(x) = \{s\}\} .$$

LEMMA 4.3. *If s_0 is an isolated point of S, then s_0 is a Poincaré center and N_{s_0} is an unbounded connected open set. If $N_{s_0} \neq R^2$, then ∂N_{s_0} is a single trajectory and N_{s_0} is a simply connected component of $R^2 - \partial N_{s_0}$.*

Proof. Let C be a simple closed curve with $S \cap \operatorname{int} C = \{s_0\}$. By virtue of Lemma 4.2 there exists a $V \in \eta(s_0)$ such that $C(V) \subset \operatorname{int} C$. Since $L^+(x) \neq \emptyset$ for each $x \in V$, $V - \{s_0\}$ consists of periodic points. If $x \in V - \{s_0\}$ then $\emptyset \neq S \cap \operatorname{int} C(x) \subset S \cap \operatorname{int} C = \{s_0\}$. Thus, V consists of s_0 and periodic orbits surrounding s_0 implying that s_0 is a Poincaré center.

Let $x \in N_{s_0} - \{s_0\}$ and $y \in (\operatorname{int} C(x)) - \{s_0\}$. Since $L^+(y) \neq \emptyset$, y is periodic. We have $\emptyset \neq S \cap \operatorname{int} C(y) \subset S \cap \operatorname{int} C(x) = \{s_0\}$ so that $y \in N_{s_0}$. Hence, $\overline{\operatorname{int} C(x)} \subset N_{s_0}$. Furthermore, N_{s_0} is connected since $N_{s_0} = \bigcup_{x \in N_{s_0}} \overline{\operatorname{int} C(x)}$ is the union of connected sets each containing the point s_0.

If $\partial N_{s_0} = \emptyset$ then $N_{s_0} = R^2$ and s_0 is a global Poincaré center. Suppose $\partial N_{s_0} \neq \emptyset$. Note that ∂N_{s_0} is invariant since N_{s_0} is invariant. We shall show that in this case ∂N_{s_0} contains no critical points or periodic points. First, suppose $s \in \partial N_{s_0} \cap S$. There is an open simply connected neighborhood U in $\eta(s)$ such that $s_0 \notin U$. By Lemma 4.2 there exists a $V_1 \in \eta(s)$ such that $C(V_1) \subset U$. Let $x \in V_1 \cap N_{s_0}$. Then $C(x) \subset U$. Since U is simply connected, $s_0 \in \operatorname{int} C(x) \subset U$ which is a contradiction. Hence, $S \cap \partial N_{s_0} = \emptyset$. Next, suppose there is a periodic point x in ∂N_{s_0}. Let $S_0 = S \cap \operatorname{int} C(x)$. There is a simply connected neighborhood $U \in \eta(C(x))$ such that $S \cap U = S_0$. By Lemma 4.2 there is a $V_2 \in \eta(C(x))$ such that $C(V_2) \subset U$. For $y \in (N_{s_0} \cap V_2) - \{s_0\}$ we have $C(y) \subset V_2$. Since U is simply connected, $\operatorname{int} C(y) \subset U$. Hence, $s_0 \in S \cap U = S_0$. The sets S_0 and $S - S_0$ are closed, so there are simple closed curves C_1 and C_2 contained in $\operatorname{int} C(x)$ and $\operatorname{ext} C(x)$, respectively, such that $S \cap (\operatorname{ext} C_1) \cap (\operatorname{int} C_2) = \emptyset$. By Lemma 4.2 there is a $V_3 \in \eta(C(x))$ such that $C(V_3) \subset (\operatorname{ext} C_1) \cap (\operatorname{int} C_2)$. Now N_{s_0} is connected with $s_0 \in N_{s_0}$ and $C(x) \subset \partial N_{s_0}$, so that we can select a point y from $N_{s_0} \cap V_3 \cap \operatorname{int} C(x)$. Thus, $S_0 \cap \operatorname{int} C(y) \neq \emptyset$, $S_0 \subset \operatorname{int} C_1$, and $C(y) \subset V_3 \subset \operatorname{ext} C_1$ imply $S_0 \subset \operatorname{int} C(y)$. Hence, $S_0 = \{s_0\}$ and $x \in N_{s_0}$. Finally, for any point $z \in V_3 \cap \operatorname{ext} C(x)$, $L^+(z) \neq \emptyset$ implying z is periodic. Since $C(z) \subset C(V_3) \subset \operatorname{int} C_2$ we have $S \cap \operatorname{int} C(z) = S_0$. The point z is in N_{s_0} and $C(x) \subset \operatorname{int} C(z) \subset N_{s_0}^0$. This contradicts $x \in \partial N_{s_0}$. Therefore, the points of ∂N_{s_0} are neither periodic nor critical.

By virtue of Lemma 4.1 and the fact that ∂N_{s_0} contains no periodic or rest points, $L^{\pm}(x) = \emptyset$ for each $x \in \partial N_{s_0}$. Thus, ∂N_{s_0} is not bounded and hence N_{s_0} is an unbounded open set.

We now show that ∂N_{s_0} is a single trajectory. Let x and y be distinct points of ∂N_{s_0}. Let C_1 and C_2 be simple closed curves such that $x \in \operatorname{int} C_1$, $y \in \operatorname{int} C_2$, and $\overline{\operatorname{int} C_1} \cap \overline{\operatorname{int} C_2} = \emptyset$. For z in $N_{s_0} \cap \operatorname{int} C_1$ we have $\overline{\operatorname{int} C(z)} \subset N_{s_0}$, and so $\operatorname{ext} C(z) \in \eta(y)$. Thus, $(\operatorname{int} C_2) \cap (\operatorname{ext} C(z)) \in \eta(y)$ and $(\operatorname{int} C_2) \cap (\operatorname{ext} (C(z)) \cap N_{s_0} \neq \emptyset$. Let $w \in (\operatorname{int} C_2) \cap (\operatorname{ext} C(z)) \cap N_{s_0}$. Then $C(z) \subset \operatorname{int} C(w) \subset N_{s_0}$. We have $z \in \operatorname{int} C(w)$ and $x \in \operatorname{ext} C(w)$. Since $x, z \in \operatorname{int} C_1$ and $\operatorname{int} C_1$ is connected, it follows that $C(w) \cap \operatorname{int} C_1 \neq \emptyset$. Hence, we can find nets (w_i) and $(w_i t_i)$ converging to y and x, respectively. In other words, $x \in D(y) = K(y) = C(y)$.

Suppose N_{s_0} is not a component of $R^2 - \partial N_{s_0}$. Since N_{s_0} is connected, it is a subset of some component B. If $N_{s_0} \neq B$, then $\partial N_{s_0} \cap B \neq \emptyset$ contradicting $B \subset R^2 - \partial N_{s_0}$. Hence, N_{s_0} is a component of $R^2 - \partial N_{s_0}$.

Finally, let $R^2 \neq N_{s_0}$. Suppose that C is a simple closed curve lying in N_{s_0} with $\operatorname{int} C \not\subset N_{s_0}$. Then $\operatorname{int} C$ connected and $N_{s_0} \cap \operatorname{int} C \neq \operatorname{int} C$ imply that $\partial N_{s_0} \cap \operatorname{int} C \neq \emptyset$. Furthermore, $\partial N_{s_0} \cap \operatorname{ext} C \neq \emptyset$ since ∂N_{s_0} is unbounded. Thus, $C \cap \partial N_{s_0} \neq \emptyset$ contradicting $C \subset N_{s_0}$. Therefore, N_{s_0} is simply connected.

LEMMA 4.4. *If $S_0 = S \cap \operatorname{int} C(x_0)$ for some periodic point x_0, then S_0 consists of exactly one Poincaré center.*

Proof. Let $N = \{x \in \operatorname{int} C(x_0)\colon x \text{ is periodic and } S_0 = S \cap \operatorname{int} C(x)\}$ and $D = \bigcap_{x \in N} \overline{\operatorname{int} C(x)}$. At least $x_0 \in N$, so that $D \neq \emptyset$. Also, D is the intersection of closed invariant sets containing S_0 so that D is a closed invariant set and $S_0 \subset D$. It also follows that ∂D is invariant.

In order to facilitate the argument we show that $V \in \eta(C(y))$ implies $V \cap N \neq \emptyset$ for all $y \in \partial D$. Suppose $V \cap N = \emptyset$ for some $V \in \eta(C(y))$. By Lemma 4.2 there is a connected set $U \in \eta(C(y))$ such that $C(U) \subset V$. For $x \in N$, $U \cap C(x) = \emptyset$. Since $y \in \operatorname{int} C(x)$ and U is connected we have $U \subset \operatorname{int} C(x)$. The point x was arbitrary, so that $U \subset D$. But this implies $y \in D^0$ which contradicts $y \in \partial D$.

Suppose that D is not a singleton. We first show that there exists a point $y \in N$ such that $D = \overline{\operatorname{int} C(y)}$. If $D = \overline{\operatorname{int} C(x_0)}$ then we are done. Assume $D \neq \overline{\operatorname{int} C(x_0)}$ and choose points x in D and y in ∂D such that $x \neq y$. Either $y \in S_0$ or y is periodic. Suppose there exists a simple closed curve C such that $x \in \operatorname{ext} C$ and $C(y) \subset \operatorname{int} C$. By Lemma 4.2 there is a $V \in \eta(C(y))$ such that $C(V) \subset \operatorname{int} C$. We have shown that $V \cap N \neq \emptyset$. Let $z \in V \cap N$. Then $C(z) \subset C(V) \subset \operatorname{int} C$. But this implies that $x \in \operatorname{int} C(z) \subset \operatorname{int} C$ contradicting $x \in \operatorname{ext} C$. Thus, y is periodic and $x \in \operatorname{int} C(y)$. Since x was an arbitrary point of D, we have $D \subset \overline{\operatorname{int} C(y)}$. Furthermore, $C(y) \subset \partial D \subset \operatorname{int} C(z)$ for each $z \in N$ implying $\overline{\operatorname{int} C(y)} \subset \bigcap_{z \in N} \overline{\operatorname{int} C(z)} = D$. Hence, $D = \overline{\operatorname{int} C(y)}$.

Since S_0 is compact there exists a simple closed curve $C \subset \operatorname{int} C(y)$ with $S_0 \subset \operatorname{int} C$. By Lemma 4.2 there is a $V \in \eta(C(y))$ such that $C(V) \subset \operatorname{ext} C$. Each point z in $V \cap \operatorname{int} C(y)$ is periodic by Lemma 4.1, and so, $S_0 \cap \operatorname{int} C(z) \neq \emptyset$. Since $C(z) \subset \operatorname{ext} C$, $\operatorname{int} C(z) \cap \operatorname{int} C \neq \emptyset$, and $\operatorname{int} C$ is connected, we have $S_0 \subset \operatorname{int} C \subset \operatorname{int} C(z)$ and $z \in N$. Thus, $D \subset \overline{\operatorname{int} C(z)}$ and $C(z) \subset \operatorname{int} C(y)$ imply that $D \subset \operatorname{int} C(y)$ which contradicts $y \in D$. Consequently, D must be a singleton.

Finally, $\emptyset \neq S_0 \subset D$ implies that D is composed of an isolated critical point. By Lemma 4.3, S_0 consists of a Poincaré center.

LEMMA 4.5. *If $S \neq \emptyset$ and $S \neq R^2$, then S consists of Poincaré centers.*

Proof. Let S_0 denote the set of Poincaré centers. We can select a point s from ∂S since $S \neq \emptyset$ and $S \neq R^2$. For any compact set $U \in \eta(s)$ there exists a $V \in \eta(s)$ such that $C(V) \subset U$ by Lemma 4.2. For any $x \in V \cap (R^2 - S)$, $L^+(x) \neq \emptyset$ implying that x is periodic. Thus, Lemma 4.4 implies $S_0 \neq \emptyset$.

Suppose $s \in \partial(S - S_0)$. Since s is bilaterally stable, $\eta(s)$ contains a compact connected simply connected invariant set V. Either V contains a regular point or a center. If it contains a regular point x, then x must be periodic so that $\operatorname{int} C(x)$, and hence V, must contain a center. Therefore, we can assume that V contains a center s_0. Now, for each $x \in N_{s_0} - \{s_0\}$, $s_0 \in \operatorname{int} C(x)$ and, by Lemma 4.4, $s \in \operatorname{ext} C(x)$. Thus, V must meet $C(x) = \partial \operatorname{int} C(x)$ since it is connected. But this implies $C(x) \subset V$ and hence $N_{s_0} \subset V$, contradicting Lemma

4.3. Therefore, $\partial(S - S_0) = \varnothing$, and so $S = S_0$.

LEMMA 4.6. *If* $S \neq \varnothing$ *and* $S \neq R^2$, *then* S *consists of at most two Poincaré centers.*

Proof. Suppose s_1, s_2, and s_3 are distinct points of S. We shall show that this supposition leads to a countable collection of mutually disjoint closed sets whose union is R^2 which is impossible. Unless explicitly stated, the remainder of the proof will be considered relative to the extended dynamical system on R^{2*}. We denote the closure of the trajectory through x in the extended system by $K^*(x)$.

Since the sets N_s are disjoint and open relative to R^2, $A = R^2 - \bigcup_{s \in S} N_s$ is nonempty. For each $x \in A$, $K^*(x) = C(x) \cup \{\infty\}$ is a simple closed curve. Let $M = \{x \in A\colon N_{s_1} \subset A_x$ and $N_{s_2} \cup N_{s_3} \subset B_x$ where A_x and B_x are the components of $R^{2*} - K^*(x)\}$. By Lemma 4.3, $M \neq \varnothing$ since $\partial N_{s_1} - \{\infty\} \subset M$. Note that $\overline{A_x} = A_x \cup K^*(x)$ and let $F_{s_1} = \overline{\bigcup_{x \in M} \overline{A_x}}$. Each set $\overline{A_x}$ is connected and contains N_{s_1}, and so F_{s_1} is connected.

For any point p_1 in $\partial F_{s_1} - \{\infty\}$ we have $F_{s_1} = \overline{A_{p_1}}$. For let p_1 and q_1 be distinct points in $\partial F_{s_1} - \{\infty\}$ and let C_1 and C_2 be simple closed curves in R^2 surrounding p_1 and q_1, respectively, such that $\overline{\operatorname{int} C_1} \cap \overline{\operatorname{int} C_2} = \varnothing$. There exists a point p for which $A_p \cap \operatorname{int} C_1$, and hence $C(p) \cap \operatorname{int} C_1$, are nonempty sets. Since $B_p \cap \operatorname{int} C_2 \in \eta(q_1)$ there exists a point q such that $A_q \cap B_p \cap \operatorname{int} C_2 \neq \varnothing$; hence, $C(q) \cap \operatorname{int} C_2 \neq \varnothing$. Now, A_q meets A_p and B_p, so that $A_p \subset A_q$. Thus, A_q is a connected set which meets both $\operatorname{int} C_1$ and $\operatorname{ext} C_1$ implying that $C(q) \cap \operatorname{int} C_1 \neq \varnothing$. We can find nets (x_i) and $(x_i t_i)$ converging to q_1 and p_1, respectively; hence, $p_1 \in D(q_1) = K(q_1) = C(q_1)$ and $\partial F_{s_1} - \{\infty\} = C(p_1)$. Now, $C(p_1) \not\subset N_s$ for any s in S since $N_s \cap F_{s_1} \neq \varnothing$ implies there exists an x in M such that $C(p_1) \subset N_s \subset A_x \subset F^0_{s_1}$ contradicting $C(p_1) \subset \partial F_{s_1}$. Thus, $C(p_1) \subset A$. Since F_{s_1} is an invariant set, either $C(p_1) \subset F_{s_1}$ or $C(p_1) \cap F_{s_1} = \varnothing$. Suppose $C(p_1) \cap F_{s_1} = \varnothing$. Then $F_{s_1} - \{\infty\}$ is the connected set $F^0_{s_1}$, and so it is a component of $R^{2*} - K^*(p_1) = R^{2*} - \partial F_{s_1}$. Also, $N_{s_2} \cup N_{s_3} \subset \bigcap_{x \in M} B_x = R^{2*} - F_{s_1}$ which means $p_1 \in F_{s_1}$, contradicting $C(p_1) \cap F_{s_1} = \varnothing$. Hence, $F_{s_1} = \bar{A}_{p_1}$.

Analogously, for s_2 and s_3 there exists points p_2 and p_3 in A and sets F_{s_2} and F_{s_3} such that $F_{s_2} = \bar{A}'_{p_2}$ and $F_{s_3} = \bar{A}''_{p_3}$. Note that $F_{s_1} = \bar{A}_{p_1}$ and $F_{s_2} \subset B_{p_1}$. If $\partial F_{s_1} = K^*(p_1) = K^*(p_2) = \partial F_{s_2}$ then $F_{s_1} \cup F_{s_2} = R^{2*}$ which contradicts $s_3 \notin F_{s_1} \cup F_{s_2}$. Hence, $F_{s_1} \cap F_{s_2} = \{\infty\}$. Similarly, $F_{s_1} \cap F_{s_3} = F_{s_2} \cap F_{s_3} = \{\infty\}$.

Let $F = \bigcup_{i=1}^{3} F_{s_i}$. Obviously, $R^{2*} \neq F$, and so $R^{2*} - F \neq \varnothing$. Suppose that $A \cap (R^{2*} - F) = \varnothing$. Then $R^{2*} - F$ must consist of periodic and rest points, so that $N_s \subset R^{2*} - F$ for some $s \in S$. Furthermore, $\partial N_s - \{\infty\} \subset A$ implies that $\partial N_s \subset \partial F = \bigcup_{i=1}^{3} K^*(p_i)$. By letting $\partial N_s = K^*(p_k)$ we have $R^{2*} = N_s \cup F_{s_k}$ since N_s and $F^0_{s_k}$ are components of

$R^{2*} - K^*(p_k)$. But this implies that $s_i \in F_{s_k}$ for $i \neq k$ which is clearly not possible. Therefore, $A \cap (R^{2*} - F) \neq \emptyset$.

For each point x in $A \cap (R^{2*} - F)$ one component $R^{2*} - K^*(x)$ contains F since $K^*(x)$ does not separate any of the sets N_{s_1}, N_{s_2}, and N_{s_3} from the other two. Denote the components of $R^{2*} - K^*(x)$ by G_x and H_x where $F \subset H_x$. For any point y in $A \cap (R^{2*} - F)$, let $M_y = \{x \in A \cap (R^{2*} - F) : G_y \subset G_x\}$. Note that $M_y \neq \emptyset$ since $y \in M_y$. Let $F'_y = \bigcup_{x \in M_y} G_x$. By arguing as we did for F_{s_1}, we can find a point w in $\partial F'_y \cap A$ such that $F'_y = \bar{G}_w$. For each point p in $A \cap (R^{2*} - F)$ for which $F'_p = F'_y$, select a point y_0 in $C(w)$ and denote F'_p by F_{y_0}. Let I' be the index set for all the F_{y_0} sets and let $I = I' \cup \{s_1, s_2, s_3\}$.

If x and z are distinct points in I, then $F_x \cap F_z = \{\infty\}$. For suppose $F_x \cap F_z \neq \{\infty\}$. The sets F_x^0 and F_z^0 are components of $R^{2*} - \partial F_x$ and $R^{2*} - \partial F_z$, respectively, where ∂F_x and ∂F_z are simple closed curves each consisting of $\{\infty\}$ and a single trajectory. Thus, either $\partial F_x - \{\infty\} \subset F_z^0$, $\partial F_z - \{\infty\} \subset F_x^0$, or $F_x^0 \cap F_z^0 = \emptyset$. The first two statements imply that $F_x = F_z$, and hence $x = z$, contradicting $x \neq z$. The third statement implies that $F_x \cup F_z = R^{2*}$ which is impossible. Therefore, $F_x \cap F_z = \{\infty\}$.

Next, $R^{2*} = \bigcup_{x \in I} F_x$. For let z belong to $R^{2*} - E$ where $E = \bigcup_{x \in I} F_x$. Since $A \subset E$, there is a point s in S such that $z \in N_s$. For some point y in E, $K^*(y) = \partial N_s$. Furthermore, there is a point x in I such that $K^*(y) = \partial F_x$ since $K^*(y) \subset \partial E$. The sets N_s and F_x^0 are disjoint components of $R^{2*} - K^*(y)$, and so $R^{2*} = N_s \cup F_x$. This implies $F_x = E$, and thus $s_i \in F_x$ for $i = 1, 2, 3$, which is clearly impossible. Hence, $R^{2*} = E$.

The set $\{F_x : x \in I\}$ is a countable collection of closed sets such that $F_x \cap F_z = \{\infty\}$ for $x \neq z$. Hence, $\{F_x - \{\infty\} : x \in I\}$ is a countable collection of mutually disjoint sets closed in R^2 and $R^2 = \bigcup_{x \in I} (F_x - \{\infty\})$. This is not possible as we indicated at the outset of our argument. Therefore, s_1, s_2, and s_3 are not distinct.

LEMMA 4.7. *Let $S \neq R^2$. Then the flow restricted to $R^2 - \bigcup_{s \in S} N_s$ is parallelizable.*

Proof. Let $Y = R^{2*} - \bigcup_{s \in S} N_s$. The set $\{\infty\}$ is compact and invariant. According to Theorem 2 p. 151 of [9], $J(x) = \infty$ for each x in $Y - \{\infty\}$. Thus, relative to $Y - \{\infty\}$, $J(x) = \emptyset$ and $D^+(x) = C^+(x)$. The result follows by Theorem 2.2.

THEOREM 4.8. *A flow (R^2, π) has characteristic 0 if and only if one of the following holds.*

(1) *$S = \emptyset$ and (R^2, π) is parallelizable.*

(2) *S consists of at most two Poincaré centers. For each $s \in S$,*

either s is a global Poincaré center or N_s is unbounded and ∂N_s is a single trajectory. The restriction of the flow to $R^2 - \bigcup_{s \in S} N_s$ is parallelizable.

(3) $S = R^2$.

Proof. The necessity of the conditions follows from the lemmas. Conversely, Theorem 2.2 shows that condition (1) is sufficient. Similarly, if condition (2) holds, we get $D(x) = K(x)$ for each $x \in R^2 - \bigcup_{s \in S} N_s$. For each $s \in S$, N_s is a component of $R^2 - \partial N_s$ since ∂N_s is a single trajectory. Thus, N_s is a connected simply connected set. Obviously, $x \in \bar{N}_s$ implies $D(x) = K(x)$. Hence, condition (2) is sufficient. Condition (3) is trivially sufficient.

COROLLARY 4.9. *A flow (R^2, π) has characteristic* 0 *if and only if $D(x) = C(x)$ for each $x \in R^2$.*

REMARK. That there are six basic types of planar flows (up to dynamical isomorphism) having characteristic 0 follows from Theorem 4.8. These are

(1) parallelizable flows,
(2) flows having a global Poincaré center,
(3) flows similar to Example 2,
(4) flows similar to Example 3,
(5) flows similar to Example 3 except that $\partial N_s = \partial N_t$ where $S = \{s, t\}$, and
(6) flows having only critical points.

Note that only the flows in (1), (2), and (6) have characteristics 0, 0^+, 0^-, and $0^\pm$ and that the flows in (3), (4), and (5) have only characteristic 0.

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