OPEN MAPPINGS ON 2-MANIFOLDS

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This paper will be concerned with the local structure of open, continuous functions $f: M^2 \to N^1$ from a 2-manifold into the real line or the circle. The two main results are:

THEOREM 1. **Let $f: M^2 \to N^1$ be an open mapping, and suppose that f has isolated branch points. Then for each point p in M^2 there are a neighborhood U of p and a positive integer d (depending on p) such that $f|_U$ is topologically equivalent to the real analytic mapping $z \to Re(z^d)$.**

THEOREM 2. **If $f: M^2 \to N^1$ is an open, real analytic mapping, then f has isolated branch points, and (hence) the conclusion of Theorem 1 holds.**

2. **Definitions and notation.** Let $f: X \to Y$ be any continuous function. Then f is an *open mapping* if $f(U)$ is an open subset of Y whenever U is an open subset of X. If $f': X' \to Y'$ is another mapping then f and f' are *topologically equivalent* if there exist homeomorphisms $\alpha: X \to X'$, $\beta: Y \to Y'$ such that $f'\alpha = \beta f$.

If A is any subset of X, then $f|_A$ will denote the restriction of f to A.

Let $f: M^n \to N^p$ be a mapping from an n-manifold into a p-manifold, $n \geq p$. The *branch set*, B_f, of f is the subset of M^n defined as follows: a point x is in $M^n - B_f$ if there exists a neighborhood U of x in M^n such that $f|_U$ is topologically equivalent to the canonical projection of n-space onto p-space.

If M^n and N^p are differentiable manifolds, and $f: M^n \to N^p$ is differentiable, then $R_q(f)$ will denote the set of points in M^n at which the rank of the Jacobian of f is at most q, $q = 0, 1, \cdots$.

The boundary of a set A will be denoted by ∂A and its closure by $\bar{A}$.

3. *Proof of Theorem 1.* We first need some lemmas.

LEMMA 3.1. *Let $f: [0, 1] \times [0, 1] \to [0, 1]$ with $\{i\} \times [0, 1] = f^{-1}(i)$, $i = 0, 1$. Suppose there exists a homeomorphism $\alpha: [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ which fixes the four vertices of $[0, 1] \times [0, 1]$ such that $\Pi\alpha = f$, where Π is the projection $(x, y) \to x$. Let $h: \{0\} \times [0, 1] \to \{0\} \times [0, 1]$ be any homeomorphism with $h(0, 0) = (0, 0)$. Then there exists a homeomorphism $\beta: [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ such that $\Pi\beta = f$ and $\beta|_{\{0\} \times [0,1]} = h$.*

Proof. The desired homeomorphism is given by $\beta(x, y) = (f(x, y),$

$h^2\alpha^{-1}(0, \alpha^2(x, y)))$, where h^2 and α^2 are the second coordinate functions of h and α, respectively.

LEMMA 3.2. *Let* $f: [0, 1] \times [0, 1] \to [0, 1]$ *be an open mapping. Suppose that the following conditions are satisfied:*

(i) $f^{-1}(0) = \{0\} \times [0, 1]$, $f^{-1}(1) = \{1\} \times [0, 1]$,

(ii)   *if* g *is the restriction of* f *to* $(0, 1) \times (0, 1)$, *then* $B_g = \varnothing$,

(iii)   *for each point* y *in the open interval* $(0, 1)$, $y \neq 1/2$ *there is a neighborhood of the point* $(1, y)$ *in* $[0, 1] \times [0, 1]$ *on which* f *is topologically equivalent to the projection of* $(0, 1] \times (0, 1)$ *onto* $(0, 1]$, *and for each point* y *in the open interval* $(0, 1)$ *there is a neighborhood of* $(0, y)$ *in* $[0, 1] \times [0, 1]$ *on which* f *is topologically equivalent to the projection of* $[0, 1) \times (0, 1)$ *onto* $[0, 1)$,

(iv)   *for each point* x *in the open interval* $(0, 1)$, *there is a neighborhood of the point* $(x, 0)$ *in* $[0, 1] \times [0, 1]$ *and a neighborhood of the point* $(x, 1)$ *in* $[0, 1] \times [0, 1]$ *on which* f *is topologically equivalent to the projection of* $(0, 1) \times [0, 1)$ *onto* $(0, 1)$, *and*

(v)   *if* $x = 0$ *or* 1, *and* $y = 0$ *or* 1, *then there exists a neighborhood of the point* (x, y) *in* $[0, 1] \times [0, 1]$ *on which* f *is topologically equivalent to the first coordinate projection of* $[0, 1) \times [0, 1)$ *onto* $[0, 1)$.

Then f *is topologically equivalent to the first coordinate projection of* $[0, 1] \times [0, 1]$ *onto* $[0, 1]$.

Proof.   Let $\Pi: [0, 1] \times [0, 1] \to [0, 1]$ denote the projection $(x, y) \to x$. Let $a_0 = [0, 1] \times \{1\}$, and let $b_0 = [0, 1] \times \{0\}$. Let $\{y_n^+\}$ be a monotone decreasing sequence in the open interval $(1/2, 1)$ converging to $1/2$, and let $\{y_n^-\}$ be a monotone increasing sequence in the open interval $(0, 1/2)$ converging to $1/2$.

In [7, p. 16 (3.1)], Timourian showed that a proper mapping with empty branch set between manifolds of codimension one is a locally trivial fiber map; hence the restriction of f to $[0, 1) \times [0, 1]$ is topologically equivalent to the projection of $[0, 1) \times [0, 1]$ onto $[0, 1)$. For each positive integer n, let a_n be an arc in $[0, 1] \times [0, 1]$ whose endpoints are $(0, y_n^+)$ and $(1, y_n^+)$, and let b_n be an arc in $[0, 1] \times [0, 1]$ with endpoints $(0, y_n^-)$ and $(1, y_n^-)$ satisfying the following conditions (these arcs exist by the above reference and condition (iii)):

(i)   the collection $\{a_1, b_1, a_2, b_2, \cdots\}$ is pairwise disjoint, and

(ii)   each a_i and each b_i is mapped by f homeomorphically onto $[0, 1]$.

For each n, let A_n' denote the closed 2-cell bounded by $a_n \cup a_{n-1} \cup (\{0, 1\} \times [y_n^+, y_{n-1}^+])$, and let B_n' be the closed 2-cell bounded by $b_n \cup b_{n-1} \cup (\{0, 1\} \times [y_{n-1}^-, y_n^-])$, where $y_0^+ = 1$, and $y_0^- = 0$.

Let $\{x_n\}$ be a monotone increasing sequence in the open interval $(0, 1)$ converging to 1, and let $A_n = A_n' \cap f^{-1}([x_{n-1}, 1])$ and $B_n = B_n' \cap$

$f^{-1}([x_{n-1}, 1])$, where $x_0 = 0$. Finally, let C_n denote the closure of

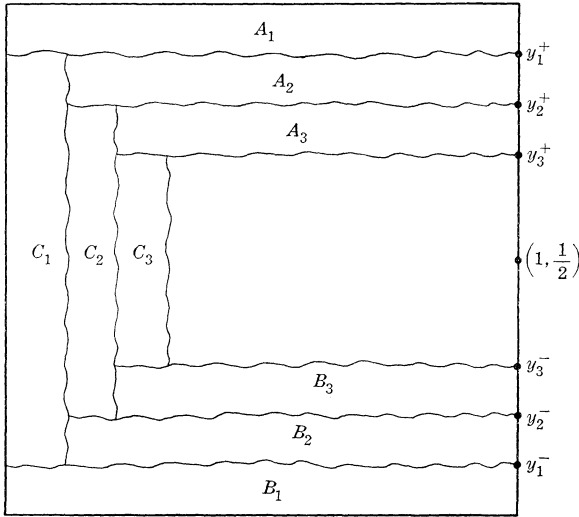$$f^{-1}([x_{n-1}, x_n]) - \left(\bigcup_{i=1}^{n} (A_i \cup B_i) \right).$$ (See figure.)

FIGURE 1.

By the aforementioned reference to [7], there exist homeomorphisms

$$h_n^A \colon A_n \longrightarrow [x_{n-1}, 1] \times [y_n^+, y_{n-1}^+] ,$$
$$h_n^B \colon B_n \longrightarrow [x_{n-1}, 1] \times [y_{n-1}^-, y_n^-] , \quad \text{and}$$
$$h_n^C \colon C_n \longrightarrow [x_{n-1}, x_n] \times [y_n^-, y_n^+]$$

such that $\Pi h_n^A = f|_{A_n}$, $\Pi h_n^B = f|_{B_n}$ and $\Pi h_n^C = f|_{C_n}$. We assume here that $h_n^A((1, y_i^+)) = (1, y_i^+)$ and $h_n^B((1, y_i^-)) = (1, y_i^-)$ for $i = n - 1, n$.

Clearly, h_n^A agrees with h_n^C on $A_n \cap C_n$, and h_n^B agrees with h_n^C on $B_n \cap C_n$, so there exists a homeomorphism

$$h_n \colon A_n \cup B_n \cup C_n \longrightarrow ([x_{n-1}, 1] \times [y_n^+, y_{n-1}^+]) \cup ([x_{n-1}, 1] \times [y_{n-1}^-, y_n^-])$$

$$\cup ([x_{n-1}, x_n] \times [y_n^-, y_n^+])$$

such that $\Pi h_n = f|_{A_n \cup B_n \cup C_n}$, $n = 1, 2, \cdots$. Clearly, h_{n+1} agrees with h_n on $A_{n+1} \cap A_n$ and on $B_{n+1} \cap B_n$; moreover, by 3.1, we may assume that h_{n+1} agrees with h_n on $C_n \cap (A_{n+1} \cup B_{n+1} \cup C_{n+1})$.

Thus, we get a continuous bijection $h; [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ by

$$h(p) = \begin{cases} h_n(p) , & \text{if } p \in A_n \cup B_n \cup C_n \\ \left(1, \dfrac{1}{2}\right) , & \text{if } p = \left(1, \dfrac{1}{2}\right) , \end{cases}$$

satisfying the condition $\Pi h = f$.

LEMMA 3.3.   *Let* $f: M^n \to N^1$ *be open, let* $U \subset M^n$ *be open, and suppose that* $f(U)$ *is contained in a closed arc on* N^1. *Let* $q \in U$, *and let* Q *be a closed subset of* $f^{-1}(f(q)) \cap U$. *Then no component of* U-Q *is separated from* ∂U. (Two sets are *separated* if their closures are disjoint.)

Proof.   If V is a component of U-Q, then V is an open connected subset of M^n, so $f(V)$ is an open interval (whose closure is a closed interval). If V is separated from ∂U, then $\partial V \subset Q$. Thus $f(\bar{V}) = f(V) \cup f(\partial V) = \{f(q)\} \cup f(V)$ is not compact.

Proof of Theorem 1.   Let U' be an open disk about p whose closure is a closed disk such that $B_f \cap \bar{U}' \subset \{p\}$ and $f(\bar{U}')$ is a closed interval on N^1. By 3.3, for each point x in U', $f^{-1}(f(x)) \cap U'$ contains no simple closed curve. Thus, the closure of each component of $(f^{-1}(f(p)) \cap \bar{U}') - \{p\}$ is a closed arc with a point of $\partial U'$ as one endpoint and either p or a point of $\partial U'$ as its other endpoint, and for each point q in $\bar{U}' - f^{-1}(f(p))$, each component of $f^{-1}(f(q)) \cap \bar{U}'$ is a (possibly degenerate) closed arc each of whose endpoints is on $\partial U'$.

Let $A_1, A_2, \cdots, A_k$ be the closures of the component arcs of $(f^{-1}(f(p)) \cap \bar{U}') - \{p\}$ which have p as an endpoint, and let a_i be the endpoint of A_i on $\partial U'$. (There are clearly finitely many such A_i.) Orient $\partial U'$, and index the a_i in order of increasing argument.

Let $R_1', \cdots, R_k'$ be the components of $U' - \bigcup_{i=1}^{k} A_i$, indexed so that $a_j \in \bar{R}_i'$ if, and only if, $j = i$ or $j \equiv (i + 1)(\mathrm{mod}\ k)$.

For each $i = 1, 2, \cdots, k$, there exist a neighborhood U_i of a_i and a homeomorphism $\alpha_i': U_i \to E^2$ such that $\Pi\alpha_i' = f|_{U_i}$, where $\Pi: E^2 \to E^1$ is projection onto the first coordinate. We may as well assume that $U_i \cap U_j = \varnothing$ when $i \neq j$, $\alpha_i'(U_i \cap \partial U')$ is the x-axis, and $\Pi\alpha_i'(q) > 0$ for each $q \in R_i' \cap U_i$.

Let $\{Y_\alpha\}$ be the collection of all components of $(f^{-1}(f(p)) \cap \bar{U}') - \{p\}$ which do not have p as a limit point, and let $Y = \cup Y_\alpha$. Clearly, p is not a limit point of Y, so there exists a neighborhood U'' of p such that

$$(*) \qquad \left(\bigcup_{i=1}^{k} U_i\right) \cap \bar{U}' = \left(\bigcup_{i=1}^{k} U_i\right) \cap \bar{U}'' \quad \text{and} \quad \bar{U}'' \cap f^{-1}(f(p)) = \bigcup_{i=1}^{k} A_i.$$

Let $R_i = R_i' \cap U''$, and fix i. We claim that there exists a point $t_i \in (\bar{R}_i \cap \partial U'' \cap U_i) - \{a_i\}$ such that if x is any point in $(\bar{R}_i \cap \partial U'' \cap U_i)$ with $0 < \Pi\alpha_i'(x) \leqq \Pi\alpha_i'(t_i)$ and X is the component arc of $f^{-1}(f(x)) \cap \bar{U}''$ containing x, then $x' \in U_j$, where $j \equiv (i + 1)(\mathrm{mod}\ k)$, and x' is the other endpoint of X. For suppose not. Then there exists a sequence $\{x_n\}$ in $(\bar{R}_i \cap \partial U'' \cap U_i) - \{a_i\}$ converging to a_i such that $x_n' \notin U_j$, for each n. There is a convergent subsequence of $\{x_n'\}$ whose limit is in

$f^{-1}(f(p)) \cap \partial U''$ but is not in $\bigcup_{i=1}^{k} A_i$. This contradicts (∗), above.

Write $f(\bar{U}'')$ as the closed interval $[a', b']$, where $a' < f(p) < b'$. We may assume that $f(p) < f(t_i) < b'$ if i is even, and $a' < f(t_i) < f(p)$ if i is odd. Note that k is even. Let $b = \min\{f(t_2), f(t_4), \cdots, f(t_k)\}$, and let $a = \max\{f(t_1), f(t_3), \cdots, f(t_{k-1})\}$. Finally, let $U = \bar{U}'' \cap f^{-1}([a, b])$, let $d = k/2$, and assume $a = 0$ and $b = 1$.

If $S_i = \bar{R}_i \cap U$, then $f|_{S_i}$ satisfies the hypotheses of 3.2. Note that $f(S_i) = [f(p), b]$ if i is even, and $f(S_i) = [a, f(p)]$ if i is odd, so by 3.2 there exists a homeomorphism $\alpha_i \colon S_i \to [-1, 1] \times [0, 1]$ such that $f|_{S_i} = \Pi' \alpha_i$, where $\Pi' \colon [-1, 1] \times [0, 1] \to [0, 1]$ is the natural projection. We may assume that $\alpha_i(p) = (0, 1)$, and that if $j \equiv (i + 1)(\mathrm{mod}\ k)$, then $\alpha_i(A_j) = \alpha_j(A_j), i = 1, 2, \cdots, k$.

Let $X_1, X_2, \cdots, X_k$ be disjoint copies of $[-1, 1] \times [0, 1]$ indexed so that $\alpha_i(S_i) = X_i$. Form the disjoint union $X = X_1 + X_2 + \cdots + X_k$. Now, make the following identifications on X: if $x_j \in \alpha_j(A_j)$ and $j \equiv (i + 1)(\mathrm{mod}\ k)$, then identify x_j with $\alpha_i \alpha_j^{-1}(x_j)$.

Let X/α denote the resulting identification space, and let $P_\alpha \colon X \to X/\alpha$ be the quotient map. Let $\alpha \colon U \to X/\alpha$ by $\alpha(u) = P_\alpha \alpha_i(u)$, if $u \in S_i$. Then α is a homeomorphism.

Let $\Pi_i \colon X_i \to [0, 1]$ denote the projection mapping, $i = 1, 2, \cdots, k$, and let $\sum_{i=1}^{k} \Pi_i \colon X \to [0, 1]$ by $(\sum_{i=1}^{k} \Pi_i)(x_j) = \Pi_j(x_j)$, where $x_j \in X_j$. Then $\sum_{i=1}^{k} \Pi_i$ can be factored through P_α, obtaining a map $G \colon X/\alpha \to [0, 1]$ such that $GP_\alpha = \sum_{i=1}^{k} \Pi_i$. But then $f|_U = G\alpha$; i.e., $f|_U$ is topologically equivalent to G. Theorem 1 follows.

In [8, Theorem 1], Tôki gives a topological characterization of mappings on a 2-manifold which are pseudo-harmonic. His results imply that if $f \colon M^2 \to N^1$ is an open map with B_f discrete, then f is topologically equivalent to a harmonic function. It is well known (see, for example, [5, pp. 8–9]) that harmonic functions have the local structure of (our) Theorem 1. The advantage of the present approach must lie in its directness and brevity.

4. **The real analytic case.** In this section we prove Theorem 2. A few facts from the theory of analytic varieties will be needed, and we include them here for completeness. For the reader who is unfamiliar with the terminology, we recommend [6].

LEMMA 4.1. *Every real analytic variety is locally connected.*

LEMMA 4.2. *If V is a k-dimensional variety in the real analytic manifold M, then for every point $p \in V$, there is a neighborhood U of p in M such that $V \cap U$ is the disjoint union $V' + V''$, where V' is a k-dimensional real analytic submanifold of U, and V'' is a variety in U of dimension at most $k - 1$.*

The proof of 4.2 is in [1].

If $f: M \to N$ is a real analytic mapping, then clearly the sets $f^{-1}(y)$, $y \in N$, are varieties, as are the sets $R_q(f)$, defined in §2. Moreover, it was proved by Church [2, p. 88 (1.3)] that $\dim(f(R_q(f))) \leqq q$, $q = 0, 1, \cdots$.

LEMMA 4.3. *Let* $f: M^n \to N^1$ *be a proper real analytic mapping. Then* $f(R_0(f))$ *is discrete.*

Proof. Let y be a limit point of $f(R_0(f))$. Let $\{y_n\}$ be a sequence in $f(R_0(f))$, converging to y, and let x_n be a point in $f^{-1}(y_n) \cap R_0(f)$, $n = 1, 2, \cdots$. By the properness of f, the sequence $\{x_n\}$ has an accumulation point; call it x. Since $R_0(f)$ is closed, $x \in R_0(f)$. Then, if U is any neighborhood of x, $U \cap R_0(f) - \{x\}) \neq \varnothing$. Then, by the 0-dimensionality of $f(R_0(f))$ (see remark, above), $U \cap R_0(f)$ is not connected; i.e., $R_0(f)$ is not locally connected at x, contradicting 4.1.

Proof of Theorem 2. Let D be an open disk in M^2 with compact closure, and let $g = f|_D$. Then $g(R_0(g))$ is discrete by 4.3. Let $q \in g(R_0(g))$. It suffices to show that $g^{-1}(q) \cap B_g$ is discrete.

Since $\dim(g^{-1}(q)) = 1$, we can remove from $g^{-1}(q)$, by 4.2, a discrete set A' such that $g^{-1}(q) - A'$ is a 1-dimensional real analytic submanifold of D. Let p be a point in $g^{-1}(q) - A'$, and let $U \subset D$ be a coodinate chart about p with real analytic coordinates (x, y) such that $g^{-1}(q)$ is the y-axis and $g(R_0(g) \cap U) = \{q\}$. Let $h = f|_U$. By Fox's Spoke Theorem [4, p. 347 (5.1)], it suffices to show that $h^{-1}(q) \cap B_h$ is discrete.

Suppose W is an open disk about a point on $h^{-1}(q) \cap U$ such that each horizontal line in W given by $y = $ constant is mapped homeomorphically by h onto $h(W)$. Then $W \cap B_h = \varnothing$. For define $F: W \to h(W) \times R$ by $F(x, y) = (h(x, y), y)$; this will give the desired equivalence.

For such fixed $\bar{y}$, consider the set $X(\bar{y})$ of points $(x, \bar{y})$ at which $h|_{y=\bar{y}}$ is not interior; i.e., the set of points at which $(\partial/\partial_x)(h|_{y=\bar{y}})$ changes sign. Now, $X = \bigcup_{\bar{y}} X(\bar{y})$ is contained in the set $A = \{(x, y): (\partial h/\partial x)(x, y) = 0\}$. A is clearly a variety, and the openness of f implies that $\dim(A) \leqq 1$. If A is discrete, then the comment in the last paragraph applies and there is nothing to prove. Hence, we may as well assume that $\dim(A) = 1$. Let E be the set of points at which A is not a 1-submanifold.

Consider the intersection of A with the y-axis. It is a variety of dimension at most one. If it has dimension zero (i.e., if it is discrete), then there is nothing to prove. So suppose it is 1-dimensional,

in which case it is the entire y-axis. Now, the openness of h implies that all points of U with positive abscissa map on one side of q, and the points with negative abscissa map on the other side of q. Hence, $\partial h/\partial x$ does not change sign at any point of the y-axis. Thus, every point on the y-axis which is not in E has a neighborhood disjoint from X, and E is discrete.

For more extensive results on the topological properties of real analytic mappings, see [3].

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