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Consider any fibration $p: E \to B$, any finite C.W. — pair (K, L), and any maps $f: K \to B$ and $h: L \to E$ such that $p \circ h = f \mid L$. A map $g: K \to E$ such that $p \circ g = f$ and $g \mid L = h$ we call a *lifting of f rel h*.

In this paper single obstruction $\Gamma(f) \in H'(K, L, f; \mathscr{E})$ is defined. $\mathscr{E}$ is a so-called B-spectrum, and $H^*(\ ;\mathscr{E})$ is cohomology in that spectrum. If a lifting of f rel h exists, $\Gamma(f) = 0$; this condition is also sufficient if the fiber of p is k-connected and $\dim(K/L) \leqq 2k + 1$.

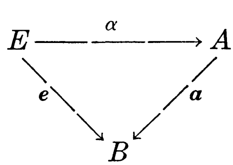
If g_0 and g_1 are liftings of f rel h, a single obstruction $\delta(g_0, g_1; h) \in H(K, L, f: \mathscr{E})$ is also defined; if g_0 and g_1 are connected by a homotopy of liftings of f rel h $\delta(g_0, g_1; h) = 0$; this condition is, also sufficient if p is k-connected and $\dim(K/L) \leqq 2k$.

In § 4, a spectral sequence is constructed for cohomology in a B-spectrum, based on the Postnikov tower of that spectrum, and the relationship between the single obstruction and the classical obstructions is defined.

For similar treatments, see Becker [1], [2], and Meyer [5].

Throughout this paper, let (K, L) be a finite $C. W.$ pair, B any space, and $f: K \to B$ any map. All spaces and maps shall be in the category CG of compactly generated spaces and maps, as described by Steenrod [7], and all constructions (i.e., function spaces, quotient space, Cartesian products) shall be as defined in that paper. When possible without confusion, we shall allow $f \mid L$ and $f \mid K \cup L$ to be denoted simply as F. A map $\pi: X \to Y$ we call a *fibration* if it has a local product structure; the polyhedral covering homotopy extension property [4] is then satisfied.

2. Basic concepts. We define a *B-bundle* to be an ordered pair (E, e) such that $e: E \to B$ is a fibration. A B-bundle map from a B-bundle $e = (E, e)$ to another B-bundle $a = (A, a)$ is defined to be a commutative diagram:



We denote this map $\alpha: e \to a$. A *pointed* B-bundle is an ordered triple $(E, \boldsymbol{e}, e')$ such that $\boldsymbol{e}: E \to B$ is a fibration and $e': B \to E$ is a *pointing*, i.e., $\boldsymbol{e} \circ e' = 1$, the identity on B. We call e' a pointing because it chooses a base-point for each fiber of e. A *bi-pointed* B-bundle is an ordered quadruple $(E, \boldsymbol{e}, e', e'')$ such that $(E, \boldsymbol{e})$ is a B-bundle and e' and e'' are both pointings. If $e = (E, \boldsymbol{e}, e')$ and $a = (A, \boldsymbol{a}, a')$ are pointed B-bundles, a B-bundle map $\alpha: e \to a$ is a *pointed* map if $\alpha \circ e' = a'$. Similarly, we can define bi-pointed maps between bi-pointed bundles. Two bundle maps (or pointed bundle maps, or bi-pointed bundle maps) are said to be homotopic if there exists a homotopy of bundle maps (or pointed bundle maps, or bi-pointed bundle maps) connecting them.

If $e = (E, \boldsymbol{e})$ is a B-bundle, $\boldsymbol{e}^{-1} b$ is called the fiber of e over b, for any $b \in B$. If $e = (E, \boldsymbol{e}, e')$ is a pointed B-bundle, each fiber, $(\boldsymbol{e}^{-1} b, e'b)$ is a pointed space. If $e = (E, \boldsymbol{e}, e', e'')$ is bi-pointed, we say that $e'b$ is the *South pole* of $\boldsymbol{e}^{-1} b$, while $e''b$ is the *North pole*.

Let $\mathscr{H}_B$ be the category of B-bundles and B-bundle maps. Let $\mathscr{H}_B{}^*$ and $\mathscr{H}_B{}^{**}$ be the categories of pointed and bi-pointed B-bundles and maps, respectively. We obviously have forgetful functors $\alpha: \mathscr{H}_B{}^{**} \to \mathscr{H}_B{}^*$ and $\beta: \mathscr{H}_B{}^* \to \mathscr{H}_B$ where $\alpha(E, \boldsymbol{e}, e', e'') = (E, \boldsymbol{e}, e')$ and $\beta(E, \boldsymbol{e}, e') = (E, \boldsymbol{e})$. We shall, whenever convenient, identify any object with its image under α, β, or $\beta \circ \alpha$. We also define functors as follows:

S: $\mathscr{H}_B \to \mathscr{H}_B{}^{**}$ two-point suspension

Σ: $\mathscr{H}_B{}^* \to \mathscr{H}_B{}^*$ one-point suspension

Ω: $\mathscr{H}_B{}^* \to \mathscr{H}_B{}^*$ looping

P: $\mathscr{H}_B{}^{**} \to \mathscr{H}_B$ paths from the South pole to the North pole

$S(E, \boldsymbol{e}) = (S_B E, \boldsymbol{s}, s', s'')$ where $S_B E$ is the quotient space of $E \times I$ obtained by identifying $(x, 0)$ with $(y, 0)$ and $(x, 1)$ with $(y, 1)$ for any $x, y \in \boldsymbol{e}^{-1} b$ for any $b \in B$. For all $[x, t] \in S_B E$, $\boldsymbol{s}[x, t] = \boldsymbol{e}x$, while $s'b = [x, 0]$ and $s''b = [x, 1]$ for all $b \in B$, where x is any element in the fiber of e over b. $\Sigma(E, \boldsymbol{e}, e) = (\Sigma_B E, \boldsymbol{s}, s')$ where $\Sigma_B E$ is the quotient space of $E \times I$ obtained by identifying $(x, 0)$ with $((e' \circ \boldsymbol{e})x, t)$ $(x, 1)$ for any $x \in E$ and any $t \in I$. Then $\boldsymbol{s}[x, t] = \boldsymbol{e}x$ for all $[x, t] \in \Sigma_B E$ and $s'b = [e'b, 0]$ for any $b \in B$.

$\Omega(E, e, e') = (\Omega_B E, \sigma, \sigma')$ where $\Omega_B E$ is the space of all loops in E based on $e'(B)$ which lie in a single fiber of e; $\sigma\alpha = (\boldsymbol{e} \circ \alpha)(0)$ for all $\alpha \in \Omega_B E$, and $(\sigma' b)t = e'b$ for all $b \in B$, and all $t \in I$. $P(E, \boldsymbol{e}, e', e'') = (P_B E, \boldsymbol{p})$ where $P_B E$ is the space of all paths from $e'(B)$ to $e''(B)$ which lie in a single fiber, and $\boldsymbol{p}\alpha = (\boldsymbol{e} \circ \alpha)(0)$ for all $\alpha \in P_B E$.

We give two adjoint constructions. First, let $e = (E, \boldsymbol{e}, e')$ and $a = (A, \boldsymbol{a}, a')$ be two pointed B-bundles. If $\alpha: e \to \Omega a$ and β; $\Sigma e \to a$ are pointed B-bundle maps, we say that α and β are *adjoints* of each other if, for any $x \in E$ and any $t \in I$, $\beta[x, t] = (\alpha x)t$. Second, let $e = (E, \boldsymbol{e})$ be a B-bundle and $a = (A, \boldsymbol{a}, a', a'')$ a bi-pointed B-bundles. We say that maps $\alpha: e \rightarrow Pa$ and $\beta: Se \rightarrow a$ (where β is bi-pointed) are *adjoints* of each other if $\beta[x, t] = (\alpha x)t$ for all $x \in E$ and all $t \in I$.

Let $[K, L, h; e]_f$ denote the set of rel L fiber-homotopy classes of liftings of f to E rel h, where $e = (E, \boldsymbol{e})$ is a B-bundle and $h: L \rightarrow E$ is a lifting of $f \mid L$. If L is empty, write $[K: e]_f$. If $e = (E, \boldsymbol{e}, e')$ is pointed, write $[K, L; e]_f$ for $[K, L, e' \mid L; e]_f$. If $\alpha: e \rightarrow a$ is a B-bundle map, let $\alpha_\sharp: [K, L, h; e]_f \rightarrow [K, L, \alpha \circ h; a]_f$ be the function where $\alpha_\sharp[g] = [\alpha \circ g]$, where $[g]$ is the fiber-homotopy rel L class of any lifting g of f rel h. If $r: (K', L') \rightarrow (K, L)$ is a map of $C.W.$ pairs, let $r^*: [K, L, h; e]_f \rightarrow [K', L, h \circ r; e]_{f \circ r}$ be the function where $r^*[g] = [g \circ r]$. We omit the proof (based in part on the *PCHEP* of $\boldsymbol{e}$) of the following lemma:

LEMMA 2.1. *If $r: (K', L') \rightarrow (K, L)$ is a homotopy equivalence of pairs, then $r^*: [K, L, h; e]_f \cong [K', L', h \circ r; e]_{f \circ r}$.*

Let $e = (E, \boldsymbol{e})$ be a B-bundle. If each fiber of e is connected, we say that e is connected. Similarly, if each fiber of e is n-connected, or n-simple, for some integer $n \geqslant 1$, we say that e is n-connected, or n-simple. If e is n-simple, define $\pi_n e$ to be the local system of Abelian groups over B such that, for every $b \in B$, $(\pi_n e)b = \pi_n(\boldsymbol{e}^{-1}b)$. We call $\pi_n e$ the n^{th} homotopy group system of e. Similarly, if e is pointed, we can define $\pi_n e$ whether e is n-simple or not, since every fiber has a base-point. Note that e is n-connected if and only if e is connected and $\pi_k e = 0$ for all $k \leqslant n$. If $\alpha: e \rightarrow a$ is any B-bundle map, where e and a are both n-simple or both pointed (and α is pointed) or e is pointed and a is n-simple, α induces a homomorphism $\alpha_\sharp: \pi_n e \rightarrow \pi_n a$ in the obvious way.

Let $\alpha: e \rightarrow a$ be any B-bundle map, where $e = (E, \boldsymbol{e})$ and $a = (A, \boldsymbol{a}, a')$. We define the *fiber* of α to be the B-bundle $c = (C, \boldsymbol{c})$ where C is the space of all ordered pairs (x, σ) such that $x \in E$ and σ is a path in A such that $\sigma(0) \in a'(B)$, $\sigma(1) = \alpha x$, and $(\boldsymbol{a} \circ \sigma)t = \boldsymbol{e}x$ for all $t \in I$; and where $\boldsymbol{c}(x, \sigma) = \boldsymbol{e}x$ for all $(x, \sigma) \in C$. If $e = (E, \boldsymbol{e}, e')$ is pointed, then $c'b = (e'b, \sigma)$ gives a pointing of c, where $\sigma t = a'b$ for all $t \in I$. The reader will note that for any $b \in B$, $\boldsymbol{c}^{-1}b$ is precisely the fiber of $\alpha: \boldsymbol{e}^{-1}b \rightarrow \boldsymbol{a}^{-1}b$. The following sequence is thus exact, if $\alpha: e \rightarrow a$ is pointed:

$$\cdots \longrightarrow \pi_n(\Omega e) \xrightarrow{(\Omega\alpha)_\sharp} \pi_n(\Omega a) \xrightarrow{j_\sharp} \pi_n c \xrightarrow{i_\sharp} \pi_n e \xrightarrow{\alpha_\sharp} \pi_n a$$

where $i(x, \sigma) = \sigma(1)$ for all $(x, \sigma) \in C$, and $j(\tau) = (c'b, \tau)$ for all $\tau \in \Omega_B A$, where $b = (\boldsymbol{a}, \tau)(1)$.

Now if α: $e \to a$ is a B-bundle map, we say that α is n-connected for any $n \geqslant 0$ if, for all $b \in B$ and $y \in \boldsymbol{a}^{-1}b$, the space

$$\{(x, \sigma) \in \boldsymbol{e}^{-1}b \times (\boldsymbol{a}^{-1}b)^I\colon \sigma(0) = y, \sigma(1) = \sigma x\}$$

is n-connected. If a is a connected pointed B-bundle, α is connected if and only if the fiber of α is n-connected.

Suppose now that α: $e \to a$ is a B-bundle map. Consider

$$\alpha_{\sharp}\colon [K, L, h; e]_f \longrightarrow [K, L, \alpha \circ h; a]_f \,.$$

LEMMA 2.2. *Suppose α is n-connected for some $n \geqslant 0$. Then:* (i) *$\alpha_{\sharp}$ is onto if* $\dim (K/L) \leqslant n$. (ii) *$\alpha_{\sharp}$ is one-to-one if* $\dim(K/L) \leqslant n - 1$.

Proof. The connectivity of α equals the connectivity of the fiber of α: $E \to A$, considered as a map of spaces. Simple application of ordinary obstruction theory enables us to complete the proof in a routine manner; we omit the details.

Suppose now that g_0, g_1: $K \to E$ are both liftings of f rel h.

LEMMA 2.3. *If α is n-connected for some $n \geqslant 1$, then g_0 and g_1 are homotopic* rel h *if and only if $\alpha \circ g_0$ and $\alpha \circ g_1$ are homotopic,* rel L; *provided* $\dim (K/L) \leqslant n - 1$.

Proof. We have a bi-pointed K-bundle map $f^{-1}\alpha$: $f^{-1} e \to f^{-1} a$, where $f^{-1} e = (f^{-1}E, f^{-1}\boldsymbol{e}, f^{-1}g_0, f^{-1}g_1)$ and

$$f^{-1}a = (f^{-1}A, f^{-1}\boldsymbol{a}, f^{-1}(\alpha \circ g^0), f^{-1}(\alpha \circ g_1)) \;;$$

and $Pf^{-1}\alpha$; $Pf^{-1}e \to Pf^{-1}a$ is $(n-1)$-connected. A section of $Pf^{-1}e$ is equivalent to a fiber homotopy, rel L, of g_0 with g_1, while a section of Pf^{-1} a is equivalent to a fiber homotopy, rel L, of $\alpha \circ g_0$ with $\alpha \circ g_1$. Apply Lemma 2.2, and we are done.

3. B-Spectra. Suppose $e = (E, \boldsymbol{e}, e')$ is a pointed B-bundle. We define an operation "+" on $[K, L, \Omega e]_f$ as follows: for any two liftings of f rel $e' \mid L$, g and g', let $g + g'$: $K \to \Omega_B E$ be the map where $((g+g')x)t = (gx)(2t)$ if $0 \leqq t \leqq 1/2$, $g'(x)(2t-1)$ if $1/2 \leqq t \leqq 1$, for all $x \in K$. Then $g + g'$ is also a lifting of f rel $e' \mid L$. We define $[g] + [g'] = [g+g]'$; it is trivial to verify that the operation is well-defined.

THEOREM 3.1. *$[K, L; \Omega e]_f$ is a group under the operation "+" with identity $[e']$.*

Proof. Let $[g]^{-1} = [g^{-1}]$ for any lifting g of f rel $e' \mid L$, where $(g^{-1}x)t = (gx)(1-t)$ for all $x \in K$ and all $t \in I$; it is routine to check that the group axioms are satisfied.

THEOREM 3.2. *$[K, L; \Omega^2 e]_f$ is an Abelian group.*

Proof. We omit the details; if g and g' are both liftings of f rel $e' \mid L$, a fiber homotopy rel L of $g + g'$ with $g' + g$ can easily be constructed in the same manner as the proof that $[X; \Omega^2 Y]$ is Abelian for pointed spaces X and Y, but the construction is done fiberwise over B.

DEFINITION 3.1. A B-spectrum is an ordered pair

$$\mathscr{E} = (\{e_i\}_{i \geqq m}, \{\varepsilon_i\}_{i \geqq m})$$

for some integer m such that:

(i) For each $i \geqq m$, e_i is a pointed B-bundle.

(ii) For each $i \geqq m$, $\varepsilon_i \colon e_i \to e_{i+1}$ is a pointed B-bundle map.

Furthermore, we say that $\mathscr{E}$ is a Ω_B-spectrum if ε_i is a homotopy equivalence (in the category $\mathscr{K}_B{}^*$) for each i, and we say that ε is a weak Ω_B-spectrum if ε_i is infinitely connected for all $i \geqq m$. We say that ε is *stabilizing* if, for each integer n, there exists an integer $N \geqq m$ such that ε_i is $(n+i)$-connected for all $i \geqq N$. The e_i are called the elements of the spectrum, the ε_i are called the connection maps, and m is called the starting value. If the first finitely many elements of a spectrum are altered, no change occurs in cohomology with coefficients in that spectrum; in that sense, the starting value is arbitrary. We define the homotopy of a spectrum $\pi_n(\mathscr{E})$ for any integer n, to be the direct limit $\mathrm{Lim}_{i \to \infty} \pi_{n+i} e_i$, under the system of homomorphisms

$$(\varepsilon)_{i\#} \colon \pi_{n+i} e_i \longrightarrow \pi_{n+i} \Omega e_{i+1} \cong \pi_{n+i+1} e_{i+1}$$

thus $\pi_n(\mathscr{E})$ is a local system of Abelian groups on B. Note that $\pi_n(\mathscr{E})$ need not be zero for negative values of n.

Henceforth, we shall assume that $\mathscr{E} = (\{e_i\}_{i \geqq m}, \{\varepsilon_i\}_{i \geqq m})$ is a B-spectrum.

DEFINITION 3.2. For any integer n, let $H^n(K, L, f; \mathscr{E})$ be the direct limit of the system of groups $\{[K, L; \Omega^{i-n} e_i]_f\}$ and homomorphisms $\{(\Omega^{i-n} \varepsilon_i)_\#\}$. (If L is empty, we write $H^n(K, f; \mathscr{E})$.) For any $i \geqq \min(n, m)$, let

$$[K, L; \Omega^{i-n} e_i]_f \longrightarrow H^n(K, L, f; \mathscr{E})$$

be called the representation. If $\mathscr{E}$ is stabilizing, the direct limit is achieved eventually, i.e., beyond some point, all representations are bijective; if $\mathscr{E}$ is a weak Ω_B-spectrum, the direct limit is achieved immediately, i.e., all representations are bijective. We call $H^*(K, L, f; \mathscr{E})$ the cohomology of the triple (K, L, f) with coefficients in the spectrum $\mathscr{E}$. If (K', L') is another $C.W.$ pair, and

$$r: (K', L') \longrightarrow (K, L)$$

is a map, an induced homomorphism

$$r^*: H^*(K, L, f; \mathscr{E}) \longrightarrow H^*(K', L', f \circ r; \mathscr{E})$$

can be defined in the obvious way.

Henceforth, let (K'', L'') be the pair $(K \times \{1\} \cup L \times I, L \times \{0\})$, and let p; $(K'', L'') \to (K, L)$ be projection onto the first factor. The reader can easily verify that p is a relative homotopy equivalence, and hence by the direct limit version of Lemma 2.1,

$$p^*: H^*(K, L, f; \mathscr{E}) \longrightarrow H^*(K'', L'', f \circ p; \mathscr{E})$$

is an isomorphism.

For any integer n, we define a connecting homomorphism

$$\delta: H^n(L, f; \mathscr{E}) \longrightarrow H^{n+1}(K, L, f; \mathscr{E})$$

as follows. For any $\alpha \in H^n(L, f; \mathscr{E})$, pick $i \geqq m$ and $[g] \in [L; \Omega^{i-n}e_i]_f$ representing α. Consider $\Omega^{i-n}e_i = \Omega\Omega^{i-n-1}e_i$. Let $p^*\delta\alpha$ be the image, in the direct limit, of $[G] \in [K'', L''; \Omega^{i-n-1}e_i]_{f\circ p}$, where $G(x, t) = (gx)t$ for all $x \in L$ and $t \in I$, and where $G(x, 1) = a'(fx)$ for all $x \in K$, where a' is the pointing of $\Omega^{i-n-1}e_i$; $\delta\alpha$ is well-defined since p^* is an isomorphisms.

The following remarks (analogous to some of the Eilenberg Steenrod axioms for a cohomology theory [3]) we state without proof:

REMARK 3.3. The following long sequence is exact, where i and j are inclusions:

$$\cdots \longrightarrow H^{n-1}(L, f; \mathscr{E}) \xrightarrow{\delta} H^n(K, L, f; \mathscr{E}) \xrightarrow{j^*} H^n(K, f; \mathscr{E})$$
$$\xrightarrow{i^*} H^n(L, f; \mathscr{E}) \xrightarrow{\delta} H^{n+1}(K, L, ; \mathscr{E}) \longrightarrow \cdots .$$

REMARK 3.5. If $r_t: (K', L') \to (K, L)$, $0 \leqq t \leqq 1$, is a homotopy of maps, where (K', L') is another $C.W.$ pair, such that $f \circ r_t = f \circ r_0$ for all t, then $r_1^* = r_0^*$.

Suppose now that $f_t: K \to B$, $0 \leqq t \leqq 1$, is a homotopy such that $f_0 = f$. Let $F: K \times I \to B$ be the map where $F(x, t) = f_t x$ for all

$(x, t) \in K \times I$. Let $i_0, i_1\colon (K, L) \to (K \times I, L \times I)$ be the inclusions along 0 and 1, respectively. According to Lemma 2.1., $(i_j)_\#$ is an isomorphism for $j = 0$ or 1. Let

$$F_\# = (i_1)_\# \circ (i_0)_\#^{-1}\colon H^*(K, L, f; \mathscr{E}) \longrightarrow H^*(K, L, f; \mathscr{E}) ,$$

clearly an isomorphism. Again without proof, we state:

REMARK 3.6. $F_\#$ depends only on the homotopy class of F, rel $K \times \{0, 1\}$.

REMARK 3.7. If G is a homotopy of f_1 with f_2, then

$$G_\# \circ F_\# = (F+G)_\#\colon H^*(K, L, f; \mathscr{E}) \longrightarrow H^*(K, L, f_2; \mathscr{E})$$

where $(F+G)(x, t) = F(x, 2t)$ if $0 \leqq t \leqq 1/2$; $G(x, 2t)$ if $1/2 \leqq t \leqq 1$, for all $x \in K$.

An immediate question one may ask is: if $f_1 = f$, is $F_\#$ the identity? The answer is generally no.

4. The associated spectrum and the single obstruction.

Let $e = (E, e)$ be a B-bundle and $h\colon L \to E$ a lifting of $f \mid L$. Let

$$\mathscr{E} = \mathscr{E}(e) = (\{e_i\}_{i \geqq 1}, \{\varepsilon_i\}_{i \geqq 1})$$

be the B-spectrum where $e_i = \sum^{i-1} Se$ for all $i \geqq 1$, and $\varepsilon_i\colon e_i \to \Omega e_{i+1}$ is adjoint to the identity on $e_{i+1} = \sum e_i$. We call $\mathscr{E}$ the B-spectrum associated to e. We shall write $e_1 = Se = (S_B E, \mathbf{s}, s', s'')$.

Recall $(K'', L'') = (K \times \{1\} \cup L \times I, L \cup \{0\})$. We define $\Gamma(f; h) \in H^1(K, L, f; \mathscr{E})$ (or simply $\Gamma(f)$ when L is empty, or when h is understood), the *single obstruction to lifting f rel h*, to be $(p^*)^{-1}$ of the representation of $[H] \in [K'', L''; Se]_{f \circ p}$, where $H\colon K'' \to S_B E$ is the map such that $H(x, t) = [hx, t]$ for all $(x, t) \in L \times I$, and $H(x, 1) = (e'' \circ f)x$, the North pole of $\mathbf{e}^{-1}fx$, for all $x \in K$. We leave it to the reader to verify that if $f_t\colon K \to B$, for $0 \leqq t \leqq 1$, is a homotopy, and if $h_t\colon L \to E$ is a homotopy such that $\mathbf{e} \circ h_t = f_t \mid L$ for all t, and if $F(x, t) = f_t x$ for all $(x, t) \in K \times I$, then $F_\# \Gamma(f_0; h_0) = \Gamma(f_1; h_1)$; i.e., $\Gamma(f; h)$ is a homotopy invariant.

THEOREM 4.2. *If f has a lifting to E* rel h, $\Gamma(f; h) = 0$.

Proof. Let $g\colon K \to E$ be such a lifting. Let $H_u\colon K'' \to S_B E$, for $0 \leqq u \leqq 1$, be the rel L'' lifting of $f \circ p$ where $H_u(x, t) = [gx, tu]$ for all $0 \leqq t, u \leqq 1$. Then $H_1 = H$, while $H_0 = s' \circ f \circ p$, and we are done.

THEOREM 4.3. *If e is $(n-1)$-connected for some $n \geqq 1$, and if*

$\dim(K/L) \leqq 2n-1$, *then* f *has a lifting to* E rel h *if and only if* $\Gamma(f; h) = 0$.

Proof. "Only if" is the previous theorem. Suppose then that $\Gamma(f; h) = 0$. Without loss of generality, we may assume that L has empty interior, whence $\dim K'' \leqq 2n-1$. By a Serre spectral sequence argument, $(\Omega^{i-1}\varepsilon_i)\colon \Omega^{i-1}e_i \to \Omega^i e_{i+1}$ is $(2n+i-1)$-connected for all $i \geqq 1$, whence, by Lemma 2.2, the representation

$$[K'', L''; e_1]_{f\circ p} \longrightarrow H^1(K'', L'', f\circ p; \mathscr{E})$$

is one-to-one and onto. Thus $[H] = [s'\circ f\circ p]$. Let $H_t\colon K'' \to S_B E$ be a fiber-homotopy rel L'' such that $H_1 = H$ and $H_0 = s'\circ f\circ p$; define $G\colon K'' \to P_B S_B E$ to be the map where $(Gy)u = H_u y$ for all $y \in K''$. Let $i\colon e \to PSe$ be adjoint to the identity on $Se = e_1$. Again, by a Serre spectral sequence argument, i is $(2n-2)$-connected. Since $[K'', L'', i\circ h\colon PSe]_{f\circ p}$ is nonempty, $[K, L, h; e]_f$ is nonempty by Lemmas 2.1 and 2.2, and we are done.

Suppose now that $f_0, g_1\colon K \to E$ are liftings of f rel h. We define $\Delta(g_0, g_1; h) \in H^0(K, L, f; \mathscr{E})$, the single obstruction to fiber homotopy, rel L, of g_0 with g_1, to be $(p^*)^{-1}$ of the representation in $H^0(K'', L'', f\circ p; \mathscr{E})$ of $[G] \in [K'', L''; \Omega Se]_{f\circ p}$, where for all (x, t) K'' and all $0 \leqq u \leqq 1$:

$$G(x,t)u = \begin{cases} [g_1 x, 2u] \text{ if } t = 0 \text{ and } 0 \leqq u \leqq 1/2 \\ [g_0 x, 2-2u] \text{ if } t = 0 \text{ and } 1/2 \leqq u \leqq 1 \\ [hx, 2u(1-t)] \text{ if } x \in L \text{ and } 0 \leqq u \leqq 1/2 \\ [hx, (2-2u)(1-)] \text{ if } x \in L \text{ and } 1/2 \leqq u \leqq 1\,. \end{cases}$$

We leave it to the reader to check that $\Delta(g_0, g_1; h)$ is a homotopy invariant in the same sense that $\Gamma(f; h)$ is.

Hence forth, we shall write $\Omega Se = (\Omega_B S_B E, \mathbf{c}, c')$.

THEOREM 4.4. *If* g_0 *and* g_1 *are fiber-homotopic* rel h, *then* $\Delta(g_0, g_1; h) = 0$.

Proof. Let g_t be a fiber homotopy rel L. Let $G_v\colon K'' \to \Omega_B S_B E$, $0 \leqq v \leqq 1$, be the rel L'' fiber homotopy, where for all $0 \leqq u, v \leqq 1$:

$$G_v(x,t)u = \begin{cases} [g_{2v-1}x, 2u] \text{ if } t = 1,\ 0 \leqq u \leqq 1/2, \text{ and } 1/2 \leqq v \leqq 1\,. \\ [g_0 x, 2-2u] \text{ if } t = 1,\ 1/2 \leqq u \leqq 1, \text{ and } 1/2 \leqq v \leqq 1\,. \\ [hx, 2u(1-t)] \text{ if } x \in L,\ 0 \leqq u \leqq 1/2, \text{ and } 1/2 \leqq v \leqq 1\,. \\ [hx, (2-2u)(1-t)] \text{ if } x \in L,\ 1/2 \leqq u \leqq 1, \text{ and } 1/2 \leqq v \leqq 1\,. \\ [g_0 x, 4uv(1-t)] \text{ if } 0 \leqq u \leqq 1/2 \text{ and } 0 \leqq v \leqq 1/2\,. \\ [g_0 x, 4(1-u)v(1-t)] \text{ if } 1/2 \leqq u \leqq 1 \text{ and } 0 \leqq v \leqq 1/2\,. \end{cases}$$

Note that $G_1 = G$ and $G_0 = c' \circ f \circ p$, and we are done.

THEOREM 4.5. *If e is $(n-1)$-connected for some $n \geqq 1$, and if* $\dim (K/L) \leqq 2n-2$, *then g_0 and g_1 are fiber homotopic if and only if* $\Delta(g_0, g_1; h) = 0$.

Proof. "Only if" is the previous theorem. Suppose, then, that $\Delta(g_0, g_1; h) = 0$. Then G is fiber homotopic, $\operatorname{rel} L''$, to c', since by Lemma 2.2, $[K'', L''; \Omega Se]_{f \circ p} \to H^0(K'', L'', f \circ p; \mathscr{E})$ is onto. A routine argument using Lemma 2.1 then shows that $i \circ g_0$ is fiber homotopic, $\operatorname{rel} i \circ h$, to $i \circ g_1$, where $i\colon e \to PSe$ is adjoint to the identity on Se. Our result follows immediately from Lemma 2.3.

THEOREM 4.6. *If g is any lifting of f* rel h, *and if* $d \in H^0(K, L, f; \mathscr{E})$, *then there exists some lifting g' of f* rel h, *such that $\Delta(g, g'; h) = d$, provided e is $(n-1)$-connected for some $n \geqq 1$ and* $\dim (K/L) \leqq 2n - 1$.

Proof. The representation $[K, L; \Omega Se]_f \to H^0(K, L, f; \mathscr{E})$ is onto by Lemma 2.2; pick a lifting, H, of f $\operatorname{rel} c^0 \circ f \mid L$ which represents d. Let s be the lifting of f to $P_B S_B E$:

$$(sx)t = \begin{cases} (Hx)(2t) & \text{if } 0 \leqq t \leqq 1/2 \\ ((i \circ g)x)(2t-1) & \text{if } 1/2 \leqq t \leqq 1 \end{cases}$$

where $i\colon e \to PSe$ is adjoint to the identity map of Se. Now by the *PCHEP* of PSe, s is fiber homotopic to a lifting s' where $s \mid L' = i \circ h$. Now $i_\sharp\colon [K, L, h; e]_f \to [K, L, i \circ h; PSe]_f$ is onto by Lemma 2.2. Choose g' to be any $\operatorname{rel} h$ lifting of f such that $i_\sharp[g'] = [s']$. We leave it to the reader to verify that $\Delta(g, g'; h) = d$.

The proof of the next theorem we omit; it is a routine homotopy argument of the type the reader should by now be familiar with.

THEOREM 4.7. *If g_0, g_1, and g_2 are liftings of f* rel h, *then*

$$\Delta(g_0, g_2; h) = \Delta(g_0, g_1; h) + \Delta(g_1, g_2; h) .$$

COROLLARY 4.8. (Becker) *If e is $(n-1)$-connected for some $n \geqq 1$, and if* $\dim(K/L) \leqq 2n - 2$, *then $[K, L, h; e]_f$ has the structure of an affine group, and, if nonempty, is isomorphic to $H^0(K, L, f; \mathscr{E})$.*

Proof. See Becker [1] for the definition of an affine group. Pick any $[g_0] \in [K, L, h; e]_f$. Let $\iota\colon [K, L, h; e]_f \to H^0(K, L, f; \mathscr{E})$ be given by $\iota[g] = \Delta(g_0, g; h)$. This function is well-defined, one-to-one, and onto, and induces an affine group structure on $[K, L, h; e]_f$ which is independent of the choice of g_0, by Theorems 4.4, 4.5, 4.6, and 4.7. We leave the details to the reader.

5. B-spectrum maps and a spectral sequence for $H^*(K, L, f; \mathscr{E})$. Let $\mathscr{E} = (\{e_i\}_{i \geqslant m}, \{\varepsilon_i\})$ and $\mathscr{A} = (\{a_i\}_{i \geqslant n}, \{\alpha_i\})$ be B-spectra. We define a *B-spectrum map* $\mathscr{f}\colon \mathscr{E} \to \mathscr{A}$ of *degree* d to be an indexed collection $\{f_i\}_{i \geqslant p}$ of pointed B-bundle maps, where $p \geqslant \max(m, n-d)$, such that for any $i \geqslant p$, $f_i\colon e_i \to a_{i+d}$ and the following diagram is commutative:

$$\begin{array}{ccc} e_i & \xrightarrow{\varepsilon_i} & e_{i+1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} \\ a_{i+d} & \xrightarrow{\alpha_{i+d}} & a_{i+d+1} \end{array} .$$

We can define $\mathscr{f}_\sharp\colon H^k(K, L, f; \mathscr{E}) \to H^{k+d}(K, L, f; \mathscr{A})$ for any integer k to be the direct limit of the $(f_i)_\sharp$; similarly we can define

$$\mathscr{f}_\sharp\colon \pi_k(\mathscr{E}) \longrightarrow \pi_{k-d}(\mathscr{A})$$

for any integer k.

Let $\mathscr{D} = (\{d_i\}_{i \geqslant p}, \{\delta_i\})$ be the *fiber* of $\mathscr{f}$, defined as follows. For any $i \geqslant p$, $d_i = (D_i, \boldsymbol{d}_i, d_i')$ where

$$\begin{aligned} D_i = \{(x, \sigma) \in E_i \times A^I_{i+d}\colon \sigma(0) &= (a'_{i+d} \circ e_i)x, \sigma(1) \\ &= f_i x, \,\&\, a_{i+d}(\sigma t) = e_i x \text{ for all } t \in I\} , \end{aligned}$$

$\boldsymbol{d}_i(x, \sigma) = e_i x$ for all $(x, \sigma) \in D_i$ and $d_i' b = (e_i' b, \langle b \rangle)$ for all $b \in B$, where $\langle b \rangle t = a'_{i+d} b$ for all $t \in I$. Let $\delta_i\colon d^i \to \Omega d_{i+1}$ be defined as follows: For any $(x, \sigma) \in D_i$ and any $t \in I$, $(\delta_i(x, \sigma))t = ((\varepsilon_i x)t, \tau)$, where $\tau u = (\alpha_{i+d}(\sigma u))t$ for all $u \in I$. Consider the sequence of B-spectra and B-spectrum maps (called the fibration sequence of $\mathscr{f}$):

(5-1)
$$\mathscr{A} \xrightarrow{\mathscr{h}} \mathscr{D} \xrightarrow{\mathscr{g}} \mathscr{E} \xrightarrow{\mathscr{f}} \mathscr{A}$$

where $\mathscr{g} = \{g_i\}_{i \geqq p}$ has degree 0 and $\mathscr{h} = \{h_i\}_{i \geqq p+d-1}$ has degree $-d+1$; defined as follows: For any $(x, \sigma) \in D_i$, $h_i(x, \sigma) = x$; and for any $y \in A_i$, $g_i y = ((e'_{i-d+1} \circ a_i)y, \alpha_i y)$. The sequence (5-1) is analogous to the fibration sequence for any map of pointed spaces (where F is the fiber of f):

$$Y \longrightarrow F \longrightarrow X \xrightarrow{f} Y .$$

As in that case, we may, in a straightforward manner, verify the exactness of the long sequences:

$$\cdots \longrightarrow \pi_{k-d+1}(\mathscr{A}) \xrightarrow{\mathscr{h}_\#} \pi_k(\mathscr{D}) \xrightarrow{\mathscr{g}_\#} \pi_k(\mathscr{E}) \xrightarrow{\mathscr{f}_\#} \pi_{k-d}(\mathscr{A}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{k+d-1}(K, L, f; \mathscr{A}) \xrightarrow{\mathscr{h}_\#} H^k(K, L, f; \mathscr{D}) \xrightarrow{\mathscr{g}_\#} H^k(K, L, f; \mathscr{E})$$
$$\xrightarrow{\mathscr{f}_\#} H^{k+d}(K, L, f; \mathscr{A}) \longrightarrow \cdots .$$

We say that $\mathscr{f}: \mathscr{E} \to \mathscr{A}$ is k-connected if $\mathscr{D}$ is k-connected, and we say that $\mathscr{f}$ is k-coconnected if $\mathscr{D}$ is k-coconnected, i.e., $\pi_r(\mathscr{D}) = 0$ for all $r \geqq k$.

Henceforth in this section, let $\mathscr{E} = (\{e_i\}_{i \geqq m}, \{\varepsilon_i\})$ be a B-spectrum. We define a *resolution* of $\mathscr{E}$ to be a commutative diagram of B-spectra, where each map has degree 0:

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathscr{E}_{k+1} & \xrightarrow{\mathscr{g}_{k+1}} & \mathscr{E}_k & \xrightarrow{\mathscr{g}_k} & \mathscr{E}_{k-1} & \longrightarrow \cdots \\ & \nwarrow {\scriptstyle \mathscr{f}_{k+1}} & & \uparrow {\scriptstyle \mathscr{f}_k} & & \nearrow {\scriptstyle \mathscr{f}_{k-1}} & \\ & & & \mathscr{E} & & & \end{array}$$

such that for any integer r, there exists an integer N such that $\mathscr{f}_k$ is r-connected for all $k \geqq N$, and an integer M such that $\mathscr{E}_k$ is r-coconnected for all $k \leqq M$. We are thus assured that $H^*(K, L, f: \mathscr{E})$ is isomorphic to the inverse limit $\mathrm{Lim}_{k \to \infty} H(K, L, f; \mathscr{E}_k)$ under the homomorphisms $(\mathscr{g}_k)_\#$. An important special case of a resolution of $\mathscr{E}$ is a Postnikov resolution: that is where $(\mathscr{f}_k)_\#: \pi_r(\mathscr{E}) \to \pi_r(\mathscr{E}_k)$ is an isomorphism for all $r \leqq k$, and where each $\mathscr{E}_k$ is $(k+1)$-coconnected. In § 6, we shall show that every B-spectrum has a Postnikov resolution.

Using a resolution of $\mathscr{E}$, (5-2), we construct a spectral sequence for $H^*(K, L, f; \mathscr{E})$. For any integer r, we have a filtration of $H^r(K, L, f; \mathscr{E})$:

$$0 \subset \cdots \subset G^{r+q,q} \subset G^{r+q-1,q-1} \subset \cdots H^r(K, L, f; \mathscr{E})$$

where $G^{p,q}$ is the kernel of

$$(\mathscr{f}_q)_\#: H^{p-q}(K, L, f; \mathscr{E}) \longrightarrow H^{p-q}(K, L, f: \mathscr{E}_q) .$$

(The conditions that $\mathscr{f}_k$ is highly connected for large k and $\mathscr{E}_k$ is highly coconnected for small k insures that the filtration has only finitely many distinct terms.) For any k, consider the fibration sequence of $\mathscr{g}_k$:

$$\mathscr{E}_{k-1} \xrightarrow{i_k} \mathscr{K}_k \xrightarrow{j_k} \mathscr{E}_k \xrightarrow{p_k} \mathscr{E}_{k-1} .$$

Recall that j_k and p_k have degree 0, and i_k has degree 1. For any integers p and q, define $E_2^{p,q} = H^{p-q}(K, L, f; \mathscr{K}_q)$ and

$$D_2^{p,q} = H^{p-q}(K, L, f; \mathscr{E}_q) .$$

Let $(p_q)_\# = i_2\colon D_2^{p,q} \to D_2^{p-1,q-1}$, $(i_{q+1})_\# = j_2\colon D_2^{p,q} \to E_2^{p+2,q+1}$, and

$$(j_q)_\# = k_2\colon E_2^{p,q} = \longrightarrow D_2^{p,q} .$$

Using general spectral sequence arguments, we can verify that

$$d_r\colon E_2^{p,q} \longrightarrow E_2^{p+r,q+r-1} \qquad \text{for all } r \geqq 2 ,$$

and that $E_\infty^{p,q} = G^{p-1,q-1}/G^{p,q}$ for all p and q.

In the special case that (5-2) is a Postnikov resolution, we can construct an E_1 term of the spectral sequence as follows. Let K^r be the r-skeleton of K, for any r: $K^r = \varnothing$ if $r < 0$. For any p and q, let $D_1^{p,q} = H^{p,q}(K^p \cup L, f; \mathscr{E})$ and $E_1^{p,q} = C^p(K, L, f^{-1}\pi_q(\mathscr{E}))$, the group of cochains with coefficients in the local system $f^{-1}\pi_q(\mathscr{E})$ over K. Let $i_1\colon D_1^{p,q} \to D_1^{p-1,q-1}$ and $k_1\colon E_1^{p,q} \to D_1^{p,q}$ be the homomorphisms induced by the appropriate inclusions, and let $j_1\colon D_1^{p,q} \to E_1^{p+1,q}$ be the connecting homomorphism of the pair $(K^{p+1} \cup L, K^p \cup L)$. The differential $d_1\colon C^p(K, L; f^{-1}\pi_q(\mathscr{E})) \to C^{p+1}(K, L; f^{-1}\pi_q(\mathscr{E}))$ is then the usual coboundary on cochains with local coefficients, hence

$$E_2^{p,q} = H^p(K, L; f^{-1}\pi_q(\mathscr{E})) .$$

We leave the rather routine verification that the above E_1, D_1, i_1, j_1, and k_1 yield the correct E_2, D_2, etc., to the reader. (Hint: If $\mathscr{E}$ is k-connected, $H^p(K, L, f; \mathscr{E}) = 0$ for all $p \geqq n - k$, where $n = \dim(K/L)$.)

We now explore the relation between the single obstruction and the classical obstructions. Let us suppose that $e = (E, e)$ is a k-connected B-bundle, for some $k \geqq 1$, and that diagram (5-2) is a Postnikov system for $\mathscr{E} = \mathscr{E}(e)$. For any integer r, let $\iota_r\colon \pi_r e \to \pi_r(\mathscr{E})$ be the composition

$$\pi_r e \longrightarrow \pi_r PSe \cong \pi_r \Omega Se \cong \pi_{r+1} e_1 \longrightarrow \pi_r(\mathscr{E}) ,$$

an isomorphism if $r \leqq 2k$. Now suppose that $f \mid K^m \cap L$ has a rel h lifting, g^m, for some integer m. Then

$$i^*\Gamma(f, h) = \Gamma(f \mid K^m \cup L; h) = 0$$

by Theorem 4.2. Consider the commutative diagram of groups and homomorphisms:

$$
\begin{array}{ccccc}
H^1(K, L, f; \mathscr{K}_m) & \xrightarrow{(\jmath_m)_\#} & H^1(K, L, f; \mathscr{E}_m) & \xrightarrow{(p_m)_\#} & H^1(K, L, f; \mathscr{E}) \\
\updownarrow = & & \downarrow (q_m)_\# & \swarrow (p_{m-1})_\# & \\
H^{k+1}(K, L, f^{-1}\pi_m(\mathscr{E})) & & H^1(K, L, f; \mathscr{E}_{m-1}) & & \\
\uparrow (\iota_m)_\# & & & & \\
H^{k+1}(K, L; f^{-1}\pi_m e) \; . & & & &
\end{array}
$$

Since $\mathscr{E}_{m-1}$ is m-coconnected,

$$i^*: H^1(K, L, f; \mathscr{E}_{m-1}) \longrightarrow H^1(K^m \cup L, L, f; \mathscr{E}_{m-1})$$

is an isomorphism. Thus $(p_{m-1})_\# \Gamma(f; h) = 0$. Since $\mathscr{K}_m$ is the fiber of q_m, $(p_m)_\# \Gamma(f; h) \in (\jmath_m)_\# H^1(K, L, \mathscr{K}_m)$. The classical obstruction to extending g^m over $K^{m+1} \cup L$, $\gamma(g^m) \in H^{k+1}(K, L; f^{-1}\pi_m e)$ up to some indeterminacy. It is a routine matter of checking definitions to verify that $(\jmath_m)_\#(\iota_m)_\#\gamma(g^m) = (p_m)_\# \Gamma(f; h)$.

6. Construction of the Postnikov resolution of $\mathscr{E}$.

For every integer, n, we define a functor K_n: $\mathscr{H}_B^* \longrightarrow \mathscr{H}_B^*$ as follows. If $n < 0$, let K_n be the identity. Otherwise, if $e = (E, e, e')$ is a pointed B-bundle, let B^{n+1} be a (topological) $(n+1)$-ball with boundary S^n and basepoint $* \in S^n$. Let $E_B^{S^n}$ be the space of all continuous maps h: $S^n \longrightarrow E$ such that $h(*) \in e'(B)$ and $e \circ h$ is constant. Let ε: $E_B^{S^n} \longrightarrow E$ be the evaluation map, and let $(K_n)_B E = E \cup_\varepsilon (E_B^{S^n} \times B^{n+1})$. We define $K_n e$ to be the pointed B-bundle $((K_n)_B E, k, k')$, where $k' = e'$, $k \mid E = e$, and $k(h, b) = (e \circ h)(*)$ for all $(h, b) \in (E_B^{S^n} \times B^{n+1})$. If α: $e \longrightarrow a$ is any pointed B-bundle map, we define $K_n\alpha$: $K_n e \longrightarrow K_n a$ in the obvious way: $K_n \alpha \mid = \alpha$, and $(K_n\alpha)(h, b) = (\alpha \circ h, b)$ for all $(h, b) \in E_B^{S^n} \times B^{n+1}$. A very simple homotopy argument shows:

REMARK 6.1. (i) For all $k < n$, $i_\#$: $\pi_k e \longrightarrow \pi_k(K_k e)$ is an isomorphism, where i: $e \longrightarrow K_n e$ is the inclusion. (ii) $\pi_n(K_n e) = 0$.

We define functors K_n^r: $\mathscr{H}_B^* \longrightarrow \mathscr{H}_B^*$ for all integers $n \leqq r$, inductively, as follows: $K_n^n = K_n$, and $K_n^{r+1} = K_{r+1}K_n^r$ for all $n \leqq r$. It is very simple to see that the "union" $\bigcup_{r=n}^\infty K_n^r$ is also a functor, which we call K_n^∞: $\mathscr{H}_B^* \longrightarrow \mathscr{H}_B^*$. We call K_n, K_n^r, and K_n^∞ *homotopy-killing* functors. The following remark is an immediate Corollary of 6.1:

REMARK 6.2. (i) $i_\#: \pi_k e \to \pi_k(K_n^\infty e)$ is an isomorphism for all $k < n$, where $i: e \to K_n e$ is the inclusion. (ii) $\pi_k(K_n e) = 0$ for all $k \geqq n$.

Thus K_n^∞ is the analogue of the $(n-1)^{\text{th}}$ stage in the Postnikov tower of a space. In order to pass to spectra, we must examine the relationship between the homotopy-killing functors and the looping functor. We define a pointed B-bundle map $T_n: K_n \Omega e \to \Omega K_{n+1} e$ for all integers n as follows: If $n \leqq -2$, T_n is the identity. If $n = -1$, $T_n = \Omega i: \Omega e \to \Omega K_0 e$, where $i: e \to K_0 e$ is the inclusion. Otherwise, let $T_n: \Omega_B E \cup_\varepsilon ((\Omega_B E)^{S^n} \times B^{n+1}) \to \Omega_B(E \cup_\varepsilon (E_B^{S^{n+1}} \times B^{n+1}))$ be the identity on $\Omega_B E$, and for any $(h, b) \in (\Omega_B E)_B^{S^n} \times B^{n+1}$, and any $t \in I$, let $(T_n(h, b))t = (h, [b, t])$. Note: $B^{n+2} = \sum B^{n+1}$ and $(\Omega_B E)_B^{S^n} = E_B^{S^{n+1}}$. We leave it to the reader to verify that $(T_n)_\#: \pi_k(K_n \Omega e) \to \pi_k(\Omega K_{n+1} e)$ is an isomorphism for all $k \leqq n$.

Similarly, we define $T_n^r: K_n^r \Omega e \to K_{n+1}^{r+1} e$ inductively for all $n \leqq r$ as follows: $T_n^n = T_n$, and $T_n^{r+1} = T_{r+1} \circ (K_{r+1} T_n^r)$ for all $r \geqq n$. In an obvious way we can then define $T_n: K_n^\infty \Omega e \to \Omega K_{n+1}^\infty e$. We leave the proof of the following to the reader:

REMARK 6.3. The B-bundle map $T_n: K_n^\infty \Omega e \to \Omega K_{n+1}^\infty e$ is a weak homotopy equivalence.

We are now ready to define the Postnikov resolution of B-spectrum $\mathscr{E} = (\{e_i\}_{i \geqq m}, \{\varepsilon_i\})$. For each integer n, let

$$\mathscr{E}_n = (\{K_{n+i+1}^\infty e_i\}_{i \geqq m}, \{T_{n+i+1}^\infty \circ (K_{n+i+1} \varepsilon_i)\}) .$$

Let $\not p_n: \mathscr{E} \to \mathscr{E}_n = \{p_i\}_{i \geqq m}$, where $p_i: e_i \to K_{n+i+1} e_i$ is the inclusion, and let $\not q_n: \mathscr{E}_n \to \mathscr{E}_{n-1} = \{q_{n,i}\}_{i \geqq m}$, where $q_{n,i} = K_{n+i+1}^\infty j: K_{n+i+1}^\infty e_i \to K_{n-i+1}^\infty e_i$, where $j: e_i \to K_{n+i} e_i$ is the inclusion. The resolution of $\mathscr{E}$ described above (see diagram (5-2)) is a Postnikov resolution, by Remarks 6.2 and 6.3.

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