SHOCK SETS FOR FIRST ORDER NONLINEAR HYPERBOLIC EQUATIONS

Donald Pollard Ballou

SHOCK SETS FOR FIRST ORDER NONLINEAR HYPERBOLIC EQUATIONS

DONALD P. BALLOU

This paper studies those values u which can be reached from u_0 across a shock or contact discontinuity which may arise in weak solutions to the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0$$
$$u(0, x) = u^0(x) ,$$

where it is not assumed that $F'' > 0$. A condition which guarantees uniqueness is used to define a set $S(u_0)$ consisting of such values u. The close relationship between $S(u_0)$ and the geometry of $F(u)$ is examined, and properties of this set are derived which give information about the interaction of curves of discontinuity.

1. **Introduction.** In this paper we shall study the discontinuities which arise in weak solutions to the Cauchy problem

(1) $$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 ,$$

(2) $$u(0, x) = u^0(x) ,$$

where $u^0(x)$ is a preassigned, bounded, measurable function. The purpose is to relate the values of a solution to the left and right of a discontinuity to the geometry of $F(u)$. To do this we require that
  (i) $F \in C^2(R^1)$,
  (ii) the inflection points of F are isolated.
The second condition is employed to avoid pathological behavior such as is found in Ballou [2].

As is well known (cf. Oleinik [4]), in order to obtain uniqueness it is necessary to impose some condition on the weak solutions. One such condition is due to Volpert (cf. [6]) and it gives uniqueness in the class BV of functions whose generalized derivatives are locally measures. As will be shown in §2, whenever the weak solution is piecewise smooth, this is equivalent to Condition E, which guarantees that characteristics cannot diverge from a curve of discontinuity. In §3 by using Condition E we define for each value u_0 of u a set $S(u_0)$, known as the shock set, which contains those values of u that can be reached from u_0 by means of either a shock or a contact discontinuity. The close connection between the properties of these

sets, such as compactness, and the geometry of F is then examined. Also, using the shock sets, conditions are given which guarantee that the interaction of two curves of discontinuity produces exactly one curve of discontinuity. (From [2] it is known that a jump in the initial data can give rise to an infinite number of discontinuities.)

We now give the definitions of some basic concepts.

DEFINITION. A *weak solution* to the Cauchy problem (1), (2) is a bounded, measurable function $u(t, x)$ which satisfies

$$\iint_{t>0}\left[u(t, x)\frac{\partial \phi}{\partial t} + F(u(t, x))\frac{\partial \phi}{\partial x}\right]dx\, dt + \int_{t=0}\phi(0, x)u^0(x)dx = 0$$

for all $\phi = \phi(t, x) \in C_0^2(R_+^2)$. $(R_+^2 = \{(t, x)\,|\,t \geqq 0$ and $-\infty < x < \infty\})$.

DEFINITION[1]. Let $x(t)$ be a smooth curve of discontinuity of a weak solution $u(t, x)$ and assume that $u^\pm = \lim_{\varepsilon \searrow 0} u(t, x(t) \pm \varepsilon)$ exist. Then $x(t)$ is a shock provided $F'(u^-) > x'(t) > F'(u^+)$.

The curve $x(t)$ is a contact discontinuity provided $F'(u^-) = x'(t)$ or $F'(u^+) = x'(t)$.

2. Equivalence of Volpert's condition and Condition E. In this section we state Volpert's uniqueness condition and Condition E and then establish their equivalence when the weak solution is piecewise smooth. We also derive some consequences of Condition E.

In the following we shall assume that the weak solutions are piecewise smooth.

Volpert's uniqueness condition. Set

$$T[v; w] = [a(v) - a(w)]\, \text{sign}\, (v - w)\,,$$

where $a(u)$ is the vector $a(u) = (u, F(u))$. Let $\gamma = \gamma(t)$ be any curve of discontinuity of the weak solution $u(t, x)$, and put $u^\pm = \lim_{\varepsilon \searrow 0} u(t, \gamma(t)\pm\varepsilon)$. Then almost everywhere

$$\langle T[u^+; c], \nu\rangle \leqq \langle T[u^-; c], \nu\rangle$$

where c is an arbitrary constant, ν is the normal to γ, considered positive to the right, and $\langle\,,\,\rangle$ denotes the scalar product of two vectors in R^2.[2]

Condition E. Set

---

[1] The definitions for shock and contact discontinuity are specializations to the case of one equation of the definitions given by Lak [3] for systems.

[2] See Volpert [6] for the definitions of u^+, u^-, and v whenever $u(t, x)$ is not piecewise smooth.

$$S[w; z] = \frac{F(w) - F(z)}{w - z} \, .$$

Let $\gamma(t)$ and $u^{\pm}$ be as above, and let v be any number lying between u^+ and u^-. Then almost everywhere

$$S[v; u^-] \geqq S[u^+; u^-] \, .$$

Volpert [6] showed that his condition guarantees uniqueness of solutions to (1), (2) in the class BV of functions whose generalized derivatives are measures locally. Wu [7] established that Condition E gives uniqueness in the class of piecewise smooth functions. We now show that these two conditions are equivalent for piecewise smooth weak solutions.

THEOREM 2.1.   *Let $u(t, x)$ be a piecewise smooth weak solution. Then Volpert's condition holds if, and only if, Condition E holds.*

Proof. We first show that Volpert's condition implies Condition E. Let $x(t)$ be a curve of discontinuity. Since $u(t, x)$ is a piecewise smooth weak solution, the Rankine-Hugoniot condition must hold along $x(t)$; i.e.,

$$\frac{dx}{dt} = \frac{F(u^+) - F(u^-)}{u^+ - u^-} \, .$$

Hence the normal can be expressed by

$$\nu = (\nu_t, \nu_x) = \left(- \frac{F(u^+) - F(u^-)}{u^+ - u^-}, 1 \right) .$$

By hypothesis

(3) $\langle T[u^+; c], \nu \rangle \leqq \langle T[u^-; c], \nu \rangle$

holds for all c. Let c lie between u^+ and u^-, and for the sake of argument suppose that $u^- < c^- < u^+$. Then

$$T[u^+; c] = (u^+ - c, F(u^+) - F(c))$$

and

$$T[u^-; c] = (c - u^-, F(c) - F(u^-)) \, .$$

From (3) it follows that

(4) $S[u^+; u^-](u^+ + u^- - 2c) \geqq F(u^+) + F(u^-) - 2F(c) \, .$

If Condition E should fail to hold for some $c, u^- < c < u^+$, then it would follow that

(5)
$$\frac{F(c) - F(u^-)}{c - u^-} < \frac{F(u^+) - F(u^-)}{u^+ - u^-}$$

or

(6)
$$-\left[- F(c) + F(u^-)\right] < -\left[(- c + u^-)\frac{F(u^+) - F(u^-)}{u^+ - u^-}\right].$$

Add (6) to (4) to get

(7)
$$\frac{F(u^+) - F(u^-)}{u^+ - u^-} \geqq \frac{F(u^+) - F(c)}{u^+ - c}.$$

But by (5) the point $(c, F(c))$ lies below the line joining $(u^-, F(u^-))$ to $(u^+, F(u^+))$ while by (7), $(c, F(c))$ lies on or above the line, giving a contradiction.   Hence Volpert's condition implies Condition E.

To establish the converse, note that if c lies between u^- and u^+, then by Condition E

$$S[u^+; c] \leqq S[u^+; u^-] \leqq S[c; u^-].$$

Suppose for the sake of argument that $u^- < c < u^+$.

Hence

(8)
$$S[u^+; u^-](u^+ - c) \geqq F(u^+) - F(c)$$

and

(9)
$$S[u^+; u^-](u^- - c) \geqq F(u^-) - F(c).$$

Using (8) and (9) it is possible to retrace the steps above to conclude that

$$\langle T[u^+, c], \nu\rangle \leqq \langle T[u^-, c]\rangle, \nu\rangle.$$

If $c \leqq \min\{u^-, u^+\}$, then

$$\langle T[u^-, c], \nu\rangle - \langle T[u^+, c], \nu\rangle$$
$$= - S[u^+; u^-](u^- - u^+) + F(u^-)F(u^+) = 0.$$

The same result holds if $c \geqq \max\{u^-, u^+\}$.

In the study of weak solutions it is desirable to allow as curves of discontinuity only those that arise naturally as a result of the intersection of characteristics, or in other words to rule out those discontinuities from which characteristics diverge.   The significance of Condition E is that it permits only the desired class of discontinuities.   This is established by the following theorem.

THEOREM 2.2. *Let $u(t, x)$ be a piecewise smooth weak solution satisfying Condition E to the Cauchy problem* (1), (2), *and let $\gamma(t)$ be a curve of discontinuity. Then*

$$F'(u^-) \geqq \frac{d\gamma(t)}{dt} \geqq F'(u^+) .^3$$

A proof of this theorem is found in [1].

Note. Using the Rankine-Hugoniot condition, the above can be written as

$$F'(u^-) \geqq S[u^+; u^-] \geqq F'(u^+) .$$

3. **Shock sets and their properties.** In this section we first define a set which contains those values of u that can be reached from u_0 across an admissible curve of discontinuity. We then study the properties of these shock sets, with emphasis on examining the relationship between the geometry of F and the sets.

DEFINITION. Let u_0 be given. Then the *shock set $S(u_0)$* contains those values u such that

$$S[v; u_0] \geqq S[u; u_0]$$

for all v lying between u_0 and u.

NOTE. If $u \in S(u_0)$, then *u can be reached from u_0* via a shock or contact discontinuity. Also the set $S(u_0)$ is closed, a fact which follows easily from the definition.

CONVENTION. We let $u_0 \in S(u_0)$. A consequence of this convention is that if $S(u_0) = \{u_0\}$, then there is no permissible solution of the Riemann problem involving a curve of discontinuity.

THEOREM 3.1. *Let L be the tangent line to $F(u)$ at $u = u_0$. Then $S(u_0) = \{u_0\}$ if, and only if, the graph of F lies above L for $u > u_0$ and below L for $u < u_0$.*

Proof. First suppose that $S(u_0) = \{u_0\}$. We shall show that the graph of F lies above L for $u > u_0$. (The case $u < u_0$ follows similarly.) Since the inflection points are isolated, there is a $u_1 > u_0$ such that either $F''(u) \geqq 0$ on $I = (u_0, u_1)$ or $F''(u) \leqq 0$ on I. If the second alternative held, then, as can be easily seen,

---

[3] Recall that the characteristics have slope $dx/dt = F'(u)$.

$$\frac{d}{dv}S[v;\, u_0] \leqq 0, \qquad v \in I\ .$$

Hence $S[v;\, u_0] \geqq S[u_1;\, u_0]\ \forall v \in I$, which implies that $u_1 \in S(u_0)$, a contradiction. Thus $F''(u) \geqq 0$ on I, and hence for $u \in I$, the graph of $w = F(u)$ cannot lie below L. It remains to show that if $\tilde{u} > u_0$, then $(\tilde{u},\, F(\tilde{u}))$ is not on L. To establish this suppose otherwise and choose

$$\tilde{u} = \inf \{u\colon (u,\, F(u)) \text{ lies on } L,\, u > u_0\}\ .$$

To see that $\tilde{u} \neq u_0$, note that otherwise there would be a $\bar{u} \in (u_0,\, u_1)$, $(\bar{u},\, F(\bar{u}))$ on L. But if $u_0 < v < \bar{u}$, then

$$S[v;\, u_0] - S[\bar{u};\, u_0] = F'(\xi) - F'(u_0) \geqq 0$$

where $u_0 < \xi < \bar{u}$. Since $\bar{u} \in S(u_0)$ is a contradiction, it follows that $\tilde{u} > u_0$. Since $(v,\, F(v))$ lies above L, $u_0 < v < \tilde{u}$,

$$F(v) > F'(u_0)(v - u_0) + F(u_0)\ ,$$

which implies

$$S[v;\, u_0] > S[\tilde{u};\, u_0]\ .$$

Hence $\tilde{u} \in S(u_0)$, a contradiction.

To establish the converse, suppose there exists $w \in S(u_0)$, $w \neq u_0$. Without loss of generality we may assume that $w > u_0$. Then by hypothesis

$$S[w;\, u_0] > F'(u_0)\ .$$

Since $S[v;\, u_0] \to F'(u_0)$ as $v \to u_0$, for v sufficiently close to u_0

$$S[w;\, u_0] > S[v;\, u_0]\ ,$$

a contradiction of Condition E.

THEOREM 3.2. *Let L_u denote the line through the points $(u_0,\, F(u_0))$ and $(u,\, F(u))$. Then $S(u_0)$ is compact if, and only if, there exist u_1, u_2, $u_1 < u_0 < u_2$, such that the graph of $F(u)$ lies below L_{u_1} for $u < u_1$ and above L_{u_2} for $u > u_2$.*

Proof. Suppose $S(u_0)$ is compact. Let $u_1 = \inf S(u_0)$ and $u_2 = \sup S(u_0)$. Then the graph of $F(u)$ lies above

$$w = S[u_2;\, u_0](u - u_0) + F(u_0) \text{ for } u > u_2\ .$$

If this were not so, then there would exist $\tilde{w} > u_2$ such that

$$F(\tilde{w}) = S[u_2; u_0](\tilde{w} - u_0) + F(u_0)$$

and

(10) $$F(w) > S[u_2; u_0](w - u_0) + F(u_0), \quad u_2 < w < \tilde{w} .$$

Since

(11) $$S[v; u_0] \geqq S[u_2; u_0] = S[\tilde{w}; u_0] \qquad \forall v \in (u_0, u_2) ,$$

it follows from (10) and (11) that

$$S[v; u_0] \geqq S[\tilde{w}; u_0] \qquad \forall v \in (u_0, \tilde{w}) .$$

Therefore $\tilde{w} \in S(u_0)$, a contradiction of the definition of u_2. The proof that $F(u)$ lies below L_{u_1} is handled similarly.

Conversely if there is a u_2 such that the graph of $F(u)$ lies above L_{u_2} for $u > u_2$, then $w > u_2$ implies $w \notin S(u_0)$. For otherwise by Condition E

(12) $$S[u_2; u_0] \geqq S[w; u_0] .$$

But from

$$F(w) > S[u_2; u_0](w - u_0) + F(u_0), \, w > u_0 ,$$

it follows that

$$S[w; u_0] > S[u_2; u_0] ,$$

which together with (12) gives a contradiction. Thus u_2 is an upper bound for $S(u_0)$. Similarly u_1 is a lower bound. Hence $S(u_0)$ is compact.

THEOREM 3.3. *Let $u_1 < u_2$ and let $u_2 \in S(u_1)$. Then $S(u_2) \cap (u_2, \infty) \subset S(u_1)$.*

NOTE. This theorem asserts that if it is possible to jump from u_1 to u_2 and from u_2 to u_3, where $u_1 < u_2 < u_3$, then it is possible to jump from u_1 to u_3. It is interesting to note that this fails to hold for systems (cf. Smoller [5]).

Proof. Let $u_3 \in S(u_2)$, where $u_3 > u_2$. Then by the note following Theorem 2.2

$$S[u_2; u_1] \geqq F'(u_2) \geqq S[u_3; u_2] .$$

From this it is easily seen that

(13) $$S[u_2; u_1] \geqq S[u_3; u_1] \geqq S[u_3; u_2] .$$

To see that $u_3 \in S(u_1)$, assume otherwise.

Then there is $w \in (u_1, u_3)$ such that

(14) $$S[w; u_1] < S[u_3; u_1] .$$

Since $S[w; u_1] \geq S[u_2; u_1] \geq S[u_3; u_1]$ holds for all $w \in (u_1, u_2)$, it follows that $w \in (u_2, u_3)$. Since $u_3 \in S(u_2)$, it follows that

(15) $$S[w; u_3] \leq S[u_3; u_2] .$$

A consequence of (14) is that

(16) $$S[w; u_3] > S[u_1; u_3] .$$

By (15) and (16)

$$S[u_3; u_2] > S[u_1; u_3] ,$$

a contradiction of (13).

For the next result two lemmas are needed.

LEMMA 1.   *If $F'' > 0$ on (u_0, u_1), then $S(u_0) \cap (u_0, u_1) = \varnothing$.*

Proof.   It is easily seen that $(d/dv)S[v; u_0] > 0$, $v \in (u_0, u_1)$. Therefore given v and w satisfying $u_0 < v < w < u_1$, it follows that $S[v; u_0] < S[w; u_0]$, and hence Condition E cannot hold.

LEMMA 2.   *If $F'' > 0$ on (u_1, u_0), then $S(u_0) \cap (u_1, u_0) = \varnothing$.*

Proof.   The proof is similar to that of Lemma 1.

THEOREM 3.4.   *The number u_0 is an interior point of $S(u_0)$ if, and only if, there exist numbers u_1, u_2 satisfying $u_1 < u_0 < u_2$ such that $F''(u) \geq 0$, $u_1 < u < u_0$, and $F''(u) \leq 0$, $u_0 < u < u_2$.*

NOTE.   This theorem gives conditions under which it is possible to jump from u_0 to any value of u sufficiently close to u_0.

Proof.   Suppose that u_0 is an interior point. Then there exist u_1, u_2, $u_1 < u_0 < u_2$, such that $(u_1, u_2) \subset S(u_0)$. Choose u_1 and u_2 such that on (u_1, u_2) the only possible point of inflection of F is u_0. If $F'' < 0$ held on (u_1, u_0), then by Lemma 2, u_0 could not be an interior point of $S(u_0)$. Hence $F'' \geq 0$ on (u_1, u_0). Similarly $F'' \leq 0$ on (u_0, u_2).

For the converse let u satisfy $u_0 < u < u_2$. Since $(d/dv)S[v; u_0] \leq 0$, $u_0 < u < u_2$, it follows that

$$S[v; u_0] \geq S[u; v_0]$$

for all $v \in (u_0, u)$. Hence $[u_0, u_2) \subset S(u_0)$. Similarly $(u_1, u_0] \subset S(u_0)$.

THEOREM 3.5.   *The point u_0 is an isolated point of $S(u_0)$ if, and only if, there is an interval $u_1 < u_0 < u_2$ such that $F'' < 0$ on (u_1, u_0) and $F'' > 0$ on (u_0, u_2).*

NOTE.   This theorem gives conditions under which it is not possible to shock from u_0 to any nearby value of u.

Proof. Suppose u_0 is an isolated point. If there did not exist $u_2 > u_0$ such that $F'' > 0$ on (u_0, u_2), then since the points of inflection are isolated, it follows that for some interval, $u_0 < v < u_2$, $F''(u) \leqq 0$. But as seen before this implies $(u_0, u_2) \subset S(u_0)$, giving a contradiction. It follows similarly that there is $u_1 < u_0$ such that $F'' < 0$ on (u_1, u_0).

Conversely since $F'' < 0$ on (u_1, u_0), by Lemma 2 $S(u_0) \cap (u_1, u_0) = \varnothing$. Similarly $S(u_0) \cap (u_0, u_2) = \varnothing$. Hence u_0 is isolated.

THEOREM 3.6.   *If u_1 is an interior point of $S(u_0)$ and if $F''(u_1) \neq 0$, then the jump from u_0 to u_1 is across a shock.*

Proof. We must show that

(17) $$F'(u_0) > S[u_1; u_0] > F'(u_1) \, .$$

For the sake of argument suppose that $u_0 < u_1$. If $F'(u_0) > S[u_1; u_0]$ did not hold, then by Theorem 2.2 $F'(u_0) = S[u_1; u_0]$. Since u_1 is an interior point, for some $\eta > 0$, $(u_1 - \eta, u_1 + \eta) \subset S(u_0)$. If $v_0 \in (u_0, u_1)$ and satisfies $S[v_0; u_0] > S[u_1; u_0]$, then $v_0 \notin S(u_0)$. To see this note that $S[\tilde{u}; u_0] \rightarrow F'(u_0) = S[u_1; u_0]$ as $\tilde{u} \rightarrow u_0$, and hence for $\tilde{u} > u_0$ sufficiently close to u_0, $S[v_0; u_0] > S[\tilde{u}; u_0]$. Thus Condition E cannot hold. Hence $S[v; u_0] = S[u_1; u_0] \; \forall v \in (u_1 - \eta, u_1)$. Thus for $v \in (u_1 - \eta, u_1)$ $w = F(u)$ coincides with the line L given by $w = S[u_1; u_0](u - u_0) + F(u_0)$, a contradiction of $F''(u_1) \neq 0$. Hence $F'(u_0) > S[u_1; u_0]$ must hold.

Next assume $S[u_1; u_0] = F'(u_1)$. Since $S[v; u_0] \geqq S[u_1; u_0]$, $u_0 < v < u_1$, the graph of $w = F(u)$ lies above the line L for $v \in (u_0, u_1)$. Since $F''(u_1) \neq 0$ and since L is tangent to $w = F(v)$ at u_1, the graph of $w = F(v)$ lies above L for $v \in (u_1, u_1 + \eta) = I$, where $\eta > 0$. Hence, as is easily seen, $S[v, u_0] > S[u_1; u_0]$, $v \in I$, and therefore $I \cap S(u_0) = \varnothing$. Thus u_1 is a boundary point of $S(u_0)$, a contradiction. Hence (17) must hold.

REFERENCES

1.   D. P. Ballou, *Solution of non-linear hyperbolic Cauchy problems without convexity conditions*, Trans. Amer. Math. Soc. **152** (1970), 441–460.

2. ———, *Weak solutions with a dense set of discontinuities*, J. Differential Equations, **10** (1971), 270-280.

3. P. D. Lax, *Hyperbolic solutions of conservation laws* II, Comm. Pure Appl. Math., **10** (1957), 537-566.

4. O. A. Oleinik, *On discontinuous solutions of non-linear differential equations*, Uspekhi Mat. Nauk., **12** (1957), 3-73. English translation, Amer. Math. Soc. Trans. Ser. 2, No. 26, pp. 95-172.

5. J. A. Smoller, *A survey of hyperbolic systems of conservation laws in two dependent variables, Hyperbolic Equations and Waves*, Battelle Seattle 1968 Rencontres, Springer-Verlag, Berlin, (1970), 51-60.

6. A. I. Volpert, *The spaces BV and quasilinear equations*, Math. USSR Sbornik, **2** (1967), 225-267.

7. Zhuo-Qun Wu, *On existence and uniqueness of generalized solutions of the Cauchy problem for quasilinear equations of first order without convexity conditions*, Acta. Math. Sinica, **13** (1963), 515-530. English translation, Chinese Math., **4** (1964), 561-577.

STATE UNIVERSITY OF NEW YORK AT ALBANY

PACIFIC JOURNAL OF MATHEMATICS

Pacific Journal of Mathematics
Vol. 42, No. 1   January, 1972