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ON A GENERALIZATION OF Σ -SPACES

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In order to simultaneously generalize the class of M -spaces and σ -spaces, K. Nagami introduced Σ -spaces. Subsequently, E. Michael defined a class of Σ^* -spaces. In this paper we will discuss the class of Σ^* -spaces which lies between Σ -spaces and Σ^* -spaces and which contains all images of Σ -spaces under closed continuous maps.

1. Introduction. Recently K. Nagami [6] has investigated a new class of spaces, called Σ -spaces, containing two different classes of generalized metric spaces; i.e. the class of M -spaces (cf. [4]) as well as the class of σ -spaces (cf. [5], [7]).

If \mathcal{K} is a cover of a space X , then a cover \mathcal{A} is called a (*mod* \mathcal{K})-*network* for X if, whenever $K \subset U$ with $K \in \mathcal{K}$ and U open in X , then $K \subset A \subset U$ for some $A \in \mathcal{A}$. According to K. Nagami [6], X is a Σ -space if it has a σ -locally finite closed (*mod* \mathcal{K})-network for some cover \mathcal{K} of X by countably compact sets.

E. Michael [2] has pointed out that the image of a paracompact, T_2 Σ -space under a closed continuous map need not be a Σ -space and also that replacing " σ -locally finite" by " σ -closure-preserving" in the definition of a Σ -space leads to a strictly larger class of spaces, which are called Σ^* -spaces.

We say that a space X is a Σ^* -space if it satisfies the definition of a Σ -space with " σ -locally finite" weakened to " σ -hereditarily closure-preserving", where we say that a collection $\mathcal{A} = \{A_\lambda: \lambda \in \Lambda\}$ is *hereditarily closure-preserving* if any collection $\{B_\lambda: \lambda \in \Lambda\}$ with $B_\lambda \subset A_\lambda$ is closure-preserving (cf. [3]).

Clearly, every Σ -space is a Σ^* -space and every Σ^* -space is a Σ^* -space. Since the image of a locally finite closed cover of the domain under a closed continuous onto map is a hereditarily closure-preserving closed cover of the range, we can easily see that the image of a Σ -space by a closed continuous map is always a Σ^* -space. As a matter of fact, E. Michael [2] has pointed out that a paracompact, T_2 Σ^* -space need not be a Σ -space, in general. Hence this fact arouses our interest in studying Σ^* -spaces comparing with Σ -spaces as well as Σ^* -spaces.

In this paper we will investigate some relationship between above spaces and obtain the following results:

- (A) Any image of a Σ^* -space under a closed continuous map is a Σ^* -space.
- (B) Any inverse image of a Σ^* -space by a perfect map (i.e. a

closed continuous map whose fibre at each point is compact) is a Σ^* -space, while this is not true for a Σ^* -space.

(C) Every Lindelöf, T_2 , Σ^* -space is a Σ -space, while this is not true for a Σ^* -space.

(D) A Σ^* -space X is a Σ -space if every open set of X is an F_σ .

(E) For a paracompact, T_2 space X the following conditions are equivalent:

(1) X is a Σ -space.

(2) $X \times I$ is a Σ -space, where I denotes the unit closed interval with usual topology.

(3) $X \times I$ is a Σ^* -space.

According to the first half of (B), the product of a Σ^* -space with I is a Σ^* -space. On the other hand, as noted above there exists a paracompact, T_2 , Σ^* -, non Σ -space. Hence statement (E) shows that the product of a paracompact, T_2 , Σ^* -, non Σ -space X with I is a Σ^* -, non Σ^* -space. Since the projection from $X \times I$ to I is perfect, this is an example for the later half of (B). Also, this shows that the class of Σ^* -spaces is strictly larger than the class of Σ^* -spaces.

Concerning (D), it raises the following question:

Is (D) true for Σ^* -spaces?

§2 is concerned with hereditarily closure-preserving closed covers of a countably compact, T_2 space, a Lindelöf, T_2 space and a T_2 space whose open sets are F_σ 's. As an immediate consequence of 2.1 and 2.3 we have the simple facts that every hereditarily closure-preserving closed cover of a countably compact, T_2 space (resp. a Lindelöf, T_2 space) has a finite (resp. a countable) subcover. In §3 we will prove main results.

We will use the following notations in §2 and §3:

For a cover \mathcal{F} of a space X and a point x of X we put

$$C(x, \mathcal{F}) = \bigcap \{F : x \in F \in \mathcal{F}\},$$

and for a sequence $\{\mathcal{F}_n : n = 1, 2, \dots\}$ of covers of X and a point x of X we put

$$C(x) = \bigcap_{n=1}^{\infty} C(x, \mathcal{F}_n).$$

Throughout this paper we assume that all spaces are T_2 and all maps are continuous.

2. Some properties of a hereditarily closure-preserving closed cover.

THEOREM 2.1. *Let $\mathcal{F} = \{F_\lambda : \lambda \in A\}$ be a hereditarily closure-preserving closed cover of a space X and C a countably compact set*

of X . Then \mathcal{F} is locally finite at almost all points of C ; i.e. there exist x_1, \dots, x_n in C such that \mathcal{F} is locally finite at any $x \in C - \{x_1, \dots, x_n\}$, and only finitely many members of \mathcal{F} meet $C - \{x_1, \dots, x_n\}$.

Proof. On the contrary, suppose \mathcal{F} is not locally finite at infinitely many points of C . Since any closure-preserving, point-finite collection of closed sets is locally finite, \mathcal{F} is not point-finite at infinitely many points of C . Then we can choose, step by step, countably many points x_1, x_2, \dots in C and countably many $\lambda_1, \lambda_2, \dots$ in A such that $x_n \in F_{\lambda_n}$ for $n = 1, 2, \dots$. Since \mathcal{F} is hereditarily closure-preserving, $\{x_1, x_2, \dots\}$ must be discrete in X . On the other hand, since C is countably compact, $\{x_1, x_2, \dots\}$ must have a cluster point in C . This is a contradiction. Hence \mathcal{F} is locally finite at all points of C but finitely many points x_1, \dots, x_n .

To complete the proof of 2.1, assume that $D = C - \{x_1, \dots, x_n\}$ is infinite. If infinitely many members of \mathcal{F} meet D , then we can again obtain a sequence $\{p_1, p_2, \dots\}$ in D and a sequence $\{F_{\lambda_1}, F_{\lambda_2}, \dots\}$ in \mathcal{F} with $p_i \in F_{\lambda_i}$ for $i = 1, 2, \dots$ by noting that \mathcal{F} is point-finite at any point of D . Since \mathcal{F} is hereditarily closure-preserving, $\{p_1, p_2, \dots\}$ must be discrete in X , therefore, in C , which is a contradiction. Hence only finitely many members of \mathcal{F} meet D . This completes the proof.

As an immediate corollary of 2.1 we have:

COROLLARY 2.2. *Every hereditarily closure-preserving closed cover of a countably compact space contains a finite subcover.*

REMARK. 2.2 does not necessarily hold for a closure-preserving closed cover even if a space is compact and metrizable; for example, let $X = \{1/n : n = 1, 2, \dots\} \cup \{0\}$ be a subspace of real line and put $\mathcal{F} = \{\{0, 1/n\} : n = 1, 2, \dots\}$. Then X is a compact, metric space and \mathcal{F} is a closure-preserving closed cover of X , but we cannot choose any finite subcover.

THEOREM 2.3. *Let $\mathcal{F} = \{F_\lambda : \lambda \in A\}$ be a hereditarily closure-preserving closed cover of a Lindelöf space X . Then the set*

$$X_0 = \{x \in X : A(x) = \{\lambda \in A : x \in F_\lambda\} \text{ is uncountable}\}$$

is countable, and the set

$$A' = \{\lambda \in A : F_\lambda \cap (X - X_0) \neq \emptyset\}$$

is countable if $X - X_0$ is uncountable.

Proof. On the contrary, suppose X_0 is uncountable. Then X_0 contains a subset $\{x_\alpha: \alpha < \omega_1\}$, where ω_1 denotes the least uncountable ordinal. For each $\alpha < \omega_1$, by transfinite induction we can obtain x_α in X_0 and a $\lambda_\alpha \in \mathcal{A}(x_\alpha)$ with $x_\alpha \in F_{\lambda_\alpha}$ and such that $\alpha \neq \beta$ implies $x_\alpha \neq x_\beta$ and $\lambda_\alpha \neq \lambda_\beta$, because for each $x \in X_0$ $\mathcal{A}(x)$ is uncountable. Since \mathcal{F} is hereditarily closure-preserving, $\{x_\alpha: \alpha < \omega_1\}$ must be discrete in X . This contradicts the assumption that X is Lindelöf, and hence the first half of 2.3 is proved.

To complete the proof, again suppose \mathcal{A}' is uncountable. From the definition of X_0 , \mathcal{F} must be point-countable at any $x \in X - X_0$. If $X - X_0$ is uncountable, by transfinite induction, we can choose an uncountable set $\{x_\alpha: \alpha < \omega_1\}$ in $X - X_0$ and a corresponding set $\{\lambda_\alpha: \alpha < \omega_1\}$ with $x_\alpha \in F_{\lambda_\alpha}$ for each $\alpha < \omega_1$ and so that $\alpha \neq \beta$ implies $x_\alpha \neq x_\beta$ as well as $\lambda_\alpha \neq \lambda_\beta$. Since \mathcal{F} is hereditarily closure-preserving, $\{x_\alpha: \alpha < \omega_1\}$ must be an uncountable discrete set in X , which contradicts the assumption that X is Lindelöf. Therefore $X - X_0$ is countable, and hence the proof is completed.

As an immediate consequence of 2.3 we have:

COROLLARY 2.4. *Every hereditarily closure-preserving closed cover of a Lindelöf space contains a countable subcover.*

REMARK. Example 3.4 in next section shows that 2.4 does not necessarily hold for a closure-preserving closed cover.

LEMMA 2.5. *Let \mathcal{F} be a closure-preserving closed cover of a space X . Then the set*

$$X_1 = \{x \in X: C(x, \mathcal{F}) = \{x\}\}$$

is discrete in X .

Proof. Let $y \in X$ be an arbitrary point and

$$U = X - \bigcup \{F \in \mathcal{F}: y \in F\}.$$

Then U is an open neighborhood of y , because \mathcal{F} is a closure-preserving closed cover. If $x \in U \cap X_1$, then we have

$$\phi \neq U \cap C(x, \mathcal{F}) = (X - \bigcup \{F \in \mathcal{F}: y \in F\}) \cap (\bigcap \{F \in \mathcal{F}: x \in F\})$$

and hence $C(y, \mathcal{F}) \subset C(x, \mathcal{F})$. Since $x \in X_1$, $C(x, \mathcal{F}) = \{x\}$ and thus we have $y = x$. This means that U contains at most one point of X_1 , which completes the proof.

THEOREM 2.6. *Let X be a space each of whose open sets is an*

F_σ , and let \mathcal{F} be a closure-preserving closed cover of X . Then the set

$$X_n = \{x \in X : |C(x, \mathcal{F})| = n\}$$

is σ -discrete in X for $n = 1, 2, \dots$, where we denote by $|A|$ the cardinality of A .

Proof. We shall prove 2.6 by induction on n . By 2.5 X_1 is discrete in X . Assume that X_n is σ -discrete in X for any $n \leq k$. We shall show that X_{k+1} is also σ -discrete.

First note that $X - \bigcup_{n=1}^k X_n$ is open in X . Let y be any point of $X - \bigcup_{n=1}^k X_n$ and let $U = X - \bigcup \{F \in \mathcal{F} : y \notin F\}$. Then U is an open neighborhood of y . If $x \in X$ belongs to U , we have $C(y, \mathcal{F}) \subset C(x, \mathcal{F})$. Since y does not belong to $\bigcup_{n=1}^k X_n$, $C(y, \mathcal{F})$ contains at least $k+1$ points of X and thus $C(x, \mathcal{F})$ also contains at least $k+1$ points. In other words, $x \notin \bigcup_{n=1}^k X_n$. This shows that $U \cap (\bigcup_{n=1}^k X_n) = \emptyset$ and hence $X - \bigcup_{n=1}^k X_n$ is open in X .

According to hypothesis, $X - \bigcup_{n=1}^k X_n$ is an F_σ ; i.e. $X - \bigcup_{n=1}^k X_n = \bigcup_{i=1}^\infty Y_i$, where each Y_i is closed in X and $Y_i \subset Y_{i+1}$ for $i = 1, 2, \dots$. Since $X_{k+1} \subset \bigcup_{i=1}^\infty Y_i$, it suffices to show that $Z_i = X_{k+1} \cap Y_i$ is discrete in X for $i = 1, 2, \dots$.

Let $y \in X$ be an arbitrary point and i fixed. If $y \notin Y_i$, then $X - Y_i$ is clearly the desired neighborhood of y . If $y \in Y_i$, put $U = X - \bigcup \{F \in \mathcal{F} : y \notin F\}$. Then $x \in U \cap Z_i$ implies $C(y, \mathcal{F}) \subset C(x, \mathcal{F})$ and $|C(x, \mathcal{F})| = k+1$. Since y belongs to Y_i , y does not belong to any X_n with $n \leq k$; i.e. $|C(y, \mathcal{F})| > k$. Hence we have $C(y, \mathcal{F}) = C(x, \mathcal{F})$. This means that x must be in $C(y, \mathcal{F})$ which is finite. Consequently, U contains at most $k+1$ points of Z_i . Since X is T_1 , we obtain the desired neighborhood of y by deleting finitely many points from U . Therefore Z_i is discrete in X . This completes the proof.

3. Some relations. Let f be a closed map from a space X onto a space Y and \mathcal{F} a hereditarily closure-preserving closed cover of X . Then $f(\mathcal{F})$ is also a hereditarily closure-preserving closed cover of Y . Since the image of any countably compact space by a map is countably compact, we have the following:

THEOREM 3.1. *Any image of a Σ^* -space under a closed map is a Σ^* -space.*

Let f be a perfect map from X onto Y and \mathcal{A} a $(\text{mod } \mathcal{K})$ -network for Y . Then we can easily see that $f^{-1}(\mathcal{A})$ is a $(\text{mod } f^{-1}(\mathcal{K}))$ -network for X . Since the inverse image of any countably compact space by a perfect map is countably compact, we have the following:

THEOREM 3.2. *Any inverse image of a Σ^* -space by a perfect map is a Σ^* -space.*

THEOREM 3.3. *Every Lindelöf Σ^* -space is a Σ -space.*

Proof. Let X be a Lindelöf Σ^* -space having a σ -hereditarily closure-preserving closed (mod \mathcal{K})-network \mathcal{F} for some cover \mathcal{K} of X by countably compact sets. Without loss of generality, we can denote \mathcal{F} by $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ such that each \mathcal{F}_n is a hereditarily closure-preserving closed cover of X . Put $\mathcal{F}_n = \{F_\lambda: \lambda \in A_n\}$ for $n = 1, 2, \dots$.

By 2.3, for each n the set

$$X_n = \{x \in X: A(x) = \{\lambda \in A_n: x \in F_\lambda\} \text{ is uncountable}\}$$

is countable. If $X - X_n$ is countable for some n , then X is countable. Since X is T_2 , X is clearly a Σ -space; more precisely, it is a cosmic space (cf. [1]). If $X - X_n$ is uncountable for $n = 1, 2, \dots$, then again by 2.3,

$$A'_n = \{\lambda \in A_n: F_\lambda \cap (X - X_n) \neq \emptyset\}$$

is countable for $n = 1, 2, \dots$. Put $\mathcal{H}_n = \{\{x\}: x \in X_n\} \cup \{F_\lambda: \lambda \in A'_n\}$ for $n = 1, 2, \dots$. Then each \mathcal{H}_n is countable and, therefore, $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ is still countable. Since each \mathcal{H}_n covers X , \mathcal{H} covers X and thus \mathcal{H} is a σ -locally finite closed cover of X . Furthermore, if we put $\mathcal{K}' = \{\{x\}: x \in \bigcup_{n=1}^{\infty} X_n\} \cup \{K \in \mathcal{K}: K \cap (X - X_n) \neq \emptyset \text{ for some } n\}$, then \mathcal{K}' is a cover of X by countably compact sets. It is easy to see that \mathcal{H} is a (mod \mathcal{K}')-network, and hence X is a Σ -space.

EXAMPLE 3.4. We shall show that in general a Lindelöf Σ^* -space need not be a Σ -space.

Let $X = \{x_\alpha: \alpha \in A\} \cup \{p\}$ be an uncountable set with a special point p . We define the topology for X as follows: each $\{x_\alpha\}$ is open; V is an open set containing p iff $X - V$ is countable. Then we can easily see that X is a regular, Lindelöf (T_2) space.

Now, put $\mathcal{F} = \{\{p, x_\alpha\}: \alpha \in A\}$. Then \mathcal{F} is a closure-preserving closed cover of X , because any subset of X missing p is open. If we put $\mathcal{K} = \mathcal{F}$, then \mathcal{K} is a cover of X by countably compact sets such that \mathcal{F} is a (mod \mathcal{K})-network for X ; i.e. X is a Σ^* -space.

Next, we shall show that X is not a Σ -space. On the contrary, suppose X is a Σ -space. Then there exists a σ -locally finite closed cover $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ of X which is a (mod \mathcal{K})-network for some cover \mathcal{K} by countably compact sets. We can assume without loss of generality that $\{\mathcal{H}_n: n = 1, 2, \dots\}$ is an increasing sequence of locally finite closed covers of X and that each \mathcal{H}_n is closed under

finite intersections. Furthermore, in case of a Σ -space we can put $\mathcal{K} = \{C(x): x \in X\}$, where $C(x) = \bigcap_{n=1}^{\infty} C(x, \mathcal{H}_n)$ as noted in the introduction. Since X is Lindelöf, each \mathcal{H}_n is countable. From the definition of the topology for X any member of \mathcal{H} missing p is a countable set. Therefore $X' = X - \bigcup \{H \in \mathcal{H}: p \notin H\}$ is an uncountable closed subspace of X , which is a Σ -space having $\mathcal{H}|X' = \{H \cap X': H \in \mathcal{H}\}$ as a σ -locally finite (mod $\mathcal{K}|X'$)-network. Consequently, we could have assumed from the beginning that each \mathcal{H}_n is finite and each member of \mathcal{H} contains p . For each $x \in X$ and n , let $H(x, n)$ be the smallest (as a subset) member of \mathcal{H}_n containing x . $H(x, n)$ exists because \mathcal{H}_n is closed under finite intersections. Since the compact sets of X are exactly the finite sets, $C(x) = \bigcap_{n=1}^{\infty} H(x, n)$ must be finite for each $x \in X$. Furthermore, for each $x \in X$ there is an n_x such that $H(x, n_x)$ is finite. To see this, suppose not. Then there is an increasing sequence $n_1 < n_2 < \dots$ with $H(x, n_{i+1}) \subsetneq H(x, n_i)$ for $i = 1, 2, \dots$. Now pick a point $x_i \in H(x, n_i) - H(x, n_{i+1})$ which is distinct from p and x . Then $F = \{x_i: i = 1, 2, \dots\}$ is a closed set in X with $F \cap C(x) = \emptyset$ but $F \cap H(x, n) \neq \emptyset$ for all n . This contradicts the fact that \mathcal{H} forms a network around $C(x)$. Hence there exists such an n_x . We denote by $n(x)$ the smallest n_x for which $H(x, n_x)$ is finite. Put

$$L_n = \{x \in X: n(x) \leq n\} \quad \text{for } n = 1, 2, \dots.$$

Then $\{L_n: n = 1, 2, \dots\}$ is an increasing cover of X . Since X is uncountable, there exists an n_0 such that L_{n_0} is an uncountable set containing p . Clearly L_{n_0} is closed in X and hence it is a Σ -space having $\mathcal{H}|L_{n_0}$ as a (mod $\mathcal{K}|L_{n_0}$)-network. But $\bigcup_{i=1}^{n_0} \mathcal{H}_i$ is finite and for each $x \in L_{n_0}$ there exists an $H(x, n(x))$ with $n(x) \leq n_0$. This means that L_{n_0} must be finite, which is a contradiction. Thus X is not a Σ -space.

LEMMA 3.5. *If X is a Σ^* -space (resp. a Σ^* -space), then X has a sequence $\{\mathcal{F}_n: n = 1, 2, \dots\}$ of hereditarily closure-preserving (resp. closure-preserving) closed covers of X such that any sequence $\{x_n: n = 1, 2, \dots\}$ with $x_n \in C(x, \mathcal{F}_n)$ for some $x \in X$ has a cluster point. In particular, X is a Σ -space iff X has a sequence $\{\mathcal{F}_n: n = 1, 2, \dots\}$ of locally finite closed covers of X such that any sequence $\{x_n: n = 1, 2, \dots\}$ with $x_n \in C(x, \mathcal{F}_n)$ for some $x \in X$ has a cluster point.*

Proof. Since all cases are proved similarly, we shall prove for a Σ^* -space, only. Let X be a Σ^* -space having a σ -closure-preserving closed (mod \mathcal{K})-network $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ for a cover \mathcal{K} of X by countably compact sets, where we can assume that each \mathcal{H}_n is a

closure-preserving closed cover of X . Put $\mathcal{F}_n = \bigcup_{k \leq n} \mathcal{H}_k$ for $n = 1, 2, \dots$. Now we shall show that $\{\mathcal{F}_n: n = 1, 2, \dots\}$ satisfies the required condition. On the contrary, suppose not. Then there exists a discrete sequence $\{x_n: n = 1, 2, \dots\}$ with $x_n \in C(x, \mathcal{F}_n)$ for some $x \in X$. Since \mathcal{H} covers X , there is a $K \in \mathcal{H}$ containing x . Since $\{x_n: n = 1, 2, \dots\}$ is discrete, there exists an n_0 such as $\{x_n: n \geq n_0\} \cap K = \emptyset$. Then $G = X - \{x_n: n \geq n_0\}$ is an open set containing K and thus, by the assumption, there exists an $F \in \mathcal{F}_m$ for some m with $K \subset F \subset G$. Hence we have $x_i \in C(x, \mathcal{F}_i) \subset C(x, \mathcal{F}_m) \subset F \subset G$ for any i with $m < i$ as well as $n_0 < i$, which is a contradiction.

The ‘if’ part in the later half is easily seen noting that any $C(x, \mathcal{F}_n)$ could have been a member of \mathcal{F}_n .

THEOREM 3.6. *Let X be a Σ^* -space for which every open set is an F_σ . Then X is a Σ -space.*

Proof. Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ be a σ -hereditarily closure-preserving closed (mod \mathcal{H})-network for a cover \mathcal{H} by countably compact sets. We can assume that each \mathcal{F}_n covers X and that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for $n = 1, 2, \dots$. Put

$$X' = \{x \in X: |C(x, \mathcal{F}_n)| \text{ is finite for some } n\}.$$

Then X' is σ -discrete in X by 2.6. Denote X' by $\bigcup_{n=1}^{\infty} P_n$, where each P_n is discrete in X and we can assume $P_n \subset P_{n+1}$ for $n = 1, 2, \dots$.

We shall show that each \mathcal{F}_n is locally finite at any $x \in X - X'$. On the contrary, suppose some \mathcal{F}_{n_0} is not locally finite at some $x \in X - X'$. Since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and since each \mathcal{F}_n is closure-preserving, $A'_n = \{\lambda \in A_n: x \in F'_\lambda\}$ must be infinite for all $n \geq n_0$. Since $x \notin X'$, $C(x, \mathcal{F}_n)$ is infinite for all $n \geq n_0$. We can choose a point $x_n \in C(x, \mathcal{F}_n)$ and a $\lambda_n \in A'_{n_0}$ with $x_n \in F'_{\lambda_n}$ for each $n \geq n_0$ and such that $n \neq m$ implies $x_n \neq x_m$ as well as $\lambda_n \neq \lambda_m$. By 3.5 $\{x_n: n = n_0, n_0 + 1, \dots\}$ has a cluster point. On the other hand, it must be discrete, because each $\{x_n\} \subset F'_{\lambda_n} \in \mathcal{F}_{n_0}$ and \mathcal{F}_{n_0} is hereditarily closure-preserving. This contradiction shows that each \mathcal{F}_n is locally finite at any $x \in X - X'$.

Next, put

$$Y_n = \{x \in X: \mathcal{F}_n \text{ is locally finite at } x\}, \quad n = 1, 2, \dots.$$

Then each Y_n is open in X and therefore an F_σ . Denote Y_n by $\bigcup_{m=1}^{\infty} Q_{nm}$, where each Q_{nm} is closed in X and $Q_{nm} \subset Q_{n,m+1}$ for $m, n = 1, 2, \dots$. Further, as was seen above, we have $X - X' \subset Y_n$ for $n = 1, 2, \dots$.

Finally, put

$$\mathcal{F}_{nm} = \{F_\lambda \cap Q_{nm} : \lambda \in A_n\} \cup \{X\} \quad \text{for } n, m = 1, 2, \dots,$$

$$\mathcal{H}_n = \{\{x\} : x \in P_n\} \cup \{X\} \quad \text{for } n = 1, 2, \dots.$$

Then each \mathcal{F}_{nm} as well as \mathcal{H}_n is locally finite closed cover of X . In order that X be a Σ -space, it suffices to show that the sequence $\{\mathcal{F}_{nm} : n, m = 1, 2, \dots\} \cup \{\mathcal{H}_n : n = 1, 2, \dots\} = \{\mathcal{G}_i : i = 1, 2, \dots\}$ satisfies the condition in 3.5. Let $x \in X$ be any point and $\{x_i : i = 1, 2, \dots\}$ a sequence with $x_i \in C(x, \mathcal{G}_i)$. If $x \in X'$, then $x \in P_k$ for some k , and since $\{P_n : n = 1, 2, \dots\}$ is increasing, we have $C(x, \mathcal{H}_n) = \{x\} \in \mathcal{H}_n$ for all $n \geq k$. Hence $\{x_i : i = 1, 2, \dots\}$ has a cluster point x . If $x \notin X'$, then $x \in Y_n$ for $n = 1, 2, \dots$ and hence, for each n , there exists a k_n with $x \in Q_{nk_n}$. Thus, for any n we have $C(x, \mathcal{F}_{nk_n}) \subset C(x, \mathcal{F}_n)$. On the other hand, by 3.5 any sequence $\{p_n : n = 1, 2, \dots\}$ with $p_n \in C(x, \mathcal{F}_n)$ has a cluster point. Hence $\{x_i : i = 1, 2, \dots\}$ must have a cluster point. This shows by 3.5 that X is a Σ -space.

THEOREM 3.7. *Let X be a paracompact space. Then the following conditions are equivalent.*

- (1) X is a Σ -space.
- (2) $X \times I$ is a Σ -space.
- (3) $X \times I$ is a Σ^* -space.

Proof. Since the property of being a paracompact Σ -space is countably productive (cf. [6]), we have (1) \Rightarrow (2). From the definition clearly (2) \Rightarrow (3).

(3) \Rightarrow (1). Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ be a σ -hereditarily closure-preserving (mod \mathcal{K})-network for some cover \mathcal{K} of $X \times I$ by countably compact sets. We assume that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for $n = 1, 2, \dots$.

At first we shall construct by induction on n a collection $\{V(\alpha_1, \dots, \alpha_n) : \alpha_1 \in A_1, \dots, \alpha_n \in A_n; n = 1, 2, \dots\}$ of open sets of X and a corresponding collection

$$\{I(\alpha_1, \dots, \alpha_n) : \alpha_1 \in A_1, \dots, \alpha_n \in A_n; n = 1, 2, \dots\}$$

of subsets of I satisfying the following conditions:

- (i) $\{V(\alpha_1, \dots, \alpha_n) : \alpha_1 \in A_1, \dots, \alpha_n \in A_n\}$ is a locally finite open cover of X for $n = 1, 2, \dots$.
- (ii) $V(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \subset V(\alpha_1, \dots, \alpha_n)$ for $\alpha_1 \in A_1, \dots, \alpha_n \in A_n, \alpha_{n+1} \in A_{n+1}; n = 1, 2, \dots$.
- (iii) If $V(\alpha_1, \dots, \alpha_n)$ is nonempty, then $I(\alpha_1, \dots, \alpha_n)$ is a closed interval.
- (iv) $I(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \subset I(\alpha_1, \dots, \alpha_n)$ for $\alpha_1 \in A_1, \dots, \alpha_n \in A_n, \alpha_{n+1} \in A_{n+1}; n = 1, 2, \dots$.
- (v) $\overline{V(\alpha_1, \dots, \alpha_n)} \times I(\alpha_1, \dots, \alpha_n)$ meets only finitely many members of \mathcal{F}_n for $\alpha_1 \in A_1, \dots, \alpha_n \in A_n; n = 1, 2, \dots$.

Assume that such collections are constructed for all $n \leq k$ and

consider $n = k + 1$.

Fix $\alpha_1 \in A_1, \dots, \alpha_k \in A_k$ with $V(\alpha_1, \dots, \alpha_k) \neq \emptyset$. For any point $x \in \overline{V(\alpha_1, \dots, \alpha_k)}$, since $\{x\} \times I(\alpha_1, \dots, \alpha_k)$ is compact and \mathcal{S}_{k+1} is hereditarily closure-preserving, by 2.1 \mathcal{S}_{k+1} is locally finite at all but finitely many points of $\{x\} \times I(\alpha_1, \dots, \alpha_k)$. Let $\{p_1, \dots, p_m\}$ be those points of $\{x\} \times I(\alpha_1, \dots, \alpha_k)$ at which \mathcal{S}_{k+1} is not locally finite. Let I_x be a closed subinterval of $I(\alpha_1, \dots, \alpha_k)$ missing p_1, \dots, p_m . Since $\{x\} \times I_x$ is compact, there exists an open neighborhood U_x (in $\overline{V(\alpha_1, \dots, \alpha_k)}$) of x such that

$$\bar{U}_x \times I_x \subset X \times I - \bigcup \{F \in \mathcal{S}_{k+1} : F \cap (\{x\} \times I_x) = \emptyset\}.$$

Since $\overline{V(\alpha_1, \dots, \alpha_k)}$ is paracompact, there is a locally finite open cover $\{V_\lambda : \lambda \in \Lambda(\alpha_1, \dots, \alpha_k)\}$ of $\overline{V(\alpha_1, \dots, \alpha_k)}$ which refines $\{U_x : x \in \overline{V(\alpha_1, \dots, \alpha_k)}\}$. Let

$$\varphi : \Lambda(\alpha_1, \dots, \alpha_k) \longrightarrow \overline{V(\alpha_1, \dots, \alpha_k)} \subset X$$

be a function which satisfies $V_\lambda \subset U_{\varphi(\lambda)}$ for $\lambda \in \Lambda(\alpha_1, \dots, \alpha_k)$.

Now varying $\alpha_1 \in A_1, \dots, \alpha_k \in A_k$, put

$$A_{k+1} = \bigcup \{\Lambda(\alpha_1, \dots, \alpha_k) : \alpha_1 \in A_1, \dots, \alpha_k \in A_k\}$$

and

$$V(\alpha_1, \dots, \alpha_k, \alpha_{k+1}) = \begin{cases} V(\alpha_1, \dots, \alpha_k) \cap V_{\alpha_{k+1}} & \text{if } V(\alpha_1, \dots, \alpha_k) \neq \emptyset \text{ and } \alpha_{k+1} \in \Lambda(\alpha_1, \dots, \alpha_k) \\ \emptyset & \text{otherwise} \end{cases}.$$

Furthermore, if $V(\alpha_1, \dots, \alpha_k, \alpha_{k+1}) \neq \emptyset$, then from the definition we have $V(\alpha_1, \dots, \alpha_k) \neq \emptyset$ and $\alpha_{k+1} \in \Lambda(\alpha_1, \dots, \alpha_k)$. By inductive hypothesis $I(\alpha_1, \dots, \alpha_k)$ is not empty. Hence we put $I(\alpha_1, \dots, \alpha_k, \alpha_{k+1}) = I_{\varphi(\alpha_{k+1})}$, which is not empty. Otherwise we put $I(\alpha_1, \dots, \alpha_k, \alpha_{k+1}) = \emptyset$. Then we can easily see that $\{V(\alpha_1, \dots, \alpha_{k+1}) : \alpha_1 \in A_1, \dots, \alpha_{k+1} \in A_{k+1}\}$ and $\{I(\alpha_1, \dots, \alpha_{k+1}) : \alpha_1 \in A_1, \dots, \alpha_{k+1} \in A_{k+1}\}$ satisfy all required conditions (i)–(v).

Consequently, for each n we can construct $\{V(\alpha_1, \dots, \alpha_n) : \alpha_1 \in A_1, \dots, \alpha_n \in A_n\}$ and $\{I(\alpha_1, \dots, \alpha_n) : \alpha_1 \in A_1, \dots, \alpha_n \in A_n\}$ satisfying (i)–(v).

Next, put

$$Y_n = \bigcup \{\overline{V(\alpha_1, \dots, \alpha_n)} \times I(\alpha_1, \dots, \alpha_n) : \alpha_1 \in A_1, \dots, \alpha_n \in A_n\}$$

and

$$Y = \bigcap_{n=1}^{\infty} Y_n.$$

Since $\{\overline{V(\alpha_1, \dots, \alpha_n)} : \alpha_1 \in A_1, \dots, \alpha_n \in A_n\}$ is locally finite in X , Y_n is

closed in $X \times I$ and thus Y is closed in $X \times I$. Also by (v) the collection

$$\mathcal{H}_n = \mathcal{F}_n|Y = \{F \cap Y: F \in \mathcal{F}_n\}$$

is a locally finite closed cover of Y for $n = 1, 2, \dots$.

Now we show that Y is a Σ -space. For this purpose it suffices to show that $\{\mathcal{H}_n: n = 1, 2, \dots\}$ satisfies the condition in 3.5. Let $y \in Y$ be any point and $\{y_n: n = 1, 2, \dots\}$ any sequence with $y_n \in C(y, \mathcal{H}_n)$. Since $C(y, \mathcal{H}_n) \subset C(y, \mathcal{F}_n)$ for each n and since $X \times I$ is a Σ^* -space, by 3.5 $\{y_n: n = 1, 2, \dots\}$ has a cluster point in $X \times I$. Since Y is closed in $X \times I$, $\{y_n: n = 1, 2, \dots\}$ must have a cluster point in Y , which shows by 3.5 that Y is a Σ -space.

Finally, let π be the restriction to Y of the projection from $X \times I$ onto Y . Since the projection is perfect and since Y is closed in $X \times I$, π is perfect. It remains to show that π is onto, because a Σ -space is preserved by a perfect map (cf. [6]). Let x be any point of X . Since $\{V(\alpha_1, \dots, \alpha_n): \alpha_1 \in A_1, \dots, \alpha_n \in A_n\}$ covers X for $n = 1, 2, \dots$, by (ii) we can choose a point $(\alpha_1, \alpha_2, \dots)$ in $A_1 \times A_2 \times \dots$ with $x \in V(\alpha_1, \dots, \alpha_n)$ for $n = 1, 2, \dots$. Since each $V(\alpha_1, \dots, \alpha_n)$ is non-empty, by (iv) $\{I(\alpha_1, \dots, \alpha_n): n = 1, 2, \dots\}$ is a decreasing sequence of nonempty closed intervals. Hence $\bigcap_{n=1}^{\infty} I(\alpha_1, \dots, \alpha_n) \neq \emptyset$. Pick a point q in this intersection. Then (x, q) belongs to $\overline{V(\alpha_1, \dots, \alpha_n)} \times I(\alpha_1, \dots, \alpha_n) \subset Y_n$ for $n = 1, 2, \dots$ and thus belongs to Y . Clearly $\pi((x, q)) = x$. This shows that π is onto and hence X is a Σ -space, which completes the proof.

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