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### A PROBABILISTIC METHOD FOR THE RATE OF CON-VERGENCE TO THE DIRICHLET PROBLEM

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The expectation  $E^p(\Phi)$  approximates the solution  $u(z) = E^w(\Phi)$  of the Dirichlet problem for a plane domain D with boundary conditions  $\phi$  on the boundary  $\gamma$  of D, where W is Wiener measure, P is the measure generated by a random walk which approximates Brownian motion beginning at z, and  $\Phi$  is the functional on paths which equals the value of  $\phi$  at the point where the path first meets  $\gamma$ . This paper develops a specific rate of convergence. If  $\gamma$  is  $C^2$ , and  $P^n$  is generated by random walks beginning at z, with independent increments in the coordinate directions at intervals 1/n, with mean zero, variance  $1/\sqrt{n}$ , and absolute third moment bounded by M, then  $|E^{P^n}(\Phi) - E^w(\Phi)| \leq (CMV/\rho(z, \gamma)) n^{-1/16} (\log n)^{9/8}$ , where V is the total variation of  $\phi$  on  $\gamma$ ,  $\rho(z, \gamma)$  is the distance from z to  $\gamma$ , and C is a constant depending only on  $\gamma$ .

Assume *D* is a Jordan region. If  $z_t = x_t + iy_t$  is Brownian motion in  $\mathbb{R}^2$  beginning at  $z_0$ , (cf. e.g., [5, p. 262]), and  $\tau =$ inf $\{t: z_t \in \gamma\}$  is the first time *z* hits the boundary  $\gamma$  of *D*, then  $\Phi$  is the functional given by  $\Phi(z_{\cdot}) = \phi(z_{\tau})$ . Let  $E^{W}(\Phi(z_{\cdot})) = \int \Phi(z_{\cdot})dW$  be the expectation of  $\Phi$  with respect to Wiener measure *W* on  $C([0, \infty),$  $\mathcal{C})$ . (See [8, pp. 218-19] for a definition of Brownian motion on the interval [0, 1] and the corresponding Wiener measure.)

Let  $g_1^1, g_2^2, g_2^1, g_2^2, \dots, g_k^1, g_k^2, \dots$  be a sequence of indendent random variables with mean zero, variance 1, and absolute third moment bounded uniformly by  $M < \infty$ , and let

$$\xi_i^lpha = g_i^lpha / \sqrt{n}, \, \zeta_0 = z_0, \, \zeta_k = z_0 + \sum_{i=1}^k \, (\hat{\xi}_i^1 + \sqrt{-1} \hat{\xi}_i^2), \, t_k = k/n \; .$$

Let  $\xi(t)$  be the continuous random broken line which has vertices  $(t_k, \zeta_k)$  and is linear between vertices. Let  $P^n$  be the measure on  $C([0, \infty), \mathcal{C})$  generated by this line, i.e.,  $P^n(S) = P(\xi(t) \in S)$ .

Now by the Central Limit Theorem  $P^n(\xi^{\alpha}(t) \leq \lambda) \to W(z_t^{\alpha} \leq \lambda)$ ,  $\alpha = 1, 2$ , where  $\xi^{\alpha}(t), z_t^{\alpha}$  are the real and imaginary parts of  $\xi(t), z_t$ respectively, (cf. e.g., [1, pp. 186-7]). More exactly one has the Barry-Esseen Theorem [3, p. 521]: For nt an integer

(1.1) 
$$\sup_{\lambda} |P(\xi^{\alpha}(t) \leq \lambda) - N(\lambda/\sqrt{t})| \leq \frac{33}{4}M/\sqrt{nt}$$

where N(x) is the normal distribution. A useful generalization of the

Central Limit Theorem is that convergence also takes place for the expectation of any functional on C[0, 1) which is continuous with respect to uniform convergence on [0, 1] and satisfies mild growth conditions, e.g.  $\Phi(x) = \int_0^1 x(t)^2 dt$ ,  $\sup_{0 \le t \le 1} x_t$ , etc. Rates of convergence have been calculated for some specific one-dimensional functionals  $\Phi$ , (e.g., [10], [11]). For an arbitrary functional  $\Phi$  satisfying a uniform Hölder condition one can get rates of convergence using Levy distance in C[0, t] ([9], see also §2 of this paper). Explicit rates of convergence are of interest for various practical problems and computer applications.

Although  $\Phi(z_{\cdot}) = \phi(z_{\tau})$  is not continuous with respect to uniform convergence, it is continuous a.s. with respect to Wiener measure, so convergence takes place. In this paper we obtain a rate of convergence.

THEOREM. There exists a universal constant  $C^* = C^*(\gamma)$  such that

(1.2) 
$$|E^{P^n}(\Phi) - E^{W}(\Phi)| \leq \frac{C^* V(\phi) M}{\rho(z_0, \gamma)} n^{-1/16} (\log n)^{9/8}$$

where  $V(\phi)$  is the total variation of  $\phi$  on  $\gamma$ , M is the bound on absolute third moments defined above,  $z_0$  is the initial point of the paths  $z_{*}$ , and  $\rho(z_0, \gamma) = \inf_s |z_0 - \gamma(s)|$  is the distance from  $z_0$  to  $\gamma$ .

2. Levy distance. We define measures  $P_t^n$ ,  $W_t$  on  $C([0, t], \mathcal{C})$  by

$$P_t^n(S) = P^n(\pi^{-1}S), \ W_t(S) = \ W(\pi^{-1}S) \ ,$$

where  $\pi: C([0, \infty), \mathcal{C}) \to C([0, t], \mathcal{C})$  is the projection  $\pi(f) = f|_{[0,t]}$ . The Levy distance L between the measures  $P_t^n$  and  $W_t$  is given by

(2.1) 
$$L(P_t^n, W_t) = \max(\varepsilon_1, \varepsilon_2),$$

where

$$egin{aligned} &arepsilon_1 &= \inf\left\{arepsilon: P_t^n(S) \leq W_t(S^{arepsilon,t}) + arepsilon ext{ for all closed sets } S 
ight\}, \ &arepsilon_2 &= \inf\left\{arepsilon: W_t(S) \leq P_t^n(S^{arepsilon,t}) + arepsilon ext{ for all closed sets } S 
ight\}, \end{aligned}$$

and

$$S^{arepsilon,t} = \{y \colon \exists x \in S 
i \sup_{0 \leq s \leq t} | \, y(s) \, - \, x(s) \, | \, < arepsilon \}$$

is an  $\varepsilon$ -neighborhood of S with respect to the sup-norm on [0, t].

The following proposition is a direct generalization of a result of Prokhorov ([9]) to two dimensions as is its proof.

**PROPOSITION 1.** There exists an absolute constant C such that

(2.2) 
$$L(P_1^n, W_1) \leq CM^{1/4} n^{-1/8} (\log n)^{15/8}$$
.

#### COROLLARY.

If t = k/n, k an integer, then  $L(P_t^n, W_t) \leq C\sqrt{t}k^{-1/8}(\log k)^{15/8}$  for some constant C.

3. Boundedness of harmonic density. Fix a point  $\gamma_0$  on  $\gamma$ , and a direction along  $\gamma$ , parametrize  $\gamma$  by arclength in the chosen direction from  $\gamma_0$ . Let *l* denote the length of  $\gamma$ , and take the argument *s* of  $\gamma = \gamma(s) \mod l$ .

Since  $\gamma$  is  $C^2$ , there exists R > 0 such that any circle of radius R will meet  $\gamma$  in at most two points. It follows that for any two points  $\gamma(a)$  and  $\gamma(a + \delta)$  on  $\gamma$  where  $0 < \delta < R$  that  $\gamma([a, a + \delta])$  will lie in the intersection of the closed disks bounded by the two circles of radius R through  $\gamma(a)$  and  $\gamma(a + \delta)$ . The case we have to eliminate is where  $\gamma$  is tangent to one of the circles at  $\gamma(a)$  and  $\gamma(a + \delta)$ , but does not cross the circle, i.e., there are neighborhoods in  $\gamma$  of  $\gamma(a)$  and  $\gamma(a + \delta)$  which do not meet the closed disk bounded by the circle except at  $\gamma(a)$  or  $\gamma(a + \delta)$ . But in this case we observe that a small rotation of the circle about one of the points  $\gamma(a)$  or  $\gamma(a + \delta)$  will result in three points of intersection, contradicting our assumption about  $\gamma$ . Furthermore, it follows from the Jordan curve theorem that the center of one of the two circles will be in D, the other center will be outside D.

We are now ready to prove the following result.

**PROPOSITION 2.** 

$$W(z_{\tau} \in \gamma([a, a + \delta])) \leq B\delta/\rho(z_0, \gamma)$$

where B is an absolute constant depending only on  $\gamma$ .

*Proof.* We may assume  $\delta < R$  and also  $\delta$  sufficiently small that

$$2(R-(R^{2}-\delta^{2}/4)^{1/2})<
ho(z_{0},\gamma)/2$$
 ,

since by addition if the proposition holds for small  $\delta$ , it holds for  $\delta$  in general.

Let C be the circle of radius R through  $\gamma(a)$  and  $\gamma(a + \delta)$ , with center not in D. Then

$$P_{z_0}(z_{\tau} \in \gamma([a, a + \delta])) \leq P_{z_0}(z_{\tau(C)} \in \delta^*)$$

where  $\tau(C) = \inf \{t: z_t \in C\}$ , and  $\delta^* = D \cap C$ . Now invert the plane with respect to the circle C, sending  $z_0$  into  $I(z_0)$ . Now  $I(z_t)$  is Brownian motion with a time change. (P. Lévy [7, p. 254], see also [5, pp. 279-80] for another proof of this.) However, where  $I(z_t)$  first hits C is independent of any time change; "Les proprietés intrinsèques de la courbe C sont invariantes par une representation conforme." Now

$$z_{ au_{(C)}} \in C \Longrightarrow I(z_{ au_{(C)}}) = z_{ au_{(C)}}, \, I(\delta^*) = \delta^*$$
 ,

so  $P_{z_0}(z_{\tau^{(C)}} \in \delta^*) = P_{z_0}((I(z))_{\tau^{\prime(C)}} \in \delta^*)$  where  $\tau^{\prime}(C) = \inf \{s: (I(z))_s \in C\}.$ 

But the harmonic density on a circle is given by the Poisson kernel (cf. e.g., [4, p. 361 ff.]); it is bounded,

$$P_{z_0}((I(z))_{ au'(C)} \in \delta^*) \leq rac{2}{2\pi} |\, \delta^* \, | / 
ho(I(z_0), \, C)$$

where  $|\delta^*|$  is the length of  $\delta^*$ . Now

$$R - 
ho(I(z_0), C) = R^2/(
ho(z_0, C) + R)$$

so

$$1/
ho(I(z_{\scriptscriptstyle 0}),\,C)\,=\,(
ho(z_{\scriptscriptstyle 0},\,C)\,+\,R)/R
ho(z_{\scriptscriptstyle 0},\,C)\,\leq rac{{\it \Delta}/R\,+\,1}{
ho(z_{\scriptscriptstyle 0},\,C)}\;,$$

where  $\Delta$  is the diameter of D. Now look at  $\rho(z_0, C)$ :

$$ho(z_0, C) \ge 
ho(z_0, \gamma) - 2(R - (R^2 - s^2/4)^{1/2})$$

where  $s = |\gamma(a + \delta) - \gamma(a)| \leq \delta$ . But  $\delta$  was sufficiently small that

$$2(R-(R^{\scriptscriptstyle 2}-\delta^{\scriptscriptstyle 2}\!/4)^{\scriptscriptstyle 1/2})<
ho(z_{\scriptscriptstyle 0},\gamma)/2$$
 ,

and since  $s \leq \delta$ ,

$$2(R-(R^2-s^2\!/4)^{\scriptscriptstyle 1/2}) \leq 2(R-(R^2-\delta^2\!/4)^{\scriptscriptstyle 1/2})$$
 .

Hence

$$ho(z_{\scriptscriptstyle 0},\,C) > 
ho(z_{\scriptscriptstyle 0},\,\gamma)/2$$
, also  $|\,\delta^*\,| \leq rac{\pi}{2}s \leq rac{\pi}{2}\delta$ 

and it follows that

$$W(z_{\tau} \in \gamma([a, a + \delta])) \leq rac{2}{2\pi} rac{\pi}{2} \delta \cdot 2(\varDelta/R + 1)/
ho(z_{0}, \gamma) = B\delta/
ho(z_{0}, \gamma) \; .$$

4. Some inequalities. We shall need the following.

LEMMA.

(4.1)  
$$W(\tau > t) \leq \frac{4}{\pi} \exp\left(-\frac{\pi^2 t}{8d^2}\right)$$
$$P^n(\tau > t) \leq \frac{4}{\pi} \exp\left(-\frac{\pi^2 t}{8d^2}\right) + AM(nt)^{-1/8}(\log nt)^{1/2},$$

where  $\Delta$  is the diameter of D and A is an absolute constant.

Proof.

$$egin{aligned} W( au > t) & \leq \Pr\left(\max_{0 \leq s \leq t} |z_s - z_0| < arDelta
ight) \ & \leq \Pr\left(\max_{0 \leq s \leq 1} |\operatorname{Re}\left(z_s - z_0
ight)| < arDelta/arVt
ight) = T(arDelta/arVt
ight) \ & \leq rac{4}{\pi} \exp\left(-\pi^2 t/8arDelta^2
ight) \,, \end{aligned}$$

where  $T(\lambda) = \Pr(\max_{0 \le s \le 1} |x_s| < \lambda)$ . The last inequality comes from the fact that the infinite series expansion for  $T(\lambda)$  [11] is alternating, with decreasing terms.

$$egin{aligned} P^n( au > t) &\leq \Pr\max_{k \leq nt} \left( |\zeta_k - z_0| < arDelta 
ight) \ &\leq \Pr\left( \max_{k \leq nt} |\operatorname{Re}\left( \zeta_k - z_0 
ight) / \sqrt{|t|} < arDelta / \sqrt{|t|} 
ight) \,. \end{aligned}$$

Now the theorem of Rosencrantz [10] applies [11] and we have

$$\Pr\left(\max_{k \leq nt} |\operatorname{Re}\left(\zeta_k - z_0
ight)/\sqrt{t}
ight| < arDelt/\sqrt{t}
ight) \ \leq A \cdot M (\log nt)^{1/2} (nt)^{-1/8} + T(arDelt/\sqrt{t})$$

where A is an absolute constant. But we saw above that

$$T(arDelta/\sqrt{t\,)} \leq rac{4}{\pi} \exp\left(- \,\pi^2 t/8 arDelta^2
ight)$$
 ,

 $\mathbf{SO}$ 

$$P^n(\tau > t) \leq rac{4}{\pi} \exp{(-\pi^2 t/8 \varDelta^2)} + AM(nt)^{-1/8} (\log{nt})^{1/2}$$
.

Now we need more notation. Let  $K_{\lambda} = \gamma([0, \lambda])$ , let  $(z_{\tau} \in K_{\lambda})^{e,\tau} \subset C([0, \infty), \mathscr{O})$  be defined by  $y \in (z_{\tau} \in K_{\lambda})^{e,\tau}$  iff  $\exists z$  such that  $z_{\tau} \in K_{\lambda}$  and (for  $\tau = \tau(z)$ )  $\sup_{0 \leq s \leq \tau} |y_s - z_s| < \varepsilon$ . Let  $\delta = \sqrt{\varepsilon}$ , and let  $K_{\lambda}^s = \gamma([0, \lambda + \delta]) \cup \gamma(l - \delta, l]$ ), where l is the length of  $\gamma$ .

**PROPOSITION 3.** 

$$W((z_{ au} \in K_{\lambda})^{arepsilon, au} \cap (z_{ au} \oplus K_{\lambda}^{s})) \leq G \sqrt{arepsilon}$$

where G is a constant depending only on  $\gamma$ .

*Proof.* Let  $\tau(\partial \varepsilon) = \inf \{t: \rho(z_i, K_\lambda) < \varepsilon\}$  where  $\rho(z_i, K_\lambda)$  is the distance from  $z_i$  to  $K_\lambda$ . Then

$$\begin{split} W((z_{\tau} \in K_{\lambda})^{\varepsilon,\tau} \cap (z_{\tau} \notin K_{\lambda}^{\delta})) \\ & \leq W(\tau(\partial \varepsilon) < \tau, \, z_{\tau} \notin K_{\lambda}^{\delta}) + W(\tau(\partial \varepsilon) > \tau, \, z_{\tau} \notin K_{\lambda}^{\delta}, \, \tau(\partial \varepsilon) < \tau(s\varepsilon)) \\ & = E^{W}(\chi_{\tau(\partial \varepsilon) < \tau} P_{z_{\tau(\partial \varepsilon)}}(z_{\tau} \notin K_{\lambda}^{\gamma}) \\ & + \chi_{[\tau(\partial \varepsilon) > \tau, z_{\tau} \notin K_{\lambda}]} P_{z_{\tau}}(\tau(\partial \varepsilon) < \tau(s\varepsilon))) \end{split}$$

by the strong Markov property [1, p. 268], where  $\tau(s\varepsilon) = \inf \{t: \rho(z_i, D) > \varepsilon\}$ . We estimate  $P_{z_{\tau}(\partial_{\varepsilon})}(z_{\tau} \notin K_{\lambda}^{\delta})$ :

Let  $\gamma(a)$  be a point in  $K_{\lambda}$  of distance  $\varepsilon$  from  $z_{\tau(\partial \varepsilon)}$ , let T be the tangent to  $\gamma$  at  $\gamma(a)$ . Let  $S_i$  (i = 1, 2) be lines perpendicular to T through the points  $\gamma(a - \delta)$  and  $\gamma(a + \delta)$ . The distance  $d_i$  from  $z_{\tau(\partial \varepsilon)}$  to each of the lines  $S_i$  will be less than  $\delta + \varepsilon$  (less than  $\delta$  unless  $\gamma(a)$  is an endpoint of  $K_{\lambda}$ ; let  $d = \min(d_1, d_2)$ . Let T' be parallel to T, at a distance  $\varepsilon \cdot \sup |\gamma''|$  on the opposite side of T from  $z_{\tau(\partial \varepsilon)}$ . I now claim  $\gamma([a - \delta, a + \delta]) \cap T' = \emptyset$  if  $2\delta < 1/\sup |\gamma''|$ . Choose coordinates such that  $\gamma(a) = 0$ ,  $\gamma'(a) > 0$ . Then by Taylor's Theorem, for each h there exists  $\theta$  such that

$$egin{array}{lll} \operatorname{Im}\gamma(a+h\delta) &= \operatorname{Im}\gamma(a)+\operatorname{Im}\gamma'(a)\!\cdot h\delta + \operatorname{Im}\gamma''(a+ heta h\delta)\!\cdot h^2\delta^2/2 \ &= \operatorname{Im}\gamma''(a+ heta h\delta)\!\cdot h^2\delta^2/2 \;. \end{array}$$

Hence for  $|h| \leq 1$ ,  $|\operatorname{Im} \gamma(a + h\delta)| \leq \sup |\delta''| \cdot \delta^2/2 < \varepsilon \cdot \sup |\gamma''|$  and  $\gamma([a - \delta, a + \delta])$  does not meet T'.

Let  $\tau_{T'}$  be the first time (after  $\tau(\partial \varepsilon)$ ) that  $z_i$  hits the line  $T', \tau_s$ the first time (after  $\tau(\partial \varepsilon)$ ) that  $z_i$  hits  $S_1 \cup S_2$ , and  $c = \rho(z_{\tau(\partial \varepsilon)}, T') \leq \varepsilon \cdot (\sup |\gamma''| + 1)$ . Note that  $\tau_{T'}$  and  $\tau_s$  are independent, since the components of Brownian motion in the direction of  $S_i$  and T' are independent. We can write

$$egin{aligned} &P_{z_{ au(\delta\epsilon)}}(z_{ au} \in K^{\delta}_{\lambda}) < P_{z_{ au(\delta\epsilon)}}( au_{T}' > au_{S}) + O(\delta) \ &= \int_{0}^{\infty} P_{z_{ au(\partial\epsilon)}}( au_{S} < t) d_{t} P_{z_{ au(\partial\epsilon)}}( au_{T'} \leq t) + O(\delta) \ . \end{aligned}$$

Now

$$egin{aligned} P_{z_{ au}(m{\partial} ec{e})}( au_{T'} &\leq t) &= P(\sup_{0 \leq s \leq t} x_s \geq c) = P(\sup_{0 \leq s \leq 1} x_s \geq c/\sqrt{t}) \ &= \sqrt{2/\pi} \int_{c/\sqrt{t}}^{\infty} e^{-u^2/2} du \end{aligned}$$

(cf. e.g., [1, p. 287] and [8, p. 227]). Hence

$$egin{aligned} P_{z_{m{ au}(\delta t)}}( au_{T'} > au_{S}) &= \int_{0}^{\infty} P_{z_{m{ au}(\delta t)}}( au_{S} < t) \sqrt{2/\pi} rac{1}{2} c t^{-3/2} e^{-c^{2}/t} dt \ &\leq \int_{0}^{\infty} P(\sup_{0 \le s \le t} |x_{s}| > d) (c/\sqrt{2\pi}) t^{-3/2} e^{-c^{2}/2t} dt \ &\leq 2 \int_{0}^{\infty} P(\sup_{0 \le s \le 1} x_{s} > d/\sqrt{t}) (c/\sqrt{2\pi}) t^{-3/2} e^{-c^{2}/2t} dt \end{aligned}$$

which by a straightforward computation, is bounded by  $2(d^2/c^2 + 1)^{-1/2}$ .

But I claim  $d \sim \delta$ : choose coordinate such that  $\gamma(a) = 0$ ,  $\gamma'(a) = 1$ . 1. Using Taylor's Theorem we get  $\delta \ge d \ge \delta - (\sup |\gamma''|/2)\delta^2 - \varepsilon$ , so  $d \sim \delta$ . And  $\delta = \sqrt{\varepsilon}$ ,  $c \le \varepsilon (\sup |\gamma''| + 1)$ , so

$$2(d^2/c^2+1)^{-1/2} \leqq G_1arepsilon/\delta = G_1\sqrt{arepsilon}$$

for some constant  $G_1$ .

Now the same argument can be applied to estimate  $P_{z_{\tau}}(\partial \varepsilon) < \tau(s\varepsilon)$ ) (i.e., the probability that a Brownian path will move a distance  $d \sim \delta = \sqrt{\varepsilon}$  in the direction tangent to the curve before it moves a distance  $c = O(\varepsilon)$  in the direction normal to the curve).

Hence  $P_{z_{\tau}}(\tau(\partial \varepsilon) < \tau(s\varepsilon)) \leq G_2 \sqrt{\varepsilon}$  for some constant  $G_2$  and our proposition follows.

5. Proof of the theorem. We are now ready to prove our theorem.

(5.1

$$egin{aligned} &|E^{P^n}(\varPhi)-E^w(\varPhi)|\ 1)&=|E^{P^n}(\varPhi\chi_{ au\leq t})-E^w(\varPhi\chi_{ au\leq t})+E^{P^n}(\varPhi\chi_{ au>t})-E^w(\varPhi\chi_{ au>t})|\ &\leq E^{P^n}(\varPhi\chi_{ au\leq t})-E^w(\varPhi\chi_{ au\leq t})+\sup_{ au}|\,\phi|\,(P^n( au>t)+W( au>t))\;. \end{aligned}$$

Looking at the first term,

$$egin{aligned} &|E^{P^n}(arPhi\chi_{ au\leq t})-E^w(arPhi\chi_{ au\leq t})|\ &=\left|\int_0^t &\phi(\gamma(\lambda))(P^n(z_ au\in\gamma(d\lambda),\, au\leq t)-W(z_ au\in\gamma(d\lambda),\, au\leq t))
ight|\ &\leq |\phi(\gamma(0))|(P( au>t)+W( au>t))+\int_0^t |P^n(z_ au\in K_\lambda,\, au\leq t)|\ &-W(z_ au\in K_\lambda,\, au\leq t)|\cdot|d\phi(\lambda)|\ . \end{aligned}$$

We estimate the integrand:

The event  $(z_{\tau} \in K_{\lambda}, \tau \leq t)$  is determined by the behavior of the path up to time t, so

$$P^n(z_{\tau} \in K_{\lambda}, \tau \leq t) - W(z_{\tau} \in K_{\lambda}, \tau \leq t) = P^n_t(z_{\tau} \in K_{\lambda}, \tau \leq t) - W_t(z_{\tau} \in K_{\lambda}, \tau \leq t)$$
.

We can use the corollary of Proposition 1 to get

$$egin{aligned} &P_t^n(z_ au\in K_\lambda,\, au\leq t)\leq W_t((z_ au\in K_\lambda,\, au\leq t)^{arepsilon,t})+arepsilon\ &\leq W_t(z_ au\in K_\lambda,\, au\leq t)+W_t((z_ au\in K_\lambda,\, au\leq t)^{arepsilon,t})\ &-(z_ au\in K_\lambda^\delta,\, au\leq t))+W_t(z_ au\in K_\lambda^\delta-K_\lambda,\, au\leq t)+arepsilon\,, \end{aligned}$$

where  $\varepsilon = \varepsilon(n, t) = CM^{1/4}n^{-1/8}t^{3/8}(\log nt)^{15/8}$ .

Now  $y \in (z_{\tau} \in K_{\lambda}, \tau \leq t)^{\varepsilon, t}$  means  $\exists z \text{ such that } \tau \leq t, z_{\tau} \in K_{\lambda}$ ,

 $\sup_{0 \le s \le t} |y_s - z_s| < \varepsilon$ . As this condition does not depend on  $y_s$  for s > t,

$$\begin{split} W_t((z_\tau \in K_\lambda, \tau \leq t)^{\varepsilon, t} - (z_\tau \in K_\lambda^{\delta}, \tau \leq t)) \\ &\leq W((z_\tau \in K_\lambda, \tau \leq t)^{\varepsilon, \tau} - (z_\tau \in K_\lambda^{\delta}, \tau \leq t)) \\ &\leq W((z_\tau \in K_\lambda))^{\varepsilon, \tau} - (z_\tau \in K_\lambda^{\delta})) + W(\tau > t) . \end{split}$$

Applying Propositions 2 and 3, we then have

$$P^n(z_{ au} \in K_\lambda, au \leq t) - W(z_{ au} \in K_\lambda, au \leq t) \ \leq (G + 2B/
ho(z_0, \gamma))\sqrt{arepsilon} + W( au > t) \; .$$

We apply the above argument to the complement  $\gamma - K_{\lambda}$  of  $K_{\lambda}$  in  $\gamma$ .

$$P^{n}(z_{ au} \in \gamma - K_{\lambda}, au \leq t) - W(z_{ au} \in \gamma - K_{\lambda}, au \leq t) \ \leq (G + 2B/
ho(z_{ au}, \gamma)) \sqrt{|arepsilon|} + W( au > t) \; .$$

It follows that

$$egin{aligned} &|P^n(z_{ au}\in K_\lambda,\, au\leq t)-W(z_{ au}\in K_\lambda,\, au\leq t)|\leq (G+2B/
ho(z_0,\,\gamma))\,\sqrt{arepsilon}\ &+W( au>t)+P^n( au>t)\;. \end{aligned}$$

We can now estimate the integral

(5.3) 
$$\int_{0}^{t} |P^{n}(z_{\tau} \in K_{\lambda}, \tau \leq t) - W(z_{\tau} \in K_{\lambda}, \tau \leq t)| \cdot |d\phi(\lambda)|$$
$$\leq ((G + 2B/\rho(z_{0}, \gamma))\sqrt{\varepsilon} + W(\tau > t) + P^{n}(\tau > t))V(\phi)$$

where  $V(\phi)$  is the total variation of  $\phi$  on  $\gamma$ .

Combining the results of (5.1), (5.2), and (5.3), (we have)

$$egin{aligned} &|E^{P^n}(arPi)-E^{\,\scriptscriptstyle W}(arPi)|&\leq \phi(\gamma(0))(P^n( au>t)+W( au>t)\ &+V(\phi)(G+2B/
ho(z_0,\,\gamma))\sqrt{\,arepsilon}+W( au>t)+P^n( au>t))\ &+\sup_{ au}|\phi|(P^n( au>t)+W( au>t))\ &\leq V(\phi)(G+2eta/
ho(z_0,\,\gamma))\sqrt{\,arepsilon}+(V(\phi)+2\sup_{ au}|\phi|)ullet(P^n( au>t)\ &+W( au>t))\ . \end{aligned}$$

This estimate is minimized by choosing t so as to balance the factors  $\sqrt{\varepsilon}$  and  $(P^{*}(\tau > t) + W(\tau > t))$ . So setting

$$t = \min\left\{s:s \ge rac{1}{2}(arDelta/\pi)^2 \mathrm{log}\;n,\,sn\; ext{ an integer}
ight\}$$
 ,

we get

$$egin{aligned} P^{*}( au > t) &+ W( au > t) \ &\leq rac{8}{\pi} n^{-1/16} + A \cdot M n^{-1/8} \Bigl(rac{1}{2} (arDelta/\pi)^2 \log n \Bigr)^{1/8} (\log nt)^{1/2} \ &\leq A_1 M n^{-1/16} (\log n)^{5/8} \ , \end{aligned}$$

and

$$egin{aligned} \sqrt{arepsilon} &= \sqrt{C}\,M^{1/8}n^{-1/16}t^{3/16}(\log\,nt)^{15/16} \ &\leq \sqrt{C}\,M^{1/8}n^{-1/16}A_2(\log\,n)^{9/8} \end{aligned}$$

where  $A_1$ ,  $A_2$  are absolute constants. Hence

$$egin{aligned} |E^{{}_{P}n}(arPhi)-E^{{}_{W}}(arPhi)| &\leq V(\phi)Mrac{GarDelta+2B}{
ho(z_{0},\,\gamma)}\sqrt{C}\,A_{2}n^{-1/16}(\log n)^{\mathfrak{g}/8}\ &+(V(\phi)+2\sup_{7}|\,\phi\,|)A_{1}Mn^{-1/16}(\log n)^{\mathfrak{g}/8} \end{aligned}$$

$$\leq (3V(\phi)+2\inf_{ au}|\phi|)M\Big(rac{GarDelta+2B}{
ho(z_{ ext{o}}, au)}\sqrt{C} \,\,A_{ ext{o}}+A_{ ext{i}}\Big)n^{-1/16}(\log n)^{ heta/8}\,.$$

But integration is linear, so we may assume  $\phi(p) = 0$  for some p in  $\gamma$ , as we are taking the difference of expectations.

Letting  $C^* = 3((G\varDelta + 2B)\sqrt{C}A_2 + \varDelta A_1)$  we have

$$|E^{{}_{F^n}}\!(arPhi) - E^{{}_W}\!(arPhi)| \!\leq \! rac{C^* V(\phi) M}{
ho(z_{\mathfrak{o}}, \gamma)} n^{-{\scriptscriptstyle 1/16}} (\log n)^{{\scriptscriptstyle 9/8}} \, .$$

COROLLARY. If O is any subset of  $\gamma$  consisting of a finite number k of intervals, then

$$|P_{z_0}^n(z_{\tau} \in O) - W_{z_0}(z_{\tau} \in O)| \leq rac{2kC^*M}{
ho(z_0, \gamma)} n^{-1/16} (\log n)^{9/8}$$
.

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