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# SOUSLIN'S CONJECTURE AS A PROBLEM ON THE REAL LINE

A. P. BAARTZ AND G. G. MILLER

**This paper is concerned with properties of real sets whose existence is related to Souslin's conjecture. One of these results is subsequently used to show that Souslin's conjecture is second order determined, i.e.,** $(\mathscr{Z} \vdash_2 SC) \vee (\mathscr{Z} \vdash_2 \sim SC)$.

By *Souslin's conjecture* (*SC*) we mean: every linearly ordered set with at most countably many pairwise disjoint intervals is separable. (A linearly ordered set $L$ is *separable* if it has a countable subset such that between any two points of $L$ there is a point of the subset). We first display a subset of the power set of the real line $R$ whose existence is equivalent to $\sim SC$. Then we reformulate the conjecture geometrically as a question concerning a single subset of $R$ of a certain type. Finally we point out that Souslin's conjecture is second order determined.

E. Miller [4] proved that $\sim SC$ is equivalent to the existence of a *Souslin tree*, i.e., an uncountable tree of countable height and countable width. A *tree* is a partially ordered set in which the set of all elements below any given element is a chain. The *height* of a partially ordered set $P$ is the least cardinal $\mathfrak{m}$ such that no chain in $P$ has cardinality greater than $\mathfrak{m}$. A is an *antichain* if no two elements of $A$ are related. The *width* of $P$ is the least cardinal $\mathfrak{n}$ such that no antichain in $P$ has cardinality greater than $\mathfrak{n}$.

PROPOSITION 1.1. *The existence of a Souslin tree is equivalent to the existence of an uncountable collection of real sets such that*

1. *any two sets in the collection are either disjoint or one of them is a subset of the other, and*

2. *if* $\mathscr{G}$ *is any uncountable subcollection, then* $\mathscr{G}$ *has two disjoint members and two nondisjoint members.*

*Proof.* Assume there is a Souslin tree $S$. Let $f$ be a one-to-one function from some uncountable subset of $S$ into $R$. For each $x \in S$, let $U(x) = \{y: x \leqq y\}$, and let $\mathscr{F} = \{f(U(x)): x \in S\}$. Then $\mathscr{F}$ has the desired properties.

Conversely, if there is such a collection $\mathscr{F}$, let $A \leqq B$ mean $B \subseteqq A$, for $A, B \in \mathscr{F}$. Then $\mathscr{F}$ is a Souslin tree.

An application of Proposition 1.1 is found in §5. In the next section we show how a Souslin tree can be represented as a single subset of the line.

**2. We first represent certain binary relations.** For this purpose let $G \subset R$ and denote by $G^*$ the set of all those points $x \in G$ which are midpoints of a nondegenerate segment whose endpoints are both in $G$. We shall call $G^*$ the *set of midpoints in* $G$. Define a relation $\alpha$ on $G^*$ by setting

$$x\alpha y \text{ iff } x \neq y \text{ and there exists } z \in G \text{ such that } y \text{ is the midpoint of the segment } xz. \tag{2.1}$$

Note that $xz$ stands for $[x, z]$ or $[z, x]$ according as $x < z$ or $z < x$.

PROPOSITION 2.1. *$\alpha$ is a (strict) partial order for $G^*$ iff for all elements $x, y, z \in G^*$ we have*

A. *(asymmetry) if $x$ and $y$ are the respective midpoints of $yv$ and $xu$, and if $u \in G$, then $v \notin G$.*

B. *(transitivity) if $y$ is the midpoint of $xu$ and $z$ is the midpoint of both $yv$ and $xw$, and if $u, v \in G$, then also $w \in G$.*

The proof is immediate since no point is both midpoint and endpoint of the same nondegenerate segment.

THEOREM 2.2. *Let $\delta$ be any antireflexive relation on a set $P$ of cardinality no larger than that of the continuum. Then there exists a subset $G$ of the real line for which the relation $\alpha$ defined by* (2.1) *is isomorphic to $\delta$.*

*Proof.* Let $f$ be a one-to-one function mapping $P$ into a Hamel basis for $R$. Let

$$U = \{2f(q) - f(p) : p, q \in P,\ p\delta q\} \tag{2.2}$$

and

$$G = U \cup 2f[P] \cup f[P] \cup \{0\}\ . \tag{2.3}$$

For each $p \in P$, $f(p)$ is the midpoint of the segment $2f(p)0$, whose endpoints belong to $G$. Thus $f[P] \subset G^*$. If $y \in G^*$, on the other hand, then there exist distinct points $x, z \in G$, such that $2y = z + x$. Also, $y \in G$. Writing $x = c_1a_1 + c_2a_2$ and $z = c_3a_3 + c_4a_4$, with $a_i \in f[P]$, we have

$$\begin{aligned} &c_1 = 2 \quad \text{and} \quad c_2 = -1 \quad \text{if} \quad x \in U\,, \\ &c_3 = 2 \quad \text{and} \quad c_4 = -1 \quad \text{if} \quad z \in U\,, \\ &c_{2k-1} \in \{0, 1, 2\} \quad \text{and} \quad c_{2k} = 0 \text{ otherwise}\,. \end{aligned} \tag{2.4}$$

Assuming now that $y \in U$, $y = 2a - b$, we have $4a - 2b = 2y = \sum c_ia_i$, and since $a \neq b$, (2.4) implies that only $c_1 = 2 = c_3$, $a_1 = a = a_3$

is possible. But this leads to $c_2 = -1 = c_4$, $a_2 = b = a_4$, and hence to $z = x$, which contradicts our assumption.

The cases $y \in 2f[P]$ and $y = 0$ similarly lead to the conclusion $z = x$. Thus by (2.3) we have $y \in f[P]$, and hence $G^* f[P]$.

To see that $f$ is an isomorphism, let $p, q \in P$, $p\delta q$. Then $x = 2f(q) - f(p)$ is a member of $U \subset G$, and $f(q)$ is the midpoint of the nondegenerate segment $xf(p)$. Thus $f(p)\alpha f(q)$. Conversely, if $z\alpha y$ in $G^* = f[P]$, say $z = f(p), y = f(q)$, then $x = 2f(q) - f(p) \in G$ by (2.1). We use (2.3) and the independence of $f[P]$ to show that $x \in U$, and again the independence of $f[P]$ to see that $p\delta q$.

*Comment* 2.3. An obvious generalization of Theorem 2.2 permits us to represent an arbitrary antireflexive relation in a vector space of sufficiently large dimension over a field of characteristic 5 or larger. Here again "$y$ is the midpoint of $xz$" means $2y = z + x, x \neq z$. For characteristic smaller than 5 we might mention that $f[P] \neq G^*$.

COROLLARY 2.4. *Let $P$ be any partially ordered set of cardinal number no larger than that of the continuum. Then there exists a subset $G$ of the real line such that $P$ is isomorphic to the partially ordered set $G^*$ of midpoints in $G$.*

This follows directly from Proposition 2.1 and Theorem 2.2.

We are now ready to apply Theorem 2.2 to trees. In a slight restatement of 2.1, A becomes: no segment with endpoints in $G$ is trisected by points of $G$; $B$ can be summarized by the phrase: $G$ is *midpoint transitive*. Henceforth we assume that $G$ has these two properties.

Chains in $G^*$ are *generating* subsets of $G^*$ in the sense that any two distinct points $x, y$ of a chain generate a segment with endpoints in $G$, one of $x$ and $y$ acting as an endpoint of the segment, the other as the midpoint; i.e. if $u = 2y - x$, $v = 2x - y$, then $u \in G$ or $v \in G$. We call a subset $X$ of $G^*$ *segment free* (antichain) if every subset of $X$ of cardinality $\geqq 2$ fails to be generating. $X$ is *free* (from above) in $G$ provided that for any two distict points $x, y \in X$ and any $u, v, z \in G$, $z$ is not the midpoint of both the segments $xu$ and $yv$.

Combining these notions with 2.1 we obtain our main result. Width bounds the cardinality of segment free sets and height that of generating sets in $G^*$.

THEOREM 2.5. *The existence of a Souslin tree is equivalent to the existence of a subset $G$ of the real line whose set $G^*$ of midpoints in $G$ is uncountable and satisfies*

1. *no segment with endpoints in $G$ is trisected by points of $G$,*
2. *$G$ is midpoint transitive,*
3. *segment free subsets of $G^*$ are free in $G$,*
4. *segment free subsets of $G^*$ are countable,*
5. *generating subsets of $G^*$ are countable.*

*Proof.* 1. and 2. imply that $\alpha$ is a partial order, by 2.1. 3. is the tree property, and 4. and 5. together with the fact that $G^*$ is uncountable make the tree $G^*$ into a Souslin tree. Thus the existence of $G$ implies the existence of a Souslin tree, and if a Souslin tree exists, then $G$ exists by 2.4.

4. In this section we conclude by applying a real line characterization of Souslin's conjecture to obtain a foundations result. In [2] and [3] the continuum hypothesis is shown to be second order determined, i.e.,

$$(\mathscr{Z} \vdash_2 CH) \vee (\mathscr{Z} \vdash_2 \sim CH)$$

where $\mathscr{Z}$ denotes Zermelo's axioms with the axiom of infinity and $CH$ the continuum hypothesis. The reader is referred to Kreisel and Krivine [3] for a detailed discussion.

A modification of the proof in Kreisel and Krivine applies to Souslin's conjecture:

PROPOSITION 4.1. *Souslin's conjecture is second order determined, i.e.,*

$$(\mathscr{Z} \vdash_2 SC) \vee (\mathscr{Z} \vdash_2 \sim SC) .$$

*Proof.* Let $C_\omega$ be the collection of all hereditarily finite sets without individuals, and for $n \in \omega$, let $C_{\omega+n+1} = C_{\omega+n} \cup \mathscr{P}(C_{\omega+n})$, where $\mathscr{P}$ denotes the power set. From Proposition 1.1, Souslin's conjecture states that any collection of real sets which under set inclusion forms a tree of countable height and countable width is countable. We may thus canonically formulate Souslin's conjecture as follows:

$$\begin{aligned} &[X \subset \mathscr{P}(C_{\omega+1}) \wedge (x \in X \wedge y \in X \rightarrow x \cap y = \phi \vee x \subset y \vee y \subset x) \\ &\wedge ((Y \subset X \wedge ((x \in Y \wedge y \in Y \rightarrow x \cap y = \phi) \\ &\vee (x \in Y \wedge y \in Y \rightarrow x \subset y \vee y \subset x))) \rightarrow \bar{\bar{Y}} \leqq \bar{\bar{C}}_\omega)] \rightarrow \bar{\bar{X}} \leqq \bar{\bar{C}}_\omega .\end{aligned}$$

This is expressed by means of quantifiers over $C_{\omega+3}$, since one-to-one correspondences between subsets of $C_{\omega+2}$ are elements of $C_{\omega+3}$. Consequently [3; p. 192] we have $(\mathscr{Z} \vdash_2 SC) \vee (\mathscr{Z} \vdash_2 \sim SC)$.

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UNIVERSITY OF VICTORIA

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# ON SOLUTIONS IN THE REGRESSIVE ISOLS

JOSEPH BARBACK

**Let $f(x)$ be a recursive function and let $D_f(X)$ denote the Nerode canonical extension of $f$ to the isols. Let $A$ and $Y$ be particular isols such that $D_f(A) = Y$. The main results in the paper deal with the following problem: if one of the isols $A$ and $Y$ is regressive, what regressive property if any will the other isol have. It is shown that if $A$ is a regressive isol then $Y$ will be also. Also, it is possible for $Y$ to be a regressive isol while $A$ is not. In this event there exist regressive isols $B$ with $D_f(B) = Y$ and $B \leqq_A A$. Extensions of these results for recursive functions of more than one variable are discussed in the last section of the paper.**

1. Introduction. We will assume that the reader is familiar with the primary definitions and results of the papers listed as references. We will cite some particular definitions and results that have a special role in the paper. $E$ will denote the set of nonnegative integers, $\Lambda$ the collection of isols, $\Lambda^*$ the collection of isolic integers, and $\Lambda_R$ the collection of regressive isols. If $f$ is a partial function from a subset of $E$ into $E$ then $\delta f$ will denote its domain. If $f: E^n \to E$ is a recursive function then $D_f$ will denote the canonical extension of $f$ to the isols. Two sets $\alpha$ and $\beta$ will be *separated*, written $\alpha | \beta$, if there exist disjoint r.e. supersets of $\alpha$ and $\beta$. $j(x, y)$ will denote the familiar recursive pairing function defined by,

$$j(x, y) = x + 1/2(x + y)(x + y + 1) ,$$

and $k$ and $l$ the associated functions with the property $j(k(x), l(x)) = x$. $[\rho_x]$ will be the canonical enumeration for the collection of all finite subsets of $E$, [6]. Associated with this enumeration is the recursive function $r(x)$ having the property $r(x) = \text{card } \rho_x$. We will use a $\sum$ to stand for union among sets (and also $a +$ for a union of two sets).

2. Recursive functions of one variable. Let $f: E \to E$ be a recursive function. If $f$ is a combinatorial function then its extension $D_f$ will map $\Lambda$ into $\Lambda$, and if $f$ is an increasing function then $D_f$ will map $\Lambda_R$ into $\Lambda_R$. Each combinatorial function of one variable will be increasing, but not conversely. The condition needed for $D_f$ to map $\Lambda_R$ into $\Lambda_R$ is that $f$ be an eventually increasing function, [1].

THEOREM 1. *Let $f: E \to E$ be a recursive function and $A$ and $Y$ be* *isols such that* $D_f(A) = Y$. *If* $A$ *is a regressive isol then* $Y$ *will be regressive also.*

*Proof.* Assume $A$ is a regressive isol. Let

$$g(0) = 0 ,$$
$$g(n + 1) = f(n) + g(n) .$$

Then $g$ will be an increasing and recursive function. Hence its canonical extension $D_g$ will map $\Lambda_R$ into $\Lambda_R$. Since

$$g(n + 1) = f(n) + g(n) ,$$

it follows from the Nerode metatheorem for such identities (combining [12, Theorem 10.1] and the representation of the canonical extension of a recursive function [11, 4]), that

$$(1) \qquad D_g(A + 1) = D_f(A) + D_g(A) .$$

Because $A$ is a regressive isol and $g$ is increasing and recursive, each of the isols $A + 1$, $D_g(A + 1)$ and $D_g(A)$ will also be regressive. In addition, $Y = D_f(A)$ is an isol and from (1) it then follows

$$(2) \qquad Y \leqq D_g(A + 1) \text{ and } D_g(A + 1) \in \Lambda_R .$$

In view of a result due to Dekker [4, P8 (a)], (2) implies that $Y$ will be a regressive isol.

REMARK. If $f$ is a recursive function of one variable then although its canonical extension may not map every isol onto an isol, its value may be an isol for some. In addition, it may also occur that the value of $D_f(A)$ will be a regressive isol for an isol $A$ which is non-regressive. An example of such a recursive function will be given in the following section. We want to show next that if this possibility does occur, then there will be a regressive isol $B$ such that $D_f(B) = D_f(A)$. The following lemma essentially gives this result, once the connection is made between the canonical extensions of recursive functions and recursive combinatorial functions.

LEMMA. *Let* $f, g \colon E \to E$ *be recursive combinatorial functions and* $A$ *and* $Y$ *be isols which satisfy the identity,*

$$(1) \qquad D_f(A) = Y + D_g(A) .$$

*If* $Y$ *is a regressive isol, then there will also exist a regressive isol* $B$ *with,*

$$(2) \qquad D_f(B) = Y + D_g(B) .$$

*Proof*. Assume that $Y$ is a regressive isol, and consider separately the following three cases.

*Case* 1. $A$ is finite. Then $A$ will be regressive and we may set $B = A$.

*Case* 2. $A$ is infinite and $Y$ is finite. Let $Y = p \in E$. Set

$$h(x) = p + g(x), \text{ for } x \in E .$$

Then $h$ will be a recursive combinatorial function, since the function $g$ is recursive and combinatorial. By a theorem of Myhill and Nerode [11, Theorem 7], we also obtain,

$$D_h(A) = Y + D_g(A) . \tag{3}$$

Combining (1) and (3) implies

$$D_f(A) = D_h(A) , \tag{4}$$

and since $A$ is an infinite isol, it follows from (4) and a theorem due to Myhill [8], that there will be infinitely many numbers $n$ that satisfy

$$f(n) = h(n) . \tag{5}$$

Let $m$ be the smallest number that satisfies (5), and let $B = m$. Then $B$ will be a regressive solution to (2), since

$$\begin{aligned} D_f(m) &= f(m) \\ &= h(m) \\ &= D_h(m) \\ &= p + D_g(m) \\ &= Y + D_g(m) . \end{aligned}$$

*Case* 3. Both $A$ and $Y$ are infinite isols. Let $\varphi_f$ and $\varphi_g$ be the normal combinatorial operators, and let $[c_i]$ and $[d_i]$ be the sequences of combinatorial coefficients that are associated with the functions $f$ and $g$ respectively. Let $\alpha \in A$ and $\eta \in Y$. Then $\alpha$ and $\eta$ will each be infinite and isolated sets, and also $\eta$ will be regressive. We will assume that

$$\eta | \alpha \text{ and } \eta | \varphi_g(\alpha) , \tag{6}$$

for otherwise an easy modification may be made in the proof. Based on their respective definitions, each of the functions $c_i$ and $d_i$ will be recursive, and also

$$\varphi_f(\alpha) = (j(x, y) \mid \rho_x \subseteqq \alpha \text{ and } y < c_{r(x)}) ,$$
$$\varphi_g(\alpha) = (j(x, y) \mid \rho_x \subseteqq \alpha \text{ and } y < d_{r(x)}) .$$

From (1) and (6) it follows also,

$$\varphi_f(\alpha) \cong \eta + \varphi_g(\alpha) . \tag{7}$$

Let $p$ be a partial recursive function that establishes (7), i.e., $p$ will be defined on $\varphi_f(\alpha)$, will be one-to-one, and will map

$$p: \varphi_f(\alpha) \rightarrow \eta + \varphi_g(\alpha) , \tag{8}$$

one-to-one and onto.

Let $y_x$ be a regressive function that ranges over the set $\eta$.

Our first aim is to define two particular sequences of subsets of $\alpha$ and of $\eta$ respectively, whose corresponding terms will share the property appearing in (8). With each number $n$ we will associate two sets $\alpha_n$ a subset of $\alpha$, and $\eta_n$ a subset of $\eta$. These sets are meant to be the collections of those members of $\alpha$ and $\eta$ respectively, that we can effectively find if we start with the value of $y_n$ and use only the regressive property of the function $y_x$, the separability property in (6), and the recursive and partial recursive properties that appear in (8). Note that the inverse function $p^{-1}$ of $p$ will be well-defined and partial recursive. The particular definition for these sets is as follows; for $n \in E$, the members of $\alpha_n$ and $\eta_n$ are determined by repeated applications of the six rules below,

(i) $y_n \in \eta_n$,
(ii) if $y_k \in \eta_n$ then $(y_0, \cdots, y_k) \subseteqq \eta_n$,
(iii) if $y_k \in \eta_n$ and $p^{-1}(y_k) = j(x, u)$, then $\rho_x \subseteqq \alpha_n$,
(iv) if $a_1, \cdots, a_k \in \alpha_n$, $\rho_x = (a_1, \cdots, a_k)$, $y < c_k$, $pj(x, y) \in \eta$ and $pj(x, y) = y_m$, then $y_m \in \eta_n$,
(v) if $a_1, \cdots, a_k \in \alpha_n$, $\rho_x = (a_1, \cdots, a_k)$, $y < c_k$ and $pj(x, y) = j(u, v)$, then $\rho_u \subseteqq \alpha_n$,
(vi) if $a_1, \cdots, a_k \in \alpha_n$, $\rho_x = (a_1, \cdots, a_k)$, $y < d_k$ and $p^{-1}j(x, y) = j(u, v)$, then $\rho_u \subseteqq \alpha_n$.

Note that each of the sets $\eta_n$ will be non-empty, in view of (i). It may occur that some of the sets $\alpha_n$ are empty, however this will be true for at most only finitely many of the $\alpha_n$. It is easy to see upon a moments reflection that from the value of the number $y_n$ one can effectively enumerate all of the members in each of the sets $\alpha_n$ and $\eta_n$. It follows that each of the sets $\alpha_n$ and $\eta_n$ (for any number $n$) will be r.e. subsets of $\alpha$ and $\eta$ respectively. Since $\alpha$ and $\eta$ are each isolated sets, we see that each of the sets $\alpha_n$ and $\eta_n$ will be finite. It will be useful to list some of these properties and also some that can be arrived at in an easy manner from the six rules above.

$$(9) \qquad (\forall n)[\eta_n \neq \emptyset] \text{ and } (\forall n)(\exists k)[\alpha_{n+k} \neq \emptyset] .$$

$$(10) \qquad (\forall n)(\exists t)[t \geqq n \text{ and } \eta_n = (y_0, \cdots, y_t)] .$$

$$(11) \qquad \alpha_0 \subseteqq \alpha_1 \subseteqq \alpha_2 \subseteqq \cdots \text{ and } \sum_0^\infty \alpha_n \subseteqq \alpha .$$

$$(12) \qquad \eta_0 \subseteqq \eta_1 \subseteqq \eta_2 \subseteqq \cdots \text{ and } \sum_0^\infty \eta_n = \eta .$$

In addition, note that the six rules (i) – (vi) have been so defined so have the following property; if one would simply know only the value of $y_n$, then the totality of those members of $\alpha$ and $\eta$ that could be found by using only the recursive and regressive features present in (8) would be the two sets $\alpha_n$ and $\eta_n$ respectively. It follows from this property that, for $n \in E$

$$(13) \qquad p\colon \varphi_f(\alpha_n) \to \eta_n + \varphi_g(\alpha_n), \text{ one-to-one and onto.}$$

For each number $n \in E$, let the

*torre* number of $\eta_n$ = the largest number $t$ with $\eta_t = \eta_n$ .

In view of (i) and the fact that each of the sets $\eta_n$ is finite, it follows that there will be infinitely many torre numbers. In addition it is easy to see that if $t$ is the torre number of $\eta_n$, then $t \geqq n$ and $\eta_t = \eta_n = (y_0, \cdots, y_t)$. Let $t_x$ denote the strictly increasing function that ranges over the set of all torre numbers. Then

$$(14) \qquad \eta_{t_x} = (y_0, \cdots, y_{t_x}) ,$$

$$(15) \qquad \eta_{t_0} \subseteqq \eta_{t_1} \subseteqq \eta_{t_2} \subseteqq \cdots ,$$

$$(16) \qquad t_x < k \leqq t_{x+1} \Longrightarrow \eta_k = \eta_{t_{x+1}}, \text{ and}$$

$$(17) \qquad \eta = \sum_0^\infty \eta_{t_n} .$$

In addition, by combining the remark prior to (13) with (16) and the fact that $y_n$ is a regressive function, we can also see that $y_{t_x}$ will be a regressive function (of $x$). This turns out to be a very useful property. Another fact that is important to note here is property $A$ given below; it follows from (13), (16), the definitions of $\eta_n$ and its torre number, and the regressive property of $y_{t_x}$.

*Property A.* If we are given the value of $y_k$ then we can effectively determine whether $k \leqq t_0$ or there is a number $x$ such that $t_x < k \leqq t_{x+1}$. In the former event we could also find the value of $y_{t_0}$, and in the latter event both of the numbers $y_{t_x}$ and $y_{t_{x+1}}$ could be found.

Combining (11), (13) and (15) gives,

$$\alpha_{t_0} \subseteqq \alpha_{t_1} \subseteqq \alpha_{t_2} \subseteqq \cdots, \text{ and} \tag{18}$$

$$p\colon \varphi_f(\alpha_{t_x}) \longrightarrow \eta_{t_x} + \varphi_g(\alpha_{t_x}) , \tag{19}$$

one-to-one and onto, for each number $x$. Since $\varphi_f$ and $\varphi_g$ are combinatorial operaors, the inclusions appearing in (18) also imply that

$$\varphi_f(\alpha_{t_x}) \subseteqq \varphi_f(\alpha_{t_{x+1}}) ,$$

and

$$\varphi_g(\alpha_{t_x}) \subseteqq \varphi_g(\alpha_{t_{x+1}}) .$$

Therefore, in view of (15) and (19), we obtain for each number $x \in E$,

$$\begin{aligned} p\colon (\varphi_f(\alpha_{t_{x+1}}) - \varphi_f(\alpha_{t_x})) \\ \longrightarrow (\eta_{t_{x+1}} - \eta_{t_x}) + (\varphi_g(\alpha_{t_{x+1}}) - (\varphi_g(\alpha_{t_x})) , \end{aligned} \tag{20}$$

one-to-one and onto.

We now begin to design a regressive set $\beta$ whose recursive equivalence type will have the desired properties of the lemma. First with each number $y_{t_x}$ a particular finite set $\beta_x$ will be associated. Let the functions $w_x$ and $e_x$ be defined by

$$\begin{aligned} w_x &= \text{cardinality of } \alpha_{t_x} , \\ e_0 &= w_0 , \\ e_{n+1} &= w_{n+1} - w_n . \end{aligned}$$

Since $y_{t_x}$ is a regressive function and since from the value of $y_{t_x}$ we can determine the complete set $\alpha_{t_x}$ (refer to the remarks appearing before (13)), we see that from the value of $y_{t_x}$ alone, each of the numbers $w_x$ and $e_x$ can be computed. Hence each of the mappings $y_{t_n} \rightarrow w_n$ and $y_{t_n} \rightarrow e_n$ will have a partial recursive extension; in the notation of [4] these properties are denoted respectively by

$$y_{t_n} \leqq^* w_n \text{ and } y_{t_n} \leqq^* e_n . \tag{21}$$

We will assume here that $e_0 \geqq 1$ (otherwise the proof would need to be slightly changed). Then, by (18), it will also follow that $e_n \geqq 1$ for each number $n$. For $n \in E$, let

$$\delta_n = [j(y_{t_n}, r) \mid r = 0, 1, \cdots, e_n - 1] . \tag{22}$$

Then $[\delta_n]$ will be a sequence of mutually disjoint nonempty sets. From

(21) and (22), we see that by knowing the value of $y_{t_n}$ we can effectively find all the members of the set $\delta_n$; this property will be denoted by writing

(23) $$y_{t_n} \leqq^* \delta_n .$$

For $n \in E$ set

(24) $$\beta_n = \delta_0 + \delta_1 + \cdots + \delta_n .$$

Then, in view of (23) and the regressiveness of $y_{t_n}$, it is possible to effectively find all the elements of $\beta_n$ from the value of $y_{t_n}$. We will denote this property by

(25) $$y_{t_n} \leqq^* \beta_n .$$

In addition, note that

(26) $$\beta_0 \subseteq \beta_1 \subseteq \beta_2 \subseteq \cdots, \text{ and}$$

(27) $$\text{card } \beta_x = \text{card } \alpha_{t_x} \text{ for every } x \in E .$$

Let

$$\beta = \sum_0^\infty \beta_n = \sum_0^\infty \delta_n .$$

We will assume here that the sets $\eta$ and $\varphi_g(\beta)$ are separated (otherwise an easy change in the proof would be made), i.e.,

(28) $$\eta \mid \varphi_g(\beta) .$$

Let $B = \text{Req } \beta$. The remainder of the discussion now is toward showing that $B$ will satisfy the desired requirements of the lemma, i.e., that $B$ is a regressive isol and that $B$ satisfies (2). Observe that by (28),

$$\eta + \varphi_g(\beta) \in Y + D_g(B) .$$

Hence in order to complete the proof, it suffices to show that

(29) $\beta$ is a regressive and isolated set, and

(30) $$\varphi_f(\beta) \cong \eta + \varphi_g(\beta) .$$

*For* (29): Note that $\beta$ will be an infinite set, since $e_n \geqq 1$ for each number $n$. Also, it is easy to see that if $\beta$ contains an infinite r.e. subset, then the set $(y_{t_0}, y_{t_1}, \cdots)$ would also then include an infinite r.e. subset. But then the set $\eta$ would contain an infinite r.e. subset, yet we know that this cannot be true since it is an isolated set. And therefore we may conclude that $\beta$ will be an isolated set. We know that the function $y_{t_x}$ is regressive. If we combine this fact with (23) and the definition of $\beta$, then it is easy to see that $\beta$ will be a regressive set, and in particular that a regressive enumeration of its members will be

$$j(y_{t_0}, 0), \cdots, j(y_{t_0}, e_0 - 1), j(y_{t_1}, 0), \cdots, j(y_{t_1}, e_1 - 1), \cdots .$$

*For* (30): Recall that

$$\beta = \sum_0^\infty \beta_n \quad \text{where} \quad \beta_n = \delta_0 + \cdots + \delta_n . \tag{31}$$

Because $\varphi_f$ and $\varphi_g$ are combinatorial operators, it follows from (26) and (31) that,

$$\varphi_f(\beta_0) \subseteqq \varphi_f(\beta_1) \subseteqq \cdots \text{ and } \varphi_f(\beta) = \sum_0^\infty \varphi_f(\beta_n) , \tag{32}$$

$$\varphi_g(\beta_0) \subseteqq \varphi_g(\beta_1) \subseteqq \cdots \text{ and } \varphi_g(\beta) = \sum_0^\infty \varphi_g(\beta_n) , \tag{33}$$

and also, in view of (19) and (27), that for $n \in E$,

$$\text{card } \varphi_f(\beta_n) = \text{card } \eta_{t_n} + \text{card } \varphi_g(\beta_n) . \tag{34}$$

Combining (15), (32), (33) and (34) gives

$$\text{card } \varphi_f(\beta_0) = \text{card } \eta_{t_0} + \text{card } \varphi_g(\beta_0), \text{ and} \tag{35}$$

$$\begin{aligned} \text{card } (\varphi_f(\beta_{k+1}) - \varphi_f(\beta_k)) &= \text{card } (\eta_{t_{k+1}} - \eta_{t_k}) \\ &+ \text{card } (\varphi_g(\beta_{k+1}) - \varphi_g(\beta_k)) . \end{aligned} \tag{36}$$

Now we can define a partial function,

$$q \colon \varphi_f(\beta) \longrightarrow \eta + \varphi_g(\beta) ,$$

based on the previous two equations. Let

$$q \colon \varphi_f(\beta_0) \xrightarrow{*} \eta_{t_0} + \varphi_g(\beta_0) ,$$

$$q \colon (\varphi_f(\beta_{k+1}) - \varphi_f(\beta_k)) \xrightarrow{*} (\eta_{t_{k+1}} - \eta_{t_k}) + (\varphi_g(\beta_{k+1}) - \varphi_g(\beta_k)) ,$$

where we write $\xrightarrow{*}$ to mean that the related mapping is to be order preserving. From (35) and (36) it follows that the mapping $q$ is well-defined, and from (12), (32) and (33) that $q$ will map $\varphi_f(\beta)$ onto $\eta + \varphi_g(\beta)$ in a one-to-one manner. To verify (30), it suffices to prove that $q$ will have a one-to-one partial recursive extension. Because the sets $\varphi_f(\beta)$ and $\eta + \varphi_g(\beta)$ are isolated, it follows from a theorem due to Dekker [4, Proposition 9(*b*)], that $q$ will have a one-to-one partial recursive extension, if both $q$ and $q^{-1}$ have partial recursive extensions. It suffices therefore to verify this latter property, and this will be our approach here. We will consider first the mapping $q$.

Let $w \in \varphi_f(\beta)$. We now describe a procedure whereby, with the possible exception of finitely many such $w$, one can effectively compute the value of $q(w)$. From $w$ first find the particular numbers $x$ and $u$ with

$$w = j(x, u),\ \rho_x \subseteqq \beta \text{ and } u < c_{r(x)}\ . \tag{37}$$

Note that if $\rho_x$ is nonempty then each of its members can also be found. Moreover, since $\varphi_f$ is a normal combinatorial operator, it follows that for all but possibly finitely many $w \in \varphi_f(\beta)$ the corresponding finite set $\rho_x$ appearing in (37) will be nonempty. From now on let us assume that $\rho_x$ is nonempty. Members of $\rho_x$ will be of the form $j(y_{t_k}, v)$, and for each such member we can find the corresponding values of $y_{t_k}$ and $v$. In addition, the values of $t_k$ and $k$ can also be determined, by using the regressive properties of $y_n$ and $y_{t_n}$. Let $k^*$ denote the largest value of $k$ such that $j(y_{t_k}, v) \in \rho_x$, for some number $v$. Then, it is easy to show that

$$\begin{aligned} &w \in \varphi_f(\beta_0) &&, \text{ if } k^* = 0, \text{ and} \\ &w \in \varphi_f(\beta_{k^*}) - \varphi_f(\beta_{k^*-1}), &&\text{ if } k^* \geqq 1\ . \end{aligned}$$

We know, by (25), that from the value of $y_{t_{k^*}}$ we can effectively find all the members of the set $\beta_{h^*}$. In addition, note that if $k^* \geqq 1$ then also the members of the set $\beta_{k^*-1}$ can be found, for we may regress down from $y_{t_{k^*}}$ to $y_{t_{k^*-1}}$ and apply (25). In a similar manner, in view of (14), it follows that from the value of $y_{t_{k^*}}$ we can find all the members in the set

$$\begin{aligned} &\eta_{t_0} &&, \text{ if } k^* = 0, \text{ and} \\ &\eta_{t_{k^*}} - \eta_{t_{k^*-1}}, &&\text{ if } k^* \geqq 1\ . \end{aligned}$$

Finally, by combining these properties with the fact that the normal operators $\varphi_f$ and $\varphi_g$ are each recursive, it can be seen that the members in each of the sets below can be effectively determined,

$$\varphi_f(\beta_0) \text{ and } \eta_{t_0} + \varphi_g(\beta_0), \qquad \text{if } k^* = 0 \text{ and },$$

$$\varphi_f(\beta_{k^*}) - \varphi_f(\beta_{k^*-1}) \quad \text{and}$$

$$(\eta_{t_{k^*}} - \eta_{t_{k^*-1}}) + (\varphi_g(\beta_{k^*}) - \varphi_g(\beta_{k^*-1})), \quad \text{if } k^* \geqq 1\ .$$

It follows directly from this property and the definition of $q$, that the value of $q(w)$ can now be computed. Therefore, there will be a procedure that is effective and which will enable one to compute $q(w)$ for all but a possible finite number of $w \in \varphi_f(\beta)$. It is readily seen that this feature implies that the mapping $q$ will have a partial recursive extension.

An approach very similar to the previous one can be employed to show that the mapping $q^{-1}$ will also have a partial recursive. For this reason we will omit the main details for doing this, and will only mention the two essentially new observatians that we would have been required to make. The first is that given any number $w \in \eta + \varphi_g(\beta)$ one can effectively determine whether $w \in \eta$ or $w \in \varphi_g(\beta)$. This property follows from the separability of the sets $\eta$ and $\varphi_g(\beta)$ given in (28). The other observation is that if $w \in \eta$, then one can effectively find the particular numbers $s$, $k^*$, $t_{k^*}$ and $y_{t_{k^*}}$ that are related to $w$ in the following way, $w = y_s$ and

$$\begin{aligned} &w \in \eta_{t_{k^*}} &&, \text{ if } k^* = 0\,, \\ &w \in (\eta_{t_{k^*}} - \eta_{t_{k^*-1}}), &&\text{ if } k^* \geqq 1\,. \end{aligned}$$

This particular property follows from (14), (16), Property A and the regressive properties of the functions $y_n$ and $y_{t_n}$. The importance of the second property lies in the fact that it means that from the value of any $w \in \eta$, one can effectively find $y_{t_{k^*}}$, and therefore also determine the appropriate sets,

$$\begin{aligned} &\beta_{t_0} \text{ and } \eta_{t_0} &&, \text{ if } k^* = 0\,, \\ &\beta_{t_{k^*}}, \beta_{t_{k^*-1}}, \eta_{t_{k^*}} \text{ and } \eta_{t_{k^*-1}}, &&\text{ if } k^* \geqq 1\,. \end{aligned}$$

It is then with these two observations that a similar approach, as with $q$, will lead to showing that $q^{-1}$ will have a partial recursive extension.

In view of the remarks made up to this point, we see that the mapping

$$q\colon \varphi_f(\beta) \longrightarrow \eta + \varphi_g(\beta)$$

will have a one-to-one partial recursive extension. This verifies (30) and complets the proof of the lemma.

THEOREM 2. *Let $f\colon E \to E$ be a recursive function and $A$ and $Y$ be isols such that*

$$(1) \qquad D_f(A) = Y\,.$$

*If $Y$ is a regressive isol, then there will also exist regressive isols $B$ such that,*

$$D_f(B) = Y\,.$$

*Proof.* Let us assume that $Y$ is a regressive isol. Let $f^+$ and $f^-$ be the *positive* and *negative* recursive and combinatorial functions that are associated with $f$ (refer to [11]). Then for every number $x \in E$, $f(x) = f^+(x) - f^-(x)$, and also

$$D_f(A) = D_f + (A) - D_f - (A) .$$

Therefore, by (1), it also follows that

$$D_f + (A) = Y + D_f - (A) .$$

If we now apply the previous lemma to this equation, we see that there will be a regressive isol $B$ such that

$$D_f + (B) = Y + D_f - (B) ,$$

and from this identity it also follows that $D_f(B) = Y$.

REMARK. Theorem 2 is our principal result and it is easy to observe that it follows almost directly from the lemma. It turns out that, as a consequence of the manner in which the lemma was proved, a slightly stronger form of both the lemma and the theorem can be established. We would like to state without a proof the particular form that is related to the theorem. It involves the Nerode canonical extension of the familiar binary relation $\leqq$ (among numbers) to the isols. The extension procedure is introduced in [12], and for the relation $\leqq$ its extension will be denoted by $\leqq_\Lambda$. It can be shown that the regressive isol $B$ constructed in the proof of the lemma (in each of the cases considered there) is related to the isol $A$ by $B \leqq_\Lambda A$. Based on this fact one can obtain the following result.

THEOREM A. *Let $f: E \to E$ be a recursive function and $A$ and $Y$ be isols such that $D_f(A) = Y$. If $Y$ is a regressive isol, then there will exist regressive isols $B$ such that $B \leqq_\Lambda A$ and $D_f(B) = Y$.*

**3. An example.** It is possible that the canonical extension of a recursive function may map an isol that is nonregressive onto an isol that is infinite and regressive. We would like to give an example of such a function. First some definitions are needed.

If $\alpha$ and $\beta$ are two sets of numbers, then $\alpha \leqq^* \beta$ will mean that either $\alpha$ is a finite set and card $\alpha \leqq$ card $\beta$, or else both $\alpha$ and $\beta$ are infinite sets and there is a partial recursive function $p$ such that, $\alpha \subseteqq \delta p$, $p(\alpha) = \beta$ and $p$ is one-to-one on $\alpha$. If $A$ and $B$ are two isols then $A \leqq^* B$ will mean that there are sets $\alpha \in A$ and $\beta \in \beta$ such that $\alpha \leqq^* \beta$. Let *min* $(a, b)$ denote the familiar recursive function minimum $(a, b)$, and let $D_{\min}$ denote its canonical extension to $\Lambda^2$. min $(a, b)$ is not an almost combinatorial function, and therefore its canonical extension will not map $\Lambda^2$ into $\Lambda$. On the otherhand, it is proved in [3] that $D_{\min}$ will map $\Lambda_R^2$ into $\Lambda_R$. In addition, by combining results in [3] and [4], one obtains for $A, B \in \Lambda_R$,

$$D_{\min}(A, B) = A \Longleftrightarrow A \leqq^* B .$$

Concerning isols and regressive isols the following property due to Dekker [4] is also needed; if $S$ and $T$ are any isols, then

$$(*) \qquad S \leqq T \text{ and } T \in \Lambda_R \Longrightarrow S \in \Lambda_R .$$

In the result below we will construct the kind of example that was described earlier. We note that the functions $j(x, y)$, $k(x)$ and $l(x)$ that appear in its proof refer to those particular recursive functions introduced in §1.

THEOREM 3. *There is a recursive function $h(x)$ and an isol $C$ such that $D_h(C) \in \Lambda_R$ and yet $C \notin \Lambda_R$.*

*Proof.* Define

$$h(x) = \min(k(x), l(x)) .$$

Then $h$ will be a recursive function, and for $a, b \in E$

$$hj(a, b) = \min(a, b) .$$

Therefore also,

$$D_h D_j(U, V) = D_{\min}(U, V), \text{ for } U, V \in \Lambda .$$

Select $A, B \in \Lambda_R$ such that

$$(1) \qquad A \leqq^* B \quad \text{and} \quad A + B \notin \Lambda_R ;$$

the existence of such a pair of regressive isols is proved in [2]. Then it follows

$$D_h D_j(A, B) = D_{\min}(A, B) = A ,$$

and in addition, if we let $C = D_j(A, B)$, then also

$$(2) \qquad D_h(C) = A \in \Lambda_R .$$

The function $j(x, y)$ is recursive and combinatorial, and therefore its canonical extension will map $\Lambda^2$ into $\Lambda$. In particular, we see that

$$(3) \qquad C = D_j(A, B) \in \Lambda .$$

Let us now verify

$$(4) \qquad C = D_j(A, B) \in \Lambda_R \Longrightarrow A + B \in \Lambda_R .$$

First consider the implications,

$$\begin{aligned} D_j(A, B) \in \Lambda_R &\Longrightarrow 2D_j(A, B) \in \Lambda_R \\ &\Longrightarrow 2A + (A + B)(A + B + 1) \in \Lambda_R \\ &\Longrightarrow A + B \in \Lambda_R \, . \end{aligned}$$

The first two implications are clear. The last one follows from (*) and the property,

$$A + B \leqq 2A + (A + B)(A + B + 1) \, .$$

Together they imply (4). In view of (1), (3) and (4) we obtain $C \in \Lambda - \Lambda_R$, and if we combine this property with (2) the desired result follows directly.

*N. B.* The fact that the familiar $j$ function is combinatorial we first learned from some unpublished notes of Erik Ellentuck. Once this property is pointed out it is easy to show, and we will leave it for the reader.

**4. Recursive functions of several variables.** We would like to describe some of the results that can be obtained for recursive functions of more than one variable that are similar to those given in § 2. First let us note some features that distinguish the one and more than one variable cases. We know that for a recursive combinatorial function of one variable, its canonical extension will map regressive isols onto regressive isols. On the otherhand, even for recursive combinatorial functions of two variables, it need not be true that their canonical extension will map pairs of regressive isols onto regressive isols. For example, Dekker showed in [4] that it is possible for both the sum and the product of two regressive isols to be an isol that is non-regressive. The characterization of those recursive functions of two variables whose canonical extensions will map regressive isols to regressive isols was given by Mathew Hassett in [9]. The following is a special case of a theorem also due to Hassett [8].

THEOREM B. (Hassett) *Let $f\colon E^n \rightarrow E$ be a recursive and combinatorial function. Let $A_1, \cdots, A_n$ be $n$ regressive isols whose sum $A_1 + \cdots + A_n$ is also regressive. Then the value of $D_f(A_1, \cdots, A_n)$ will be a regressive isol.*

Note that when $n = 1$ in Theorem B one obtains the earlier result mentioned about recursive combinatorial functions of one variable. Based upon the procedure for representing the canonical extension of a recursive function (in terms of the canonical extensions of recursive combinatorial functions) and applying Theorem *B*, analogues of Theorems

1 and 2 can be obtained for functions of more than one variable. We conclude the paper with statements of these two theorems.

THEOREM C. *Let* $f: E^n \rightarrow E$ *be a recursive function and* $A_1, \cdots, A_n$ *and* $Y$ *be isols with* $D_f(A_1, \cdots, A_n) = Y$. *If the sum* $A_1 + \cdots + A_n$ *is regressive, then the isol* $Y$ *will also be regressive.*

THEOREM D. *Let* $f: E^n \rightarrow E$ *be a recursive function and* $A_1, \cdots, A_n$ *and* $Y$ *be isols with* $D_f(A_1, \cdots A_n) = Y$. *If* $Y$ *is regressive, then there will be regressive isols* $B_1, \cdots, B_n$ *such that the sum* $B_1 + \cdots + B_n$ *will be regressive and also* $D_f(B_1, \cdots, B_n) = Y$.

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# HOMOTOPY AND ALGEBRAIC $K$-THEORY

BARRY DAYTON

**A notion of homotopy is described on a category of rings. This is used to induce a notion of equivalence on the categories of projective modules and to construct a $K$-theory exact sequence. The topological $K$-theory exact sequence is then obtained from the algebraic $K_0$, $K_1$ sequence.**

**1. Homotopy.** In this section we describe the homotopy notion and the notion of equivalence it induces on the categories of projective modules.

A cartesian square of rings is a commutative diagram of rings

$$(*)\qquad \begin{array}{ccc} A & \xrightarrow{h_2} & A_2 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow{f_1} & A_0 \end{array}$$

where $A = \{(a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2)\}$ and $h_1$, $h_2$ are restrictions of the coordinate projections. We will further assume that $f_1$ is surjective. If $\mathscr{K}$ is a category of rings and $F: \mathscr{K} \to \mathscr{K}$ is a functor we call $F$ cartesian square preserving if the functor applied to a cartesian square gives a cartesian square.

DEFINITION 1.1. Let $\mathscr{K}$ be a category of rings. A homotopy theory $\mathscr{H}$ for $\mathscr{K}$ is an ordered quadruple $(I, \iota_0, \iota_1, \pi)$ where $I$ is a cartesian square preserving functor and $\iota_0, \iota_1: I \to 1_{\mathscr{K}}$, $\pi: 1_{\mathscr{K}} \to I$ are natural transformations such that $\iota_0(A)\pi(A) = 1_A = \iota_1(A)\pi(A)$ for $A \in \mathscr{K}$.

For a homotopy theory $\mathscr{H} = (I, \iota_0, \iota_1, \pi)$ on $\mathscr{K}$ and $f, g: B \to A$ morphisms in $\mathscr{K}$ define $f \sim g$ if there exists a morphism $h: B \to IA$ in $\mathscr{K}$ such that $f = \iota_0 h$, $g = \iota_1 h$; $h$ is called a homotopy of $f$ to $g$. Let $\cong$ be the smallest equivalence relation on $\mathscr{K}(B, A)$ containing $\sim$; if $f \cong g$ we say $f$ is homotopic to $g$.

Note that a homotopy theory gives rise to a homotopy category, i.e. a category whose objects are those of $\mathscr{K}$ and whose morphisms are homotopy classes of morphisms.

Let $\mathscr{L}$ be an arbitrary category and $G: \mathscr{K} \to \mathscr{L}$ be a covariant functor A homotopy theory $\mathscr{H} = (I, \iota_0, \iota_1, \pi)$ on $\mathscr{K}$ is called compatible with $G$ if $G(\pi(A))$ is an isomophism for each $A \in \mathscr{K}$. Note that if $\mathscr{H}$ is compatible with $G$ then $G(\iota_0) = G(\iota_1) = G(\pi)^{-1}$ consequently if $f \cong g$, then $G(f) = G(g)$.

For any ring $A$ let $\underline{\underline{P}}(A)$ denote the category of finitely generated projective right $A$-modules. Given a ring homomorphism $f\colon A \to B$ denote by $\hat{f}\colon \underline{\underline{P}}(A) \to \underline{\underline{P}}(B)$ the covariant additive functor defined by $\hat{f}(M) = M \otimes_A B$ on objects $M$ of $\underline{\underline{P}}(A)$ and $\hat{f}(\alpha) = \alpha \otimes 1$ on morphisms of $\underline{\underline{P}}(A)$. It is well known that if $M$ is $A$-projective then $M \otimes_A B$ is $B$-projective.

If $A_0, A_1, \cdots, A_n, B_0, \cdots, B_e$ are rings, if $f_i\colon A_{i-1} \to A_i$ and $g_i\colon B_{i-1} \to B_i$ are ring homomorphisms, if $A_0 = B_0 = A$, $A_n = B_e = B$ and if $f_n f_{n-1} \cdots f_1 = g_e g_{e-1} \cdots g_1$, we denote by $\langle f_1, \cdots, f_n / g_1, \cdots, g_e \rangle$ the canonical natural equivalence $\hat{f}_n \cdots \hat{f}_1 \to \hat{g}_e \cdots \hat{g}_1$; it is straightforward to verify that

$$\left\langle \frac{g_1, \cdots, g_e}{h_1, \cdots, h_k} \right\rangle \left\langle \frac{f_1, \cdots, f_n}{g_1, \cdots, g_e} \right\rangle = \left\langle \frac{f_1, \cdots, f_n}{h_1, \cdots, h_k} \right\rangle ,$$

that

$$\left\langle \frac{f_1, \cdots, f_n, h}{g_1, \cdots, g_e, h} \right\rangle = \hat{h} \left\langle \frac{f_1, \cdots, f_n}{g_1, \cdots, g_e} \right\rangle$$

whenever $h\colon B \to C$ and that

$$\left\langle \frac{h, f_1, \cdots, f_n}{h, g_1, \cdots, g_e} \right\rangle_M = \left\langle \frac{f_1, \cdots, f_n}{g_1, \cdots, g_e} \right\rangle_{\hat{h}M}$$

for $h\colon C \to A$ where the subscript $M$ means that the natural equivalence is evaluated at the module $M \in \underline{\underline{P}}(C)$.

Definition 1.2. A homotopy theory $\mathscr{H} = (I, \iota_0, \iota_1, \pi)$ in $\mathscr{K}$ induces an $\mathscr{H}$-equivalence $\cong_{\mathscr{H}}$ in each category $\underline{\underline{P}}(A)$, $A \in \mathscr{K}$ as follows: given $M, N \in \underline{\underline{P}}(A)$ write $M \sim_{\mathscr{H}} N$ if there is a $Q \in \underline{\underline{P}}(IA)$ such that $M \approx \iota_0 Q$, $N \approx \iota_1 Q$ and let $\cong$ be the smallest equivalence relation on the set of isomorphism classes of objects in $\underline{\underline{P}}(A)$ containing $\sim_{\mathscr{H}}$. If $M \cong_{\mathscr{H}} N$ we say that the modules are equivalent mod-$\mathscr{H}$.

The homotopy theory $\mathscr{H}$ in $\mathscr{K}$ also induces an equivalence relation $\cong_{\mathscr{H}}$ in the set $\mathrm{Iso}(M, N)$ of isomorphisms $M \to N$ of $A$-projectives by letting $\phi_0 \sim_{\mathscr{H}} \phi_1$ denote that there is an isomorphism $\theta\colon \hat{\pi}M \to \hat{\pi}N$ such that

$$\phi_j = \left\langle \frac{\pi, \iota_j}{1} \right\rangle_N (\hat{\iota}_j \theta) \left\langle \frac{1}{\pi, \iota_j} \right\rangle_M$$

for $j = 0, 1$ and letting $\cong_{\mathscr{H}}$ be the smallest equivalence relation containing $\sim_{\mathscr{H}}$ on the set $\mathrm{Iso}(M, N)$. If $\phi_0 \cong_{\mathscr{H}} \phi_1$ we say the isomorphisms are equivalent mod $\mathscr{H}$.

Note that if $M' \xrightarrow{\omega} M \underset{\phi_1}{\overset{\phi_0}{\rightrightarrows}} N \xrightarrow{\mu} N'$ are isomorphisms and if $\phi_0 \cong \phi_1$ mod $\mathscr{H}$ then also $\mu\phi_0\omega \cong \mu\phi_1\omega$ mod $\mathscr{H}$. It is not difficult to show that if $f: A \to B$ is a morphism in $\mathscr{K}$ then $M \cong N \bmod \mathscr{H}$ in $\underline{\underline{P}}(A)$ implies $\hat{f}M \cong \hat{f}N \bmod \mathscr{H}$ in $\underline{\underline{P}}(B)$ and $\phi_0 \cong \phi_1 \bmod \mathscr{H}$ implies $\hat{f}\phi_0 \cong \hat{f}\phi_1 \bmod \mathscr{H}$ in $\underline{\underline{P}}(B)$. It is also easily seen that if $f \cong g: A \to B$ and $M \in \underline{\underline{P}}(A)$ then $\hat{f}M \cong \hat{g}M \bmod \mathscr{H}$ in $\underline{\underline{P}}(B)$.

Given a ring with unit $R$, an $R$-algebra will mean a unitary $R$-algebra. If $A$ is an $R$-algebra, then $a: R \to A$ will denote the unique $R$-algebra homomorphism such that $a(1) = 1$. In addition to the above results we then have:

LEMMA 1.3. *Let $\mathscr{K}$ be a category of $R$-algebras and $R$-algebra homomorphisms and let $\mathscr{H} = (I, \iota_0, \iota_1\pi)$ be a homotopy theory on $\mathscr{K}$. Let $f \cong g: A \to B$ in $\mathscr{K}$, let $M, N \in \underline{\underline{P}}(R)$ and let $\phi \in \operatorname{Iso}(\hat{a}M, \hat{a}N)$. Then*

$$\left\langle \frac{a,\, f}{b} \right\rangle_N (\hat{f}(\phi)) \left\langle \frac{b}{a,\, f} \right\rangle_M \cong \left\langle \frac{a,\, g}{b} \right\rangle_N (\hat{g}(\phi)) \left\langle \frac{b}{a,\, g} \right\rangle_M \bmod \mathscr{H}$$

*in* $\operatorname{Iso}(\hat{b}M, \hat{b}N)$.

*Proof.* We may assume $f \sim g$. Letting $h: A \to IB$ be a homotopy from $f$ to $g$, define $\omega: \hat{\pi}\hat{b}M \to \hat{\pi}\hat{b}N$ by

$$\omega = \left\langle \frac{a,\, h}{b,\, \pi} \right\rangle_N (h(\phi)) \left\langle \frac{b,\, \pi}{a,\, h} \right\rangle_M .$$

It is easily verified that $\omega$ shows that the two isomorphisms are equivalent mod $\mathscr{H}$.

Equivalence mod $\mathscr{H}$ works well with cartesian squares. If (*) is a cartesian square we can construct the fiber product category $\underline{\underline{P}}(A) \times_{\underline{\underline{P}}(A_0)} \underline{\underline{P}}(A_2)$, [2, p. 358] in which objects are triples $(M, \phi, N)$ where $M \in \underline{\underline{P}}(A_1)$, $N \in \underline{\underline{P}}(A_2)$ and $\phi: \hat{f}_1 M \to \hat{f}_2 N$ is an isomorphism in $\underline{\underline{P}}(A_0)$; and the morphisms $(M, \phi, N) \to (M', \phi', N')$ are pairs $(\alpha, \beta)$ where $\alpha: M \to M' \in \underline{\underline{P}}(A_1)$, $\beta: N \to N' \in \underline{\underline{P}}(A_2)$ and $\phi'(\hat{f}\alpha) = (\hat{f}_2\beta)\phi$. By Milnor's theorem [2, p. 479] the functor $F: \underline{\underline{P}}(A) \to \underline{\underline{P}}(A_1) \times_{\underline{\underline{P}}(A_0)} \underline{\underline{P}}(A_2)$ given by $F(M) = (\hat{h}_1 M, \langle h_1 f_1 / h_2 f_2 \rangle_M, \hat{h}_2 M)$ and $F(\alpha) = (\hat{h}_1\alpha, \hat{h}_2\alpha)$ is an equivalence of categories. Making this identification, the following is a projective module analogue of a theorem on vector bundles. [1, Lemma 1.4.6].

PROPOSITION 1.4. *Let $\mathscr{H} = (I, \iota_0, \iota_1\pi)$ be a homotopy theory on $\mathscr{K}$ and (*) a cartesian square in $\mathscr{K}$. Let $M \in \underline{\underline{P}}(A)$, $N \in \underline{\underline{P}}(A)$ and $\phi_0 \cong \phi_1: \hat{f}_1 M \to \hat{f}_2 N \bmod \mathscr{H}$. Then $(M, \phi_0, N) \cong (M, \phi_1, N) \bmod \mathscr{H}$ in $\underline{\underline{P}}(A)$.*

*Proof.* Assume $\phi_0 \sim_{\mathscr{H}} \phi_1$ and let $\omega: \hat{\pi}\hat{f}_1 M \to \hat{\pi}\hat{f}_2 N$ show $\phi_0 \sim_{\mathscr{H}} \phi_1$.

Define $\omega'\colon \widehat{If_1}\tilde{\pi}M \to \widehat{If_2}\tilde{\pi}N$ by

$$\omega' = \left\langle \frac{f_2,\ \pi}{\pi,\ If_2} \right\rangle_N (\omega) \left\langle \frac{\pi,\ If_1}{f_1,\ \pi} \right\rangle_M .$$

Since

$$\begin{array}{ccc} IA & \xrightarrow{Ih_2} & IA_2 \\ {\scriptstyle Ih_1}\downarrow & & \downarrow{\scriptstyle If_2} \\ IA_1 & \xrightarrow{If_1} & IA_0 \end{array}$$

is by hypothesis also a cartesian square we have $(\tilde{\pi}M, \omega', \tilde{\pi}N) \in \underline{\underline{P}}(IA)$ and direct calculation shows that $\hat{\iota}_j(\tilde{\pi}M, \omega', \tilde{\pi}N) \approx (M, \phi_j, N)$ for $j = 0, 1$.

**2. A connecting homomorphism.** In this section we obtain an explicit formula for a connecting homomorphism useful in constructing algebraic $K$-theory exact sequences.

Let $K_0$, $K_1$ be the algebraic $K_i$ functors [2, p. 445]. If $\mathscr{K}$ is a category of $R$-algebras and $R$-algebra homomorphisms define $\tilde{K}_i(A) = K_i(A)/\mathrm{Im}\, K_i(a)$. If $f\colon A \to B$ is a morphism in $\mathscr{K}$ then $f \circ a = b$ and we let $\tilde{K}_i(f)\colon \tilde{K}_i(A) \to \tilde{K}_i(B)$ be the induced map. It is simple to verify that $\tilde{K}_0$, $\tilde{K}_1$ are functors on $\mathscr{K}$ and moreover that $\tilde{K}_i(A)$ is isomorphic to the usual reduced group whenever $A$ is an augmented $R$-algebra.

THEOREM 2.1. *Let $\mathscr{H}$ be a homotopy theory on a category $\mathscr{K}$ of $R$-algebras compatible with $\tilde{K}_0$. Let*

$$\begin{array}{ccc} B & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle a_0} \\ B_1 & \xrightarrow{g} & A_0 \end{array} \qquad\qquad \begin{array}{ccc} A & \longrightarrow & R \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle a_0} \\ A_1 & \xrightarrow{f} & A_0 \end{array}$$

*be cartesian squares in $\mathscr{K}$, $h\colon B_1 \to A_1$ such that $fh \cong g$ and $\hat{K}_0(B_1) = 0$. Then there is a unique group homomorphism $\delta\colon \hat{K}_0(B) \to \hat{K}_0(A)$ such that*

$$\delta[(\hat{b}_1 M, \phi, N)] = \left[\left(\hat{a}_1 M, \phi\left\langle \frac{a_1,\ f}{b_1,\ g} \right\rangle_M, N\right)\right]$$

*for $M, N \in \underline{\underline{P}}(R)$.*

*Proof.* For $Q = (\hat{b}_1 M, \phi, N) \in \underline{\underline{P}}(B)$ define

$$DQ = \left(\hat{a}_1 M, \phi\left\langle \frac{a_1,\ f}{b_1,\ g} \right\rangle_M, N\right) \in \underline{\underline{P}}(A) .$$

Once one has established

(i) If $Q_1 \approx Q_2$ then $DQ_1 \cong DQ_2 \bmod \mathscr{H}$.

(ii) $D(Q_1 \oplus Q_2) \approx DQ_1 \oplus DQ_2$

(iii) $D(\hat{b}M) = \hat{a}M$

(iv) every element of $\hat{K}_0(B)$ is of the form $[Q]$

it follows easily that $\delta$ is well defined, unique and a group homomorphism. Because proofs of assertions (ii)—(iv) are themselves straightforward and do not depend on homotopy, we will prove only (i). Suppose then that $(\alpha, \beta)$: $(\hat{b}_1M, \phi, N) \to (\hat{b}M', \phi', N')$ is an isomorphism. Then we have $\phi' = \hat{a}_0(\beta)(\phi)g(\alpha^{-1})$. By Lemma 1.3

$$\left\langle\frac{b_1, g}{a_0}\right\rangle_M \hat{g}(\alpha^{-1})\left\langle\frac{a_0}{b_1, g}\right\rangle_{M'} \cong \left\langle\frac{b_1, fh}{b_1, g}\right\rangle_M \widehat{fh}(\alpha^{-1})\left\langle\frac{a_0}{b_1, h}\right\rangle_{M'} \bmod \mathscr{H}.$$

A direct computation gives

$$\hat{g}(\alpha^{-1})\left\langle\frac{a_1, f}{b_1, g}\right\rangle_M \cong \left\langle\frac{a_1, f}{b_1, g}\right\rangle_M \hat{f}\left(\left\langle\frac{b_1, h}{a_1}\right\rangle_M \hat{h}(\alpha^{-1})\left\langle\frac{a_1}{b_1, h}\right\rangle_{M'}\right) \bmod \mathscr{H},$$

so

$$\phi'\left\langle\frac{a_1, f}{b_1, g}\right\rangle_{M'} \cong \hat{a}_0(\beta)(\phi)\left\langle\frac{a_1, f}{b_1, g}\right\rangle_M \hat{f}(\gamma)$$

where

$$\gamma = \left\langle\frac{b_1, h}{a_1}\right\rangle_M (\hat{h}(\alpha^{-1}))\left\langle\frac{a_1}{b_1, h}\right\rangle_{M'}.$$

Therefore (using Proposition 1.4)

$$\left(\hat{a}_1M', \phi'\left\langle\frac{a_1, f}{b_1, g}\right\rangle_{M'}, N'\right) \cong \left(\hat{a}_1M', a_0(\beta)(\phi)\left\langle\frac{a_1, f}{b_1, g}\right\rangle_M \hat{f}(\gamma), N'\right) \bmod \mathscr{H}.$$

Since $(\gamma, \beta^{-1})$ is an isomorphism from this latter module to

$$\left(\hat{a}_1M, \phi\left\langle\frac{a_1, f}{b_1, g}\right\rangle_{M'}, N\right)$$

the assertion (i) is proved.

**3. An exact sequence.** In this section we use the homomorphism of 2.1 and the standard $K_0$, $K_1$ exact sequence to construct a 5-term exact sequence.

An $R$-algebra $A$ is called proper if the morphism $K_0(a)$: $K_0(R) \to K_0(A)$ is injective. We note that either of the following two conditions is sufficient to insure that an $R$-algebra $A$ is proper:

(i) $A$ has as an augmentation, i.e. there is a $e$: $A \to R$ such that $ea = 1_R$

(ii) $R$ is a principal ideal domain and $A$ is a commutative $R$ algebra.

LEMMA 3.1. *Let* (*) *be a cartesian square of proper $R$-algebra. Then there is an exact sequence*

$$\begin{aligned}\tilde{K}_1(A) &\longrightarrow \tilde{K}_1(A_1) \oplus \tilde{K}_1(A_2) \longrightarrow \tilde{K}_1(A_0) \xrightarrow{\tilde{\partial}} \tilde{K}_0(A)\\ &\longrightarrow \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) \longrightarrow \tilde{K}_0(A_0)\end{aligned}$$

*which is functorial with respect to transformations of cartesian squares.*

*Proof.* Since

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R \arrow[d] \\ R \arrow[r] & R \end{tikzcd}$$

is a cartesian square, by [2, p. 481] we have the commutative diagram

$$\begin{tikzcd}[column sep=small] & & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ K_1(R) \arrow[r] \arrow[d] & K_1(R) \oplus K_1(R) \arrow[r] \arrow[d] & K_1(R) \arrow[r] \arrow[d] & K_0(R) \arrow[r] \arrow[d] & K_0(R) \oplus K_0(R) \arrow[r] \arrow[d] & R_0(R) \arrow[d] \\ K_1(A) \arrow[r] \arrow[d] & K_1(A_1) \oplus K_1(A_2) \arrow[r] \arrow[d] & K_1(A_0) \arrow[r, "\partial"] \arrow[d] & K_0(A) \arrow[r] \arrow[d] & K_0(A_1) \oplus K_0(A_2) \arrow[r] \arrow[d] & K_0(A_0) \arrow[d] \\ \tilde{K}_1(A) \arrow[r] \arrow[d] & \tilde{K}_1(A_1) \oplus \tilde{K}_1(A_2) \arrow[r] \arrow[d] & \tilde{K}_1(A_0) \arrow[r, "\tilde{\partial}"] \arrow[d] & \tilde{K}_0(A) \arrow[r] \arrow[d] & \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) \arrow[r] \arrow[d] & \tilde{K}_0(A_0) \arrow[d] \\ 0 & 0 & 0 & 0 & 0 & 0 \end{tikzcd}$$

where the columns and the first two rows are exact. An easy chase shows that the third row is exact.

We wish to give an explicit formula for the morphism $\tilde{\partial}$. For this we have:

LEMMA 3.2. *Let $A$, $A_0$ and $A_1$ be proper $R$-algebras and*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f'"'] & R \arrow[d, "a_0"] \\ A_1 \arrow[r, "f"] & A_0 \end{tikzcd}$$

*be a cartesian square. Then the connecting homomorphism of* 3.1 *is* *given by*

$$\tilde{\partial}[\hat{a}_0 M, \alpha] = \left[\left(\hat{a}M, \alpha\left\langle\frac{a_1, f}{a_0}\right\rangle_M, M\right)\right] \quad \textit{for} \quad M \in \underline{\underline{P}}(R) .$$

*Proof.* Since the full subcategory of $\underline{\underline{P}}(A_0)$ with objects $\hat{a}_0 M$, $M \in \underline{\underline{P}}(R)$ is cofinal, $K_1(A_0)$ and hence $\tilde{K}_1(A_0)$ is generated by elements of the form $[\hat{a}_0 M, \alpha]$ [2, p. 355]. But

$$\begin{aligned}
\partial[\hat{a}_0 M, \alpha] &= \partial\left[\hat{f}\hat{f}'\hat{a}M, \left\langle\frac{a_0}{a, f, f'}\right\rangle_M \alpha\left\langle\frac{a, f', f}{a_0}\right\rangle_M\right] \\
&= \left[\left(\hat{f}'\hat{a}M, \left\langle\frac{a_0}{a, \varepsilon, a_0}\right\rangle_M \alpha\left\langle\frac{a, f', f}{a_0}\right\rangle_M, \hat{\varepsilon}\hat{a}M\right)\right] - [\hat{a}M] \\
&= \left[\left(\hat{a}, M, \alpha\left\langle\frac{a_1, f}{a_1}\right\rangle_M, M\right)\right] + 0
\end{aligned}$$

from [2, 4.3 p. 365] since $[\hat{a}M] \in \operatorname{Im} K_0(a)$.

In order to apply 2.1 we need

LEMMA 3.3. *Under the hypotheses of Theorem* 2.1 *the diagram*

$$\begin{CD}
\tilde{K}_1(A_0) @>{\tilde{\partial}'}>> \tilde{K}_0(B) @>>> \tilde{K}_0(B_1) = 0 \\
@VV{1}V @VV{\delta}V @VVV \\
\tilde{K}_1(A_0) @>{\tilde{\partial}}>> \tilde{K}_0(A) @>{\tilde{K}_0(f')}>> \tilde{K}_0(A_1)
\end{CD}$$

*commutes.*

*Proof.*

$$\begin{aligned}
\delta\tilde{\partial}'[\hat{a}'M, \alpha] &= \delta\left[\left(\hat{b}, M, \alpha\left\langle\frac{b_1, g}{a'}\right\rangle_M, M\right)\right] = \left[\left(\hat{a}_1 M, \alpha\left\langle\frac{b_1, g}{a_0}\right\rangle_M \left\langle\frac{a_1, f}{b_1, g}\right\rangle_M, M\right)\right] \\
&= \left[\left(\hat{a}_1 M, \alpha\left\langle\frac{a_1, f}{a'}\right\rangle, M\right)\right] = \tilde{\partial}[\hat{a}_0 M, \alpha] .
\end{aligned}$$

Also since $\tilde{K}_0(B_1) = 0$ it can be seen that if

$$[N] \in \tilde{K}_0(B),\ [N] = [(\hat{b}_1 M, \phi, N)],\ M, N \in P(R) .$$

Thus

$$\tilde{K}_0(f')\delta[(\hat{b}_1 M, \phi, N)] = \tilde{K}_0(f')\left[\left(\hat{a}_1 M, \phi\left\langle\frac{a_1, f}{h_1, g}\right\rangle_M, N\right)\right] = [\hat{a}_1 M] = 0 .$$

THEOREM 3.4. *Let $\mathscr{K}$ be a category of proper $R$-algebras and $\mathscr{H}$ be a homotopy theory on $\mathscr{K}$ compatible with $\tilde{K}_0$. Let*

(**)
$$\begin{tikzcd}
C \arrow[rr, "f'"] \arrow[dd] \arrow[dr] & & B \arrow[dd] \arrow[dr] & & A \arrow[dd] \arrow[dr] & \\
& R \arrow[rr, "1"] \arrow[dd] & & R \arrow[rr, "1"] \arrow[dd] & & R \arrow[dd] \\
C_1 \arrow[rr] \arrow[dr] & & B_1 \arrow[rr, "h"] \arrow[dr, "g"] & & A_1 \arrow[dr, "f"] & \\
& A_1 \arrow[rr, "f"] & & A_0 \arrow[rr, "1"] & & A_0
\end{tikzcd}$$

*be a diagram in* $\mathscr{K}$ *where* $fh \cong g$, *all other squares commute and the vertical squares are cartesian. If* $\tilde{K}_0(C_1) = \tilde{K}_0(B_1) = 0$ *then*

$$\tilde{K}_0(C) \xrightarrow{\tilde{K}_0(f')} \tilde{K}_0(B) \xrightarrow{\delta} \tilde{K}_0(A) \longrightarrow \tilde{K}_0(A_1) \longrightarrow \tilde{K}_0(A_0)$$

*is exact*

*Proof.* From 3.1 and 3.3 we get a commutative diagram

$$\begin{tikzcd}
& \tilde{K}_1(A_1) \arrow[r] \arrow[d, "\tilde{K}_1(f)"] \arrow[dl, "1"'] & \tilde{K}_0(C) \arrow[r] \arrow[d] & \tilde{K}_0(C_1)=0 \arrow[d] & \\
& \tilde{K}_1(A_0) \arrow[r] \arrow[d, "1"] & \tilde{K}_0(B) \arrow[r] \arrow[d, "\delta"] & \tilde{K}_0(B_1)=0 \arrow[d] & \\
\tilde{K}_1(A_1) \arrow[r, "K_1(f)"] & \tilde{K}_1(A_0) \arrow[r] & \tilde{K}_0(A) \arrow[r] & \tilde{K}_0(A_1) \arrow[r] & \tilde{K}_0(A_1)
\end{tikzcd}$$

where the rows are exact. A diagram chase gives the result.

**4. The topological $K$-theory exact sequence.** In this section we use 3.4 to construct the topological $K$-Theory exact sequence.

Let $R$ denote the real or complex numbers. For a compact Hausdorff space $X$ let $CX$ be the ring of continuous $R$-valued functions and for a continuous function $f: X \to Y$ let $f^*: CY \to CX$ be the induced ring homomorphism. Denote the one point space by $*$ and take $\mathscr{K}$ to be the category of rings $CX$ and ring homomorphisms. We will consider $\mathscr{K}$ to be a category of $C^* = R$ algebras. Define $J: \mathscr{K} \to \mathscr{K}$ by $JCX = C(X \times I)$ where $I$ denotes the unit interval and $J(f) = (f \times 1)^*$. Define $\iota_0, \iota_1, \pi$ by $i_0^*, i_1^*, \pi^*$ where $i_j: X \to I$ is given by $i_j(x) = (x, j)$ and $\pi(x, t) = x$, $\pi: X \times I \to X$. It follows easily that $\mathscr{H} = (J, \iota_0, \iota_1, \pi)$ is a homotopy theory on $\mathscr{K}$. We recall that $K_0^T(X) = K_0(CX)$ where $K_0^T$ is topological $K_0$ functor. If $X$ is a pointed space the reduced group as defined above coincides with the usual reduced group. It follows from standard results on vector bundles [1, Lemma 1.4.3] and on the correspondence between vector bundles over $X$ and projective modules over $CX$ that $\mathscr{H}$ is compatible with $K_0^T$. Alternatively it can be easily proved directly that if $M, N \in \underline{\underline{P}}(X)$ then $M \cong$

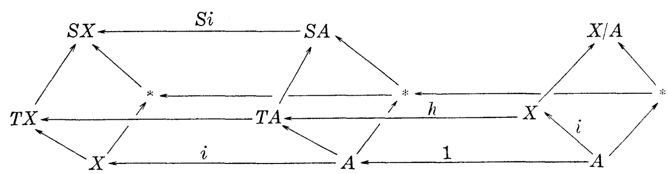
$N \bmod \mathscr{H}$ if and only if $M \approx M$.

We then have

THEOREM 4.1. *Let $X$ be a compact Hausdorff space, $A \subset X$ a closed subspace. Let $SA$, $SX$ denote the suspensions of $A$, $X$ respectively. Then there is an exact sequence*

$$\tilde{K}_0^T(SX) \longrightarrow \tilde{K}_0^T(SA) \longrightarrow \tilde{K}_0^T(X/A) \longrightarrow \tilde{K}_0^T(X) \longrightarrow \tilde{K}_0^T(A)$$

*Proof.* Consider the diagram




where $TX$ denotes the cone on $X$ and $h$ is any continuous function. Applying the functor $C$ we get a diagram of the form (*) and it is not hard to show that the vertical squares are cartesian. Since $TA$ is contractible $hi \cong j$ so $i^*h^* \cong j^*$. Thus theorem (3.4) applies to give the desired exact sequence.

The long exact $K$-theory sequence follows in the usual manner by splicing sequences of this form together.

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HARVEY MUDD COLLEGE
AND
NORTHEASTERN ILLINOIS UNIVERSITY

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# WEIGHTED CONVERGENCE IN LENGTH

WILLIAM R. DERRICK

**This paper studies the lower semicontinuity of weighted length**

$$(*)\qquad \liminf_{n\to\infty}\int_{\gamma_n} f\,ds \geqq \int_{\gamma} f\,ds\ ,$$

**where the sequence of curves $\{\gamma_n\}$ converges uniformly to the curve $\gamma$, and $f$ is a nonnegative lower semicontinuous function. Necessary and sufficient conditions for equality in (*) are obtained, as well as conditions which prevent $\gamma$ from being rectifiable. Requirements are given for the attainment of the weighted distance, from a point to a set, and the families of functions, for which weighted distance is attained or (*) is satisfied, are shown to be monotone closed from below. Finally, the solutions to the integral inequality**

$$(**)\qquad |\gamma(t)-\gamma(0)| \geqq \int_{\gamma[0,t]} f\,ds\ ,$$

**are shown to be compact if the initial values $\gamma(0)$ lie in a compact set.**

Let $\gamma$ be a curve in Euclidean $m$-space $E^m$ and $f$ be a real-valued function on $E^m$. The $(f)$-weighted length of $\gamma$, $\int_\gamma f ds$, has proved of fundamental importance in establishing the path-cut inequality for condensers [2], [3] and the relationship between capacity and extremal length [5], [8]. Theorem (2.4) provides necessary and sufficient conditions for weighted convergence in length, and (2.10) gives conditions under which the weighted distance, from a point to a set, is attained. Corollary (2.6) is a useful special case of [8, Lemma 3.3]. In (3.1) the family of functions, for which weighted distance is attained, is shown to be monotone closed from below, and Theorem (3.2) establishes the compactness of the set of solutions to the contingent equation (**), similar to a result of Filippov [4].

## 2. Convergence theorems.

NOTATION 2.1. Let $E^m$ denote *Euclidean m-space* consisting of all $m$-tuples $x=(x_1,\cdots,x_m)$ of real numbers with inner product $\langle x, y\rangle=\sum_{i=1}^m x_iy_i$, for all $x$, $y$ in $E^m$ and norm $|x|=\langle x, x\rangle^{1/2}$. Throughout this paper, points in $E^m$ will often be denoted by the letters $x$ and $y$, whereas the letters $s$, $t$ will be reserved for real numbers. The complex plane is designated by the symbol $\mathscr{C}$.

Let Int $A$, cl $A$, $\partial A$ denote the *interior, closure,* and *boundary* of the set $A$, respectively. The open ball of radius $t$ centered at $x$ will be indicated by the expression $B(x, t)$.

A function $f: E^m \to E^n$ is *Lipschitz* on the set $A$ in $E^m$ if there is a constant $M$ such that

$$|f(x) - f(y)| \leqq M|x - y| ,$$

for all $x, y$ in $A$. If $n = 1$, the *gradient* of $f$, grad $f$, will exist $L_m$ – a.e. in $A$, where $L_m$ is the $m$-dimensional *Lebesgue measure*. The *Hausdorff* 1-*dimensional measure* in $E^m$ will be denoted by $H^1$ (for its definition and properties see [1]). Then $H^1(A)$ represents the *length* of the set $A$ in $E^m$.

DEFINITIONS 2.2. Two functions $\gamma: [a, b] \to E^m$, $\gamma^* = [c, d] \to E^m$ are *Fréchet-equivalent* if

$$\inf_h \sup_t |\gamma(t) - \gamma^*(h(t))| = 0 ,$$

where $h: [a, b] \to [c, d]$ is a homeomorphism. A Fréchet equivalence class $\gamma$ of continuous functions into $E^m$ is called a *curve* in $E^m$, and each member of the class is called a *parametrization* of $\gamma$.

The *length* of a curve $\gamma$ is given by

$$H^1_*(\gamma) = \sup_\pi \sum_i |\gamma(t_{i-1}) - \gamma(t_i)| ,$$

where $\gamma: [a, b] \to E^m$ is any parametrization of $\gamma$ and $\pi$ is a partition of $[a, b]$. Note that $H^1(\gamma) < H^1_*(\gamma)$, unless the set of multiple points of $\gamma$ has $H^1$-measure zero (see [7, p. 125]). A curve $\gamma$ is *rectifiable* if $H^1_*(\gamma) < \infty$. In this case we can write

$$H^1_*(\gamma) = \int_\gamma dH^1 = \int_a^b |\text{grad}\, \gamma(t)|\, dt .$$

A rectifiable curve can be parametrized with respect to arc-length (see [6, p. 259]); we denote this parametrization by $\gamma(s)$. Note that $|\text{grad}\, \gamma(s)| = 1$, $H^1$ – a.e. in $[0, H^1_*(\gamma)]$, since $|\gamma(s) - \gamma(s^*)| \leqq |s - s^*|$ implies that $|\text{grad}\, \gamma(s)| \leqq 1$, $H^1$ – a.e., and

$$H^1_*(\gamma) = \int_0^{H^1_*(\gamma)} |\text{grad}\, \gamma(s)|\, ds .$$

If $f: E^m \to E^1$ is a Borel-measurable function and $\gamma$ is a rectifiable curve, define (as above)

$$\int_\gamma f dH^1 = \int_a^b f(\gamma(t))\, |\text{grad}\, \gamma(t)|\, dt ,$$

then in the event $\gamma$ is parametrized by arc length,

$$\int_{\gamma} f dH^1 = \int_0^{H_*^1(\gamma)} f(\gamma(s))ds \ .$$

In particular, if $0 \leqq S \leqq H_*^1(\gamma)$, we define

$$\int_{\gamma[S]} f dH^1 = \int_0^S f(\gamma(s))ds \ .$$

A curve $\gamma$ is *locally rectifiable* if $H_*^1(\gamma \cap \mathrm{cl}\, B(0, k)) < \infty$, for all $k = 1, 2, 3, \cdots$, where $\gamma \cap \mathrm{cl}\, B(0, k)$ are the subcurves of $\gamma$ with images in $\mathrm{cl}\, B(0, k)$.

THEOREM 2.3. *Let $\{\gamma_n(s)\}$ be a sequence of rectifiable curves in $E^m$, such that $H_*^1(\gamma_n) \geqq L > 0$ and $\gamma_n(0) \rightarrow \gamma_0$. Let $\gamma_S$ be an accumulation point of the set $\{\gamma_n(S)\}$, $0 < S \leqq L$. Then some subsequence $\{\gamma_{n_j}\}$ converges uniformly on $[0, S]$ to a curve $\gamma$ containing $\gamma_0$ and $\gamma_S$ such that for every nonnegative lower semicontinuous function $f: E^m \rightarrow E^1$,*

$$\liminf_{j\to\infty} \int_{\gamma_{n_j}[S]} f dH^1 \geqq \int_{\gamma} f dH^1 \ . \tag{1}$$

*Proof.* Since all but finitely many points of $\{\gamma_n(S)\}$ lie in $B(\gamma_0, S + 1)$, so does $\gamma_S$. By selecting a subsequence and reindexing we can assume $\gamma_n(S) \rightarrow \gamma_S$. Each $\gamma_n$ is Lipschitzian with constant 1, so $\{\gamma_n\}$ is equicontinuous on $[0, S]$, and uniformly bounded by $|\gamma_0| + S + 1$. By Ascoli's Theorem, some subsequence $\{\gamma_{n_j}\}$ converges uniformly on $[0, S]$ to a function $\gamma: [0, S] \rightarrow E^m$. Clearly $\gamma$ is a curve from $\gamma_0$ to $\gamma_S$, and is Lipschitzian with constant 1. Thus, $|\mathrm{grad}\, \gamma| \leqq 1$, $H^1$ – a.e., and by Fatou's lemma and the lower semicontinuity of $f$

$$\begin{aligned}\liminf_{j\to\infty} \int_{\gamma_{n_j}[S]} f dH^1 &\geqq \int_0^S \liminf_{j\to\infty} f(\gamma_{n_j}(t))dt \ , \\ &\geqq \int_0^S f(\gamma(t))dt \geqq \int_{\gamma} f dH^1 \ .\end{aligned}$$

COROLLARY 2.4. *Assuming the hypotheses in Theorem (2.3), the condition*

$$\lim_{j\to\infty} \int_0^S |\mathrm{grad}\, \gamma_{n_j}(t) - \mathrm{grad}\, \gamma(t)| dt = 0 \ , \tag{2}$$

*holds if and only if*

$$\lim_{j\to\infty} \int_{\gamma_{n_j}[S]} f dH^1 = \int_{\gamma} f dH^1 \ , \tag{3}$$

*for every continuous function $f: E^m \rightarrow E^1$.*

*Proof.* Let $M$ be a bound for $f$ on $B(\gamma_0, S+1)$. Given $\varepsilon > 0$, the uniform convergence of $\{\gamma_{n_j}\}$ and (2) imply that, for sufficiently large $j$,

$$\left|\int_{\gamma_{n_j}} f dH^1 - \int_\gamma f dH^1\right| \leqq \int_0^S \left|f\circ\gamma_{n_j} - f\circ\gamma\right|\left|\operatorname{grad}\gamma_{n_j}\right| dt$$
$$+ \int_0^S |f\circ\gamma|\,|\operatorname{grad}\gamma_{n_j} - \operatorname{grad}\gamma|\,dt \leqq \varepsilon(S+M) .$$

Thus (2) implies (3). Conversely, if $f \equiv 1$, then

$$S = \lim_{j\to\infty}\int_{\gamma_{n_j}[S]} dH^1 = \int_0^S |\operatorname{grad}\gamma(t)|\,dt \leqq S ,$$

and it follows that $|\operatorname{grad}\gamma| = 1$, $H^1 -$ a.e. By the triangle inequality,

$$|\operatorname{grad}\gamma_{n_j} - \operatorname{grad}\gamma|^2$$
$$= 4 - |\operatorname{grad}\gamma_{n_j} + \operatorname{grad}\gamma|^2 \leqq 4(2 - |\operatorname{grad}\gamma_{n_j} + \operatorname{grad}\gamma|) ,$$

so Schwarz's inequality yields

$$(4)\quad \left[\int_0^S |\operatorname{grad}\gamma_{n_j} - \operatorname{grad}\gamma|\,dt\right]^2 \leqq 4S\left(2S - \int_0^S |\operatorname{grad}\gamma_{n_j} + \operatorname{grad}\gamma|\,dt\right) .$$

But $\{\gamma_{n_j} + \gamma\}$ converges uniformly to $2\gamma$ on $[0, S]$, so Theorem (2.3) implies

$$(5)\qquad \liminf_{j\to\infty}\int_0^S |\operatorname{grad}\gamma_{n_j} + \operatorname{grad}\gamma|\,dt \geqq 2\int_0^S |\operatorname{grad}\gamma|\,dt = 2S .$$

Combining equations (4) and (5) we find

$$0 \leqq \liminf_{j\to\infty}\left(\int_0^S |\operatorname{grad}\gamma_{n_j} - \operatorname{grad}\gamma|\,dt\right)^2$$
$$\leqq \limsup_{j\to\infty}\left(\int_0^S |\operatorname{grad}\gamma_{n_j} - \operatorname{grad}\gamma|\,dt\right)^2$$
$$= 4S\left(2S - \liminf_{j\to\infty}\int_0^S |\operatorname{grad}\gamma_{n_j} + \operatorname{grad}\gamma|\,dt\right) \leqq 0 ,$$

which yields (2).

EXAMPLE 2.5. Let $\gamma_n\colon [0, 2\pi] \to \mathscr{C}$ be given by $\gamma_n(s) = (e^{ins})/n$. Note these functions converge uniformly to the constant function $\gamma(t) = 0$. Although (1) holds, (3) clearly does not, and

$$\int_0^{2\pi} |\operatorname{grad}\gamma_n(t) - \operatorname{grad}\gamma(t)|\,dt = \int_0^{2\pi} |e^{int}|\,dt = 2\pi .$$

COROLLARY 2.6. *Let $\{\gamma_n(s)\}$ be a sequence of rectifiable curves in $E^m$ such that $\gamma_n(0) \to \gamma_0$ and $\gamma_n(s_n) \to \gamma_S$, $0 < s_n \leqq S < \infty$. Then there* *is a subsequence* $\{\gamma_{n_j}\}$ *and a curve* $\gamma$ *containing* $\gamma_0$ *and* $\gamma_S$ *such that*

$$\liminf_{j\to\infty} \int_{\gamma_{n_j}[s_{n_j}]} f dH^1 \geqq \int_\gamma f dH^1 , \tag{6}$$

*for every nonnegative lower semicontinuous function* $f\colon E^m \to E^1$.

*Proof.* Let $\gamma_n^*$ be the restriction of $\gamma_n$ to $[0, s_n]$. Extend $\gamma_n^*$ to $[0, S]$ by setting $\gamma_n^*(t) = \gamma_n^*(s_n)$, for $s_n \leqq t \leqq S$. Each $\gamma_n^*$ is Lipschitzian with constant 1, so, as in Theorem (2.3), some subsequence converges uniformly on $[0, S]$ to a curve $\gamma$ containing $\gamma_0$ and $\gamma_S$, and having Lipschitz constant 1. By passing to a subsequence, we can assume $s_{n_j} \to s^*$ in $[0, S]$. Then $\gamma(s^*) = \gamma_S$ since

$$|\gamma(s^*) - \gamma_{n_j}^*(s_{n_j})| \leqq |\gamma(s^*) - \gamma_{n_j}^*(s^*)| + |s^* - s_{n_j}| \to 0 .$$

For every $\varepsilon > 0$, $s_{n_j} > s^* - \varepsilon$ for large $j$, so by Fatou's Theorem

$$\begin{aligned}\liminf_{j\to\infty} \int_{\gamma_{n_j}[s_{n_j}]} f dH^1 &\geqq \liminf_{j\to\infty} \int_0^{s^*-\varepsilon} f(\gamma_{n_j}^*(t))dt \\ &\geqq \int_0^{s^*-\varepsilon} f(\gamma(t)dt \geqq \int_{\gamma[s^*-\varepsilon]} f dH^1 ,\end{aligned}$$

from which the result follows.

THEOREM 2.7. *Let* $\{\gamma_n(t)\}$ *be a sequence of curves in* $E^m$ *such that* $H_*^1(\gamma_n \cap \operatorname{cl} B(0, k)) \leqq L_k < \infty$, *for all* $n$, $k = 1, 2, \cdots$, *and* $\gamma_n(0) \to \gamma_0$. *Then some subsequence* $\{\gamma_{n_j}\}$ *converges uniformly on compact subsets to a curve* $\gamma$ *containing* $\gamma_0$ *such that*

$$\liminf_{j\to\infty} \int_{\gamma_{n_j}} f dH^1 \geqq \int_\gamma f dH^1 , \tag{7}$$

*for every nonnegative lower semicontinuous function* $f\colon E^m \to E^1$.

*Proof.* There exists an integer $K$ such that $\gamma_0$ and all $\gamma_n(0)$ lie in $B(0, K)$. In each closed ball $\operatorname{cl} B(0, k)$, $k \geqq K$, reparametrize a restriction of $\gamma_n$ by arc length

$$\gamma_{kn}\colon [0, s_{kn}] \to \gamma_n ,$$

where $0 < s_{kn} \leqq L_k$ is either the first real number such that $\gamma_{kn}(s_{kn})$ lies in $\partial B(0, k)$ or $H_*^1(\gamma_n)$, if no such number exists. If denumerably many $\gamma_n$ lie in some $B(0, k)$ the proof follows by Corollary 2.6. Otherwise, delete all $\gamma_n$ which lie in $B(0, k + 1)$. Then a subsequence of $\{\gamma_{(K+1)n}(1)\}$ converges to a point $p_1$ in $\operatorname{cl} B(0, K + 1)$, and Theorem 2.3 yields a subsequence $\{\gamma_{(K+1)n_i}\}$ converging uniformly on $[0, 1]$ to a curve $\gamma^1$ containing $\gamma_0$ and $p_1$. Delete all $\gamma_{n_i}$ lying in $B(0, K + 2)$. A subsequence of $\{\gamma_{(K+2)n_i}(2)\}$ converges to a point $p_2$ in cl $B(0+2)$, implying a subsequence, which we also denote by $\{\gamma_{(K+2)n_i}\}$, converges uniformly on $[0, 2]$ to a curve $\gamma^2$ containing $\gamma_0$ and $p_2$. Continuing in this manner we note that $\gamma^k$ is an extension of $\gamma^j$, for $k > j$, hence there is a $\gamma\colon [0, \infty) \to E^m$ and a subsequence $\{\gamma_{n_j}\}$ obtained by Cantor's diagonalization process such that $\{\gamma_{n_j}\}$ converges uniformly to $\gamma$ on compact subsets of $[0, \infty)$. By Theorem 2.3 we have that for every real number $S > 0$,

$$\liminf_{j\to\infty} \int_{\gamma_{n_j}[S]} f dH^1 \geqq \int_{\gamma[S]} f dH^1 ,$$

for every nonnegative lower semicontinuous function, hence the proof is complete.

REMARK 2.8. Observe, from the construction above, that $\gamma$ is bounded if denumerably many $\gamma_n$ lie in some $B(0, k)$, and is unbounded otherwise, as a consequence of the hypothesis $H^1_*(\gamma_n \cap \mathrm{cl}\, B(0, k)) \leqq L_k < \infty$.

Theorem 2.7 is true if we replace this condition by the requirement that $H^1_*(\gamma_n \cap \mathrm{cl}\, B(0, k)) < \infty$, for all positive integers $n$ and $k$, since if denumerably many $\gamma_n$ lie in some $B(0, k)$ and no uniform bound exists on their lengths, an argument similar to the rest of the proof above, using curves of length $\geqq j$, sequences of points $\{\gamma_n(j)\}$, $j = 1, 2, \cdots$, and diagonalization, yields a subsequence $\{\gamma_{n_j}\}$ converging uniformly on compact subsets of $[0, \infty)$ to a curve $\gamma$ for which (7) holds. Of course, $\gamma$ might then be a constant function as in Example 2.5. Moreover, it is no longer true that $\gamma$ is unbounded if only finitely many $\gamma_n$ lie in each $B(0, k)$, as is seen in the next example.

EXAMPLE 2.9. In $E^2$, select the points

$$a_n = \left(\frac{n-1}{n}, \frac{n-1}{n}\right),\ b_n = \left(1, \frac{n}{n+1}\right),$$
$$c_n = \left(n+1, \frac{n}{n+1}\right),\ n = 1, 2, \cdots .$$

Let $\gamma_n$ be the polygonal arc obtained by joining the points $a_1, b_1, a_2, b_2, \cdots, a_n, b_n, c_n$ be straight line segments in their given order. Clearly $H^1_*(\gamma_n \cap \mathrm{cl}\, B(0, k)) < \infty$, for all $n$, and $\gamma_n$ lies in $B(0, k)$ iff $n \leqq k - 2$. However, if we parametrize these arcs by arc length, then $\{\gamma_n\}$ converges uniformly on compact subsets of $[0, \infty)$ to the polygonal arc $\gamma$ joining the points $a_1, b_1, a_2, b_2, \cdots$.

LEMMA 2.10. *Let $K$ be a closed subsets of the bounded arcwise connected set $A$ in $E^m$, $y$ a point in $A - K$, $\Gamma$ the family of curves* *joining $y$ to $K$ in $A$, and $f\colon E^m \to E^1$ a positive lower semicontinuous function. Then there is a curve $\gamma_f$ in $\Gamma$ such that*

$$(8) \qquad \int_{\gamma_f} f dH^1 = \inf_{\Gamma} \int_{\gamma} f dH^1 .$$

*Proof.* Assume the right side of (8) equals $M < \infty$ as otherwise any $\gamma$ will do. Let $\{\gamma_n\}$ be a minimizing sequence of curves in $\Gamma$. Since $f(x) \geqq a > 0$ on $\operatorname{cl} A$, for sufficiently large $n$ we have $H^1_*(\gamma_n) \leqq 2M/a$. Parametrizing these rectifiable curves by arc length so that $\gamma_n(0) = y$ and $\gamma_n(s_n)$ belongs to $K$, for $s_n \leqq 2M/a$, Corollary (2.6) and the compactness of $K$ imply the existence of a curve $\gamma_f$ in $\Gamma$ such that

$$M \leqq \int_{\gamma_f} f dH^1 \leqq \liminf_{j\to\infty} \int_{\gamma_{n_j}} f dH^1 = M .$$

REMARK 2.11. If $A$ is unbounded, the same result may be obtained by requiring that the lower semicontinuous function $f$ be bounded below, by a positive constant, on $A$.

One may also weaken the requirement on the lower semi-continuous function $f$ by asking that it be nonnegative and satisfy

$$(9) \qquad H^1(\{x\colon f(x) < \varepsilon\}) = o(1) .$$

Then $M > 0$ and a minimizing sequence $\{\gamma_n\}$ can be chosen, for $M < \infty$ and sufficiently small $\varepsilon$, such that

$$H^1_*(\gamma_n) < o(1) + 2M/\varepsilon .$$

The proof follows as before. Condition (9) can not be removed entirely as is seen by letting $A$ be the closed unit disk in $\mathscr{D}$, $K = \partial A$, $y = 0$, and $f$ be the characteristic function on the complement of the set

$$\{z(t)\colon z(t) = (1 - t^{-1})e^{i\pi t}, 1 \leqq t < \infty\} .$$

**3. Some compactness theorems.** Let $\mathfrak{A}$ be the set of functions $f\colon E^m \to E^1$ for which Theorem 2.3 (2.6, or 2.7) holds, and $\mathfrak{B}$ the set of functions which permit the verification of Lemma 2.10. Clearly $\mathfrak{A}$ and $\mathfrak{B}$ properly contain the nonnegative and positive lower semicontinuous function respectively, since the function values may be changed on sets of $H^1$-measure zero without affecting (1) or (8).

THEOREM 3.1. *Let $\{f_k\}$ be a nondecreasing sequence of functions in $\mathfrak{A}$ and $f(x) = \lim_k f_k(x)$. Then $f$ is also in $\mathfrak{A}$. The same result also holds for $\mathfrak{B}$ provided $f_1(x) \geqq a > 0$ on $\operatorname{cl} A$.*

*Proof.* Let $\{\gamma_n\}$ be a sequence of curves satisfying the hypothesis of Theorem (2.3). Then by the Lebesgue monotone convergence theorem and (2.3), we have

$$(10)\qquad \int_\gamma f dH^1 \leqq \limsup_{k\to\infty}\left(\liminf_{n\to\infty}\int_{\gamma_n} f_k dH^1\right) \leqq \liminf_{n\to\infty}\int_{\gamma_n} f dH^1 ,$$

implying that $f$ lies in $\mathfrak{A}$. Let $M$ equal the right side in equation (8). There is nothing to prove if $M = \infty$, so let $M < \infty$. For each $f_k$ there is a curve $\gamma_k$ such that

$$\int_{\gamma_k} f_k dH^1 = \inf_{\Gamma}\int_\gamma f_k dH^1 = M_k \leqq M .$$

Since

$$(11)\qquad M_j = \int_{\gamma_j} f_j dH^1 \leqq \int_{\gamma_k} f_j dH^1 \leqq M_k,\ j \leqq k ,$$

the sequence $\{M_k\}$ has a limit $M^* \leqq M$. Moreover

$$aH^1_*(\gamma_k) \leqq \int_{\gamma_k} f_k dH^1 \leqq M ,$$

so the curves $\{\gamma_k\}$ satisfy the hypothesis in Corollary (2.6). Hence there is a curve $\gamma$ such that (6) holds for each $f_k$. Thus by (10) and (11)

$$\begin{aligned} M \leqq \int_\gamma f dH^1 &\leqq \limsup_{j\to\infty}\left(\liminf_{k\to\infty}\int_{\gamma_k} f_j dH^1\right) \\ &\leqq \limsup_{j\to\infty}\,(\liminf_{k\to\infty} M_k) = M^* \leqq M . \end{aligned}$$

Now let $A$ be a subset of $E^m$, $0 < S < \infty$, $f\colon E^m \to E^1$ a nonnegative lower semicontinuous function, and

$$\mathfrak{D}_A = \mathfrak{D}_A(f, S) = \left\{\gamma\colon \gamma(0) \in A,\ |\gamma(s) - \gamma(0)| \geqq \int_{\gamma[s]} f dH^1,\ 0 \leqq s \leqq S\right\} .$$

Then $\mathfrak{D}_A$ is a subset of the Banach space of all continuous functions on $[0, S]$ with the sup norm.

THEOREM 3.3. *If $A$ is compact, then $\mathfrak{D}_A$ is compact.*

*Proof.* Let $\{\gamma_n\}$ be a sequence of curves in $\mathfrak{D}_A$. By Theorem (2.3), some subsequence, which will also be denoted be $\{\gamma_n\}$, converges uniformly on $[0, S]$ to a curve $\gamma$, with $\gamma(0)$ in $A$, and satisfies

$$\int_{\gamma[0,s]} f dH^1 \leqq \liminf_{n\to\infty}\int_{\gamma_n[s]} f dH^1 \leqq |\gamma(s) - \gamma(0)|,\ 0 \leqq s \leqq S .$$

Reparametrizing $\gamma$ by arc length $(H^1_*(\gamma) \leqq S)$ and extending it to $[0, S]$, as in the proof of (2.6) shows that $\gamma$ belongs to $\mathfrak{D}_A$.

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# COLLECTIVELY COMPACT AND SEMI-COMPACT SETS OF LINEAR OPERATORS IN TOPOLOGICAL VECTOR SPACES

M. V. DESHPANDE AND N. E. JOSHI

**A set of linear operators from one topological vector space to another is said to be collectively compact (resp. semi-compact) if and only if the union of images of a neighbourhood of zero (respectively every bounded set) is relatively compact. In this paper sufficient conditions for a set of operators to be collectively compact or semi-compact are obtained. It is proved that if $T_n \to T$ asymptotically, where $X$ is quasi-complete and $T_n$ are $W$-compact then $\{T_n - T\}$ is collectively compact. The final section deals with collectively weakly compact sets. It is proved that in a reflexive locally convex space a family of continuous endomorphisms is collectively weakly compact if and only if**

$$\mathscr{K}^* = \{K^*: E_s^* \longrightarrow E_{w^*}^*\}$$

**is collectively compact.**

The concept of collectively compact sets of linear operators on normed linear spaces was introduced by Anselone and Moore [3]. This concept was studied in greater detail by Anselone and Palmer [1, 2]. Some of the results in these papers were extended to more general spaces in [4]. In this paper some further generalizations are obtained.

2. Let $X$ and $Y$ be topological vector spaces and $\mathscr{L}[X, Y]$, the set of continuous linear operators on $X$ to $Y$. The underlying scalar field will be assumed to be the field of complex numbers, unless otherwise stated.

DEFINITION 2.1. A subset $\mathscr{K} \subset \mathscr{L}[X, Y]$ is said to be collectively compact (respectively, weakly compact, totally bounded) if and only if there exists a neighbourhood $V$ of zero in $X$ such that $\mathscr{K}V = \{Kx: K \in \mathscr{K}, x \in V\}$ is relatively compact (respectively weakly compact, totally bounded) in $Y$.

REMARK. It is obvious that $\mathscr{K}$ collectively compact $\Rightarrow \mathscr{K}$ collectively weakly compact. However, if $Y$ is a Montel space, the reverse implication is also true.

PROPOSITION 2.1. *Let $\mathscr{K} \subset \mathscr{L}[X, Y]$ be collectively compact and $Y$, a quasi-complete locally convex space. Then the following statements hold.*

(a) *The convex hull of $\mathscr{K}$ is collectively compact.*

(b) *The balanced hull of $\mathscr{K}$ is collectively compact.*

(c) *The absolutely convex hull of $\mathscr{K}$ is collectively compact.*

(d) *The closure of $\mathscr{K}$ in the pointwise topology, and therefore in $\mathscr{L}_b{}^*[X, Y]$ is collectively compact.*

(e) *The set $\{\sum_{n=1}^{N} \lambda_n K_n\colon K_n \in \mathscr{K}, \sum_{n=1}^{N} |\lambda_n| \leqq b, b > 0, N \leqq \infty\}$ is collectively compact, the convergence of the series being in $\mathscr{L}_b[X, Y]$.*

*Proof.* (a) Let $\mathfrak{C}\mathscr{K}$ be the convex hull of $\mathscr{K}$. As $\mathscr{K}$ is collectively compact, there exists a neighbourhood $V$ of zero in $X$ such that $\mathscr{K}V$ is relatively compact in $Y$. Now,

$$(\mathfrak{C}\mathscr{K})V \subset \mathfrak{C}(\mathscr{K}V) \subset \overline{\mathfrak{C}(\overline{\mathscr{K}V})}\,,$$

where bar denotes the closure. Since $\overline{\mathscr{K}V}$ is compact and $Y$ is quasi-complete, $\overline{\mathfrak{C}(\overline{\mathscr{K}V})}$ is compact [9, 20.6(3)]. It follows that $\mathfrak{C}\mathscr{K}$ is collectively compact. The proofs of (b) – (e) are similar to those in [1].

PROPOSITION 2.2. *Let $X$, $Y$ and $Z$ be topological vector spaces and $\mathscr{K} \subset \mathscr{L}[X, Y]$, $\mathscr{M} \subset \mathscr{L}[Z, X]$, $\mathscr{N} \subset \mathscr{L}[Y, Z]$ then:*

(a) *$\mathscr{K}$ collectively compact and $\mathscr{M}$ equicontinuous $\Rightarrow \mathscr{K}\mathscr{M}$ collectively compact.*

(b) *$\mathscr{K}$ collectively compact and $\mathscr{N}$ relatively compact in the $\mathscr{L}_b[X, Y] \Rightarrow \mathscr{N}\mathscr{K}$ is collectively compact.*

*Proof.* (a) Since $\mathscr{K}$ is collectively compact, there exists a neighbourhood $V$ of zero in $X$ such that $\mathscr{K}V$ is relatively compact in $Y$. Further, by the equicontinuity of $\mathscr{M}$, there exists a neighbourhood $W$ of zero in $Z$ such that $\mathscr{M}W \subset V$. Hence

$$(\mathscr{K}\mathscr{M})W \subset \mathscr{K}V\,.$$

From this the assertion follows.

(b) See [4], Prop. 2.3 (b).

COROLLARY. *If $\mathscr{K} \subset \mathscr{L}[X, Y]$ is collectively compact and $\mathscr{M} \subset \mathscr{L}[Z, X]$ is bounded where $Z$ is barreled and $X$ locally convex, then $\mathscr{K}\mathscr{M}$ is collectively compact.*

* $\mathscr{L}[X, Y]$ with the topology of uniform convergence on bounded sets of $X$.

*Proof*. For, if $Z$ is barreled and $X$ is locally convex then $\mathscr{M} \subset \mathscr{L}_b[Z, X]$ is bounded if and only if it is equicontinuous.

It is proved in [1] that a compact set of compact operators on a Banach space is collectively compact. We shall prove a similar but slightly weaker result for topological vector spaces. For this, we introduce the following definitions.

DEFINITION 2.2. A linear operator $K \in \mathscr{L}[X, Y]$, where $X$ and $Y$ are topological vector spaces, is said to be semi-compact if it maps every bounded subset of $X$ into a relatively compact subset of $Y$.

It is obvious that a compact operator is semi-compact. The converse is also true if $X$ is a quasinormed space.

DEFINITION 2.3. A set of linear operators $\mathscr{K} \subset \mathscr{L}[X, Y]$ is said to be collectively semi-compact, if and only if, for every bounded set $B \subset X$, $\mathscr{K}B$ is relatively compact in $Y$.

It is clear that a collectively compact set of operators is collectively semi-compact and the propositions proved so far, for collectively compact sets, are also true for collectively semi-compact operators if $X$ is bornological and $Y$ locally convex, because, a semi-compact operator is bounded on bounded sets and therefore continuous if the domain space is bornological.

We prove the following

LEMMA 2.1. *Let $\mathscr{F}$ be an equicontinuous family of operators on a compact set $\mathscr{K}$ into a topolological vector space $Y$. Let $\mathscr{F}$ be compact with respect to the topology of pointwise convergence. Then the set $\mathscr{F}(\mathscr{K}) = \{f(K): f \in \mathscr{F}, K \in \mathscr{K}\}$ is compact.*

*Proof*. $\mathscr{F}$ is equicontinuous, therefore, $f(K)$ is jointly continuous, in the sense, that the map $(\mathscr{F} \times \mathscr{K}) \to Y$ defined by $(f, K) \to fK$ is continuous relative to the product topology [8, 8.14]. Now $\mathscr{F} \times \mathscr{K}$ is compact, hence $\mathscr{F}\mathscr{K}$, the continuous image of $\mathscr{F} \times \mathscr{K}$ is compact.

The following proposition generalizes the theorem 3.6 in [4].

PROPOSITION 2.3. *Let $X$, $Y$ be locally convex spaces, $X$ bornologic. Let $\mathscr{K}$ be a set of semi-compact operators, compact in $\mathscr{L}_b[X, Y]$. Then $\mathscr{K}$ is collectively semi-compact.*

*Proof*. Define a map $f_x$: $\mathscr{L}[X, Y] \to Y$ by $f_x(K) = Kx$ for $K \in \mathscr{L}[X, Y]$ and each $x \in B$, a bounded set in $X$. Consider the set

$\mathscr{F} = \{f_x\colon x \in B\}$. We prove that $\mathscr{F}$ is equicontinuous. Let $V$ be a neighbourhood of zero in $Y$. Then the set $W = \{K\colon KB \subset V\}$ is a neighbourhood of zero in $\mathscr{L}_b[X, Y]$. Now,

$$\begin{aligned}\mathscr{F}W &= \{f_x(K)\colon f_x \in \mathscr{F}, K \in W\} \\ &= \{Kx\colon K \in W, x \in B\} \\ &= W(B) \subset V.\end{aligned}$$

This proves the equicontinuity of $\mathscr{F}$. Now, the closure $\overline{\mathscr{F}}$ in the pointwise topology is also equicontinuous. Also, $\overline{\mathscr{F}}K \subset \overline{\mathscr{F}K} = \overline{KB}$ which is compact by hypothesis on $\mathscr{K}$. Hence $\overline{\mathscr{F}}K$ is relatively compact in $Y$, for each $K \in \mathscr{K}$. From this follows the compactness of $\overline{\mathscr{F}}$ [8, § 8, Problem H]. From Lemma 2.1 we deduce that $\mathscr{F}\overline{\mathscr{F}}\mathscr{K}$ is compact. But $\mathscr{K}B = \mathscr{F}\mathscr{K} \subset \overline{\mathscr{F}}\mathscr{K}$. Hence $\mathscr{K}B$ is relatively compact. This implies that $\mathscr{K}$ is collectively semi-compact.

COROLLARY. *If $Y$ is complete, then every totally bounded set $\mathscr{K}$ of semi-compact operators in $\mathscr{L}_b[X, Y]$ is collectively semi-compact.*

*Proof.* In this case $\mathscr{L}_b[X, Y]$ is complete. Hence $\overline{\mathscr{K}}$ is compact. By the proposition $\overline{\mathscr{K}}$ is collectively semi-compact. Then so is $\mathscr{K}$.

PROPOSITION 2.4. *Suppose $X$, $Y$ are locally convex spaces. Let $Y$ be reflexive. Then every set $\mathscr{K}$ of semi-compact operators bounded in $\mathscr{L}_b[X, Y]$ is collectively weakly semi-compact.*

*Proof.* Since $\mathscr{K}$ is bounded in $\mathscr{L}_b[X, Y]$, $\mathscr{K}B$ is bounded for every bounded set $B \subset X$. Since $Y$ is reflexive, every closed bounded set is weakly compact. [10, Th. 36.5]. The conclusion follows.

3. Convergence properties of collectively compact sequences of operators.

PROPOSITION 3.1. *Let $X$ and $Y$ be topological vector spaces, $Y$ sequentially complete. Let $T, T_n \in \mathscr{L}[X, Y]$ for all $n$. Then:*

(a) *$T_n \to T$ in $\mathscr{L}_b[X, Y]$ if and only if $T_n \to T$ in pointwise topology and $\{T_n - T\}$ is totally bounded in $\mathscr{L}_b[X, Y]$.*

(b) *If, in addition, $X$ is bornologic and $Y$ locally convex, then $T_n \to T$ in $\mathscr{L}_b[X, Y]$ and each $T_n - T$ semi-compact $\Rightarrow \{T_n - T\}$ is collectively semi-compact.*

*Proof*. (a) It is evident that $T_n \to T$ in $\mathscr{L}_b[X, Y]$ $T_n \to T$ pointwise and $\{T_n - T\}$ is relatively compact, and hence, totally bounded.

For the reverse implication assume that $T_n - T \nrightarrow 0$ in $\mathscr{L}_b[X, Y]$. This implies that for a given neighbourhood $V$ of zero in $Y$, and bounded set $B$ in $X$, there exists a sequence $n_i$ such that $(T_{n_i} - T)(B) \not\subset V$, for each $i = 1, 2, \cdots$. Since $\{T_n - T\}$ is totally bounded, there exists a Cauchy subsequence $\{T_{n_{ij}} - T\}$ which must converge in $\mathscr{L}_b[X, Y]$ by completeness of $Y$. Because $T_n - T \to 0$ pointwise, it follows that $T_{n_{ij}} - T \to 0$ in $\mathscr{L}_b[X, Y]$. Therefore $(T_{n_{ij}} - T)(B) \subset V$, $j > N$, a positive integer. This is a contradiction.

(b) This follows from the fact that a totally bounded set of semi-compact operators is collectively semi-compact if $Y$ is a complete locally convex space and $X$ is bornologic (Cor. Prop. 2.3).

REMARKS. If $T_n \to T$ pointwise and $X$ is of second category, the Banach-Steinhaus theorem implies that the $\{T_n\}$ is equicontinuous, and hence, the pointwise convergence is uniform on the compact sets of $X$. On the other hand, as proved in (a) above, $\{T_n - T\}$ totally bounded and $T_n \to T$ pointwise imply convergence in $\mathscr{L}_b[X, Y]$, i.e. uniform convergence on bounded sets. This leads to the following propositions.

PROPOSITION 3.2. *Suppose $T_n \to T$ pointwise on $X$, where $X$ is bornologic and of second category. Suppose $\mathscr{K} \subset \mathscr{L}[X, X]$ is collectively semi-compact. Then $(T_n - T)K \to 0$ in $\mathscr{L}_b[X, X]$ uniformly for $K \in \mathscr{K}$.*

PROPOSITION 3.3. *Let $T_n \to T$ pointwise and $\mathscr{K} \subset \mathscr{L}[X, X]$ be totally bounded in the pointwise topology. Suppose $X$ is complete and of second category. Then $T_nK \to TK$ pointwise uniformly for $K \in \mathscr{K}$.*

*Proofs*. Similar to Propositions 3.1 and 3.2 in [2].

4. Asymptotic convergence and collectively compact sequences of operators.

The concept of convergence of operator sequences in the uniform operator topology in the normed spaces, is generalized in the following manner in [5].

DEFINITION 4.1 A linear operator $K$ on a topological vector space $E$ into itself is said to be the asymptotic limit of a sequence $K_n$ of linear operators, if and only if, there exists a neighbourhood $V$ of zero in $E$, a sequence $\alpha_n$ of scalars $\to 0$ as $n \to \infty$ and a bounded set $B \subset E$ such that $(K - K_n) V \subset \alpha_n B$, for $n = 1, 2, \cdots$. This mode of convergence will be denoted by $K_n \twoheadrightarrow K$, and $K$ will be called the $V$-asymptotic limit of $K_n$.

DEFINITION 4.2. A linear operator $K$ on $E$ to itself is said to be $V$-totally bounded if and only if $V$ is a neighbourhood of zero and $KV$ is totally bounded in $E$.

DEFINITION 4.3. If $K$ is the $V$-asymptotic limit of $K_n$ and if each $K_n$ is $V$-totally bounded, $K$ is said to be asymptotically $V$-totally bounded.

PROPOSITION 4.1. *If $K$ is asymptotically $V$-totally bounded, then $K$ is $V$-totally bounded.*

*Proof.* [5, 4.2–1].

PROPOSITION 4.2. *Let $T$, $T_n \in \mathscr{L}[X, X]$ and let $T$ be the $V$-asymptotic limit of $T_n$ where each $T_n$ is $W$-totally bounded. Then $\{T_n - T\}$ is collectively totally bounded.*

*Proof.* $T_n \twoheadrightarrow T$ and each $T_n$ $W$-totally bounded implies $T$ is $W$-totally bounded (Prop 4.1.). Now, $T_n \twoheadrightarrow T \Rightarrow$ there exists a sequence $\alpha_n$ of scalars $\to 0$ as $n \to \infty$, a bounded set $B \subset X$ such that

$$(T_n - T)(W) \subset \alpha_n B \qquad \text{for all } n .$$

Let $V$ be any neighbourhood of zero. Choose a balanced neighbourhood $V_1$ of zero such that $V_1 + V_1 \subset V$. Since $B$ is bounded, $B \subset \alpha V_1$ for some scalar $\alpha$. Therefore, $(T_n - T)(W) \subset \alpha_n \alpha V_1$. We can choose $N$ such that $|\alpha\alpha_n| < 1$ for $n > N$. Hence $(T_n - T)(W) \subset V_1$ for $n > N$. It follows that

$$\bigcup_n (T_n - T)(W) \subset \bigcup_{i=1}^{N} (T_i - T)(W) + V_1 .$$

As $(T_i - T)(W)$ is totally bounded for each $i$, so is their finite union. Hence, $\bigcup_{i=1}^{N} (T_i - T)(W) \subset \bigcup_{i=1}^{M} (x_i + V_1)$ for some $M$, a positive integer, and $x_i \in E$. Hence,

$$\bigcup_n (T_n - T)(W) \subset \bigcup_{i=1}^{M} (x_i + V) .$$

This proves the proposition.

COROLLARY 1. *Let* $T, T_n \in \mathscr{L}[X, X]$ *where* $X$ *is quasi-complete. Suppose* $T_n$ *is W-compact* i.e. $T_n(W)$ *is relatively compact for some neighbourhood* $W$ *of zero in* $X$. *If* $T_n \twoheadrightarrow T$, *then* $\{T_n - T\}$ *is collectively compact.*

*Proof.* $T_n \twoheadrightarrow T$ and each $T_n$ $W$-compact $\Rightarrow T$ is $W$-compact because $X$ is quasi-complete [5, 4.2-1 Cor. 3]. From the above proposition it follows that $\bigcup_n(T_n - T)(W)$ is totally bounded. Hence, the closure $\overline{\bigcup_n(T_n - T)(W)}$ is bounded and closed and, therefore, complete by the quasi-completeness of $X$. Thus, $\overline{\bigcup_n(T_n - T)(W)}$ is totally bounded and complete, and therefore compact.

COROLLARY 2. *If* $T_n \twoheadrightarrow T$ *on a neighbourhood* $W$ *of zero in* $X$, *and each* $T_n$ *is W-totally bounded, then* $\{T_n - T\}$ *is collectively compact if* $X$ *is a Montel space.*

*Proof.* From the Proposition 4.2 it follows that $\{T_n - T\}$ is collectively $W$-totally bounded, and, therefore $W$-collectively compact, as $X$ is a Montel space.

PROPOSITION 4.3. *Let* $T_n \twoheadrightarrow T$, *where* $T_n, T \in \mathscr{L}[X, X]$. *If* $\mathscr{K} \subset \mathscr{L}[X, X]$ *is collectively compact, then* $(T_n - T)K \twoheadrightarrow 0$ *uniformly on* $\mathscr{K}$.

*Proof.* Since $\mathscr{K}$ is collectively compact, there exists a neighbourhood $A$ of zero in $X$ such that $\overline{\mathscr{K}A}$ is compact in $X$, and hence bounded. Now, $T_n \twoheadrightarrow T \Rightarrow$ there exists a neighbourhood $W$ of zero in $X$, bounded set $B \subset X$, and a sequence $\alpha_n$ of scalars $\to 0$ such that $(T_n - T)(W) \subset \alpha_n B$ for all $n$. As $\mathscr{K}A$ is bounded, $\mathscr{K}A \subset rW$ for some scalar $r$. Hence, $(T_n - T)(\mathscr{K}A) \subset (T_n - T)(rW) \subset (r\alpha_n B)$, for all $n$. Since $\alpha_n$ and $B$ are independed of $\mathscr{K}$, $(T_n - T)K \twoheadrightarrow 0$, uniformly on $\mathscr{K}$.

5. **Collectively compact sets in weak topology.** In this section we consider the inter-relation between a collectively compact set of operators and its dual family.

PROPOSITION 5.1. *Let* $E$ *be a locally convex topological vector space and* $\mathscr{K}$ *a family of continuous endomorphisms, uniformly bounded on a neighbourhood* $V$ *of zero in* $E$. *Let* $\mathscr{K}^*$ *be the family of dual operators. Then* $\mathscr{K}^*$ *considered as the set of mappings* $\{K^*: E_s^* \to E_{w^*}^*\}$ *is collectively compact, where* $E_s^*$ *is the strong dual and* $E_{w^*}^*$ *the* $w^*$*-dual of* $E$.

*Proof.* By hypothesis, $\mathscr{K}V = B$ is a bounded set in $E$. Consider neighbourhood $W$ of zero in $E_s^*$ defined by

$$W = \left\{f\colon\ f \in E_s^*,\ \operatorname*{Sup}_{y \in B}\ |\langle y, f\rangle| < 1\right\}$$
$$= \left\{f\colon\ f \in E_s^*,\ \operatorname*{Sup}_{\substack{x \in V \\ K^* \in \mathscr{K}^*}}\ |\langle x, K^*f\rangle| < 1\right\}.$$

It then follows that

$$\begin{aligned} f \in W &\Longrightarrow |\langle x, K^*f\rangle| < 1 \quad \text{for all } x \in V, \text{ and } K^* \in \mathscr{K}^* \\ &\Longrightarrow K^*f \in V^\circ, \text{ the polar of } V \text{ in } E, \text{ for all } K^* \in \mathscr{K}^* \\ &\Longrightarrow \mathscr{K}^* W \subset V^\circ\ . \end{aligned}$$

Now, by the Banach-Alaoglu theorem [8, Th. 17.4], $V^\circ$ is $w^*$-compact in $E^*$. Hence $\mathscr{K}^*W$ is relatively compact in $E_{w^*}^*$. This completes the proof.

PROPOSITION 5.2. *Let $E$ be a semi-reflexive locally convex space and $\mathscr{K}$, a family of continuous endomorphisms on $E$. If $\mathscr{K}$ is uniformly bounded on a neighbourhood $V$ of zero in $E$, then $\mathscr{K}$ considered as a family of operators from $(E_s^*)_s^* \to (E_s^*)_{w^*}^*$ is collectively compact.*

*Proof.* From Proposition 5.1 it follows that the family $\mathscr{K}^*$ of operators from $E_s^* \to E_{w^*}^*$ is collectively compact. Therefore, there exists a neighbourhood $W$ of zero in $E_s^*$ such that $B = \mathscr{K}^*W$ is relatively compact in $E_{w^*}^*$ and, hence, bounded in $w^*$-topology. From semi-reflexivity and from the fact, that a weakly bounded set is also bounded in the initial topology [8, Th. 17.5], it follows that $B$ is bounded in $E$. From Proposition 5.1, it follows that

$$\mathscr{K}^{**} = \{K^{**}\colon\ (E_s^*)_s^* \longrightarrow (E_s^*)_{w^*}^*\}$$

is collectively compact. Also $\mathscr{K} = \mathscr{K}^{**}$ by the continuity of each $K \in \mathscr{K}$. Hence the result.

COROLLARY. *Let $K$ be a continuous linear endomorphism on $E$, a locally convex space. Suppose $K$ is bounded on a neighbourhood of zero in $E$. If $E$ is reflexive, then $K$ is weakly compact.*

PROPOSITION 5.3. *Let $E$ be a locally convex, reflexive space, and $\mathscr{K}$ a family of continuous endomorphisms on $E$. Let $\mathscr{K}^*$ be the corresponding dual family of endomorphisms on $E^*$. Then $\mathscr{K}$ is collectively weakly compact if and only if $\mathscr{K}^*$ as the family of* *operators* $\{K^*\colon E_s^* \to E_{w^*}^*\}$ *is collectively compact.*

*Proof.* Suppose $\mathscr{K}^*$ is collectively compact as the family of operators $\{K^*\colon E_s^* \to E_{w^*}^*\}$. Then there exists a neighbourhood $W$ of zero in $E_s^*$, such that $\mathscr{K}^*(W)$ is relatively $w^*$-compact. This implies, since $E$ is reflexive and, therefore barreled, that $\mathscr{K}^*(W)$ is equicontinuous, [10, Th. 33.2]. Hence, there exists a neighbourhood $V$ of zero in $E$, such that $\mathscr{K}^*(W) \subset V^\circ$, the polar of $V$. [10, Prop. 32.7]. Therefore,

$$|\langle K^*w, x\rangle| \leqslant 1 ,$$
$$\text{for all } x \in V,\ K^* \in \mathscr{K}^*,\ w \in W \Longrightarrow \mathscr{K}(V) \subset W^\circ .$$

From the reflexivity of $E$ and the Banach-Alaoglu theorem, $\mathscr{K}(V)$ is relatively $w$-compact. This proves that $\mathscr{K}$ is collectively weakly compact.

The converse follows from Proposition 5.1.

COROLLARY. *Let $K$ be a continuous endomorphism on a reflexive locally convex space $E$. Then $K$ is weakly compact if and only if $K^*\colon E_s^* \to E_{w^*}^*$ is compact.*

This is a partial generalization of the Theorem 2.13.7 in [7].

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NAGPUR UNIVERSITY, NAGPUR,
(MAHARASHTRA) INDIA

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# SOME $H^p$ SPACES WHICH ARE UNCOMPLEMENTED IN $L^p$

SAMUEL E. EBENSTEIN

**Let $T^j$ denote the compact group which is the Cartesian product of $j$ copies of the circle where $j$ is a positive integer or $\omega$. If $1 \leqq p \leqq \infty$ let $L^p(T^j)$ denote the space of complex valued measurable functions which are integrable with respect to Haar measure on $T^j$. If $j$ is finite we shall write $n$ instead of $j$. The subspaces $H^p(T^n)$ of $L^p(T^n)$, i.e. the Hardy spaces of $T^n$, have many well-known properties. A family of subspaces $H^p(T^\omega)$ of the $L^p(T^\omega)$ is defined and they are shown to have many of the same properties as the $H^p(T^n)$. However a major difference between $H^p(T^\omega)$ and $H^p(T^n)$ is observed. If $1 < p < \infty$ then $H^p(T^n)$ is complemented in $L^p(T^n)$, but $H^p(T^\omega)$ is uncomplemented in $L^p(T^\omega)$ for $1 < p < \infty$ unless $p = 2$.**

Special properties of homogeneous functions in $H^1(T^\omega)$. Let $j$ be a positive integer or $\omega$. If $j$ is finite we shall write $n$ in place of $j$. We shall let $T^n$ denote the compact group which is the Cartesian product of $n$ circles, and $T^\omega$ the compact group which is the Cartesian product of countably many circles. The dual of $T^n$ is the direct sum of $n$ copies of the integers, and the dual of $T^\omega$ is the direct sum of countably many copies of the integers. If $g \in T^n$, then we write

$$g = (z_1, z_2, \cdots, z_n)$$

where each $z_i$ is a complex number of unit modulus. If $g \in T^\omega$ it has a similar representation, but we must take a countable family, i.e.

$$g = (z_1, z_2, z_3, \cdots) .$$

By abuse of notation if $i \leqq n \leqq \infty$, we let $z_i$ denote that $g \in T^n$ or $g \in T^\omega$ which has the following representation:

$$g = (1, \cdots, 1, z_i, 1, \cdots)$$

where $z_i$ occurs in the $i$th place. We shall write $m_n$ for the normalized Haar measure on $T^n$ and $m$ for the normalized Haar measure on $T^\omega$.

The dual of $T^n$ can be written as $\sum_{i=1}^n Z$, and if $x \in \sum_{i=1}^n Z$ then we write

$$x = (x_1, x_2, \cdots, x_n)$$

where each $x_i$ is an integer. The dual of $T^\omega$ can be written as $\sum_{i=1}^\infty Z$, and if $x \in \sum_{i=1}^\infty Z$, then we write

$$x = (x_1, x_2, x_3, \cdots)$$

where each $x_i$ is an integer, and for any particular $x$, only finitely many $x_i$ are nonzero.

We define $A_n \subset \sum_{i=1}^{n} Z$ and $A \subset \sum_{i=1}^{\infty} Z$ by

$$A_n = \{x\colon x_i \geqq 0 \text{ for all } i\}$$
$$A = \{x\colon x_i \geqq 0 \text{ for all } i\} .$$

We need the following definitions to define $H^p(T^j)$. Although the definitions could be stated in terms of $T^j$ it is easier to state them in the context of arbitrary compact abelian groups.

DEFINITION 1.1. Suppose $G$ is a compact abelian group with dual group $\Gamma$. If $1 \leqq p \leqq \infty$ let $L^p(G)$ denote the space of complex valued measurable functions which are $p^{\text{th}}$ power integrable with respect to Haar measure on $G$. If $E$ is a subset of $\Gamma$, $f$ will be called an *$E$-function* if $f \in L^1(G)$ and $\hat{f}(\gamma) = 0$ if $\gamma \in \Gamma \sim E$, where $\hat{f}(\gamma)$ is the Fourier transform of $f$ evaluated at $\gamma$.

DEFINITION 1.2. Suppose $1 \leqq p \leqq \infty$ then $L_E^p(G) = \{f\colon f \in L^p(G)$ and $f$ is an $E$-function$\}$.

DEFINITION 1.3.

$$H^p(T^n) = L^p_{A_n}(T^n)$$
$$H^p(T^\omega) = L^p_A(T^\omega) .$$

The properties of $H^p(T^n)$ are discussed in [7]. These spaces are related to analytic functions in several complex variables which are defined on the interior of the $n$-polydisc in $C^n$, and are subject to certain growth conditions near the distinguished boundary $T^n$. If $j = \omega$, there is no analogue of the interior of the $n$-polydisc. However we still have many of the nice properties of $H^p(T^n)$.

It is possible to imbed $H^p(T^n)$ in $H^p(T^\omega)$ in a natural way. We have the following homomorphisms

$$\begin{aligned} \pi_n : T^\omega & \longrightarrow T^n \\ (z_1, z_2, \cdots, z_n, z_{n+1} \cdots) & \longmapsto (z_1, z_2, \cdots z_n) \end{aligned}$$

and $\pi_n$ induces an isometry $I_n$.

$$\begin{aligned} I_n : H^p(T^n) & \longrightarrow H^p(T^\omega) \\ f & \longmapsto f \circ \pi_n . \end{aligned} \tag{1}$$

DEFINITION 1.4. Suppose $f \in H^1(T^n)$ and $s$ is a positive integer or

0. Then the *s homogeneous component of* $f = {}_nP_s(f)$, where ${}_nP_s(f)$ is defined by its Fourier transform

$$\widehat{{}_nP_s(f)}(x) = \begin{cases} \hat{f}(x) & \text{if } \sum x_i = s \\ 0 & \text{otherwise} \end{cases} .$$

That is if $f$ has Fourier series

$$f(g) \sim \sum_{x \in A_n} a_x(g, x) ,$$

then ${}_nP_s(f)$ has the following Fourier series:

$$ {}_nP_s(f)(g) \sim \sum_{\substack{x \in A_n \\ \Sigma x_i = s}} a_x(g, x) .$$

Then ${}_nP_s(f)$ is a trigonometric polynomial since $\widehat{{}_nP_s(f)}$ has finite support.

DEFINITION 1.5. Suppose $f \in H^1(T^\omega)$ and $f = {}_nP_s(f)$ for some $s$. Then we say *$f$ is homogeneous of degree $s$*. The previous definition is motivated by the following fact: If $\lambda$ is a complex number of unit modulus and we write $\lambda$ to mean the point $(\lambda, \lambda, \lambda, \cdots, \lambda)$ of $T^n$, then

$$f(\lambda g) = \lambda^s f(g) \quad \text{for all} \quad g \in T^n$$

if $f$ is homogeneous of degree $s$. Clearly if $f$ is homogeneous of degree $s$ its Fourier transform has finite support, so $f$ is a trigonometric polynomial and hence $f \in H^p(T^\omega)$ for $1 \leqq p \leqq \infty$. It is easy to show that ${}_nP_s$ is a bounded linear operator from $H^1(T^n)$ into $H^p(T^n)$ for each $p$. However it is not obvious that we can define an operator $P_s$ on $H^1(T^\omega)$ which is analogous to ${}_nP_s$ on $H^1(T^n)$ because the sum that should define $P_s(f)$ for $f \in H^1(T^\omega)$ is not necessarily finite. The following lemma helps show that $P_s$ can be defined as a bounded linear operator from $H^1(T^\omega)$ into $H^p(T^\omega)$.

LEMMA 1.6. *Suppose $s$ is a positive integer or* 0, *and* $1 \leqq p \leqq \infty$. *Then there exists a projection $P_s$ on $H^p(T^\omega)$ with $\|P_s\| = 1$ satisfying*:

$$\widehat{P_s f(x)} = \begin{cases} \hat{f}(x) & \text{if } \Sigma x_i = s \\ 0 & \text{otherwise} \end{cases} , \quad f \in H^p(T^\omega) .$$

*That is if $f$ has Fourier series*

$$f(g) \sim \sum_{x \in A} a_x(g, x) ,$$

*then $P_s(f)$ has the following Fourier series*:

$$P_s(f)(g) \sim \sum_{\substack{x \in A \\ \Sigma x_i = s}} a_x(g, x) .$$

*Proof.* Consider the following subgroup $H$ of $\sum_{i=1}^{\infty} Z$:

$$H = \left\{x\colon x \in \sum_{i=1}^{\infty} Z \quad \text{and} \quad \Sigma x_i = 0\right\} .$$

But $(\sum_{i=1}^{\infty} Z)/H$ is a quotient group of $\sum_{i=1}^{\infty} Z$ and hence its dual which we shall call $D$, is a compact subgroup of $T^{\omega}$. Let $m_D$ be normalized Haar measure on $D$. Since $D \subset T^{\omega}$, we can calculate the Fourier coefficients of $m_D$ with respect to $\sum_{i=1}^{\infty} Z$. It is easy to calculate that

$$\hat{m}_D(x) = \chi_H(x) \quad \text{for all} \quad x \in \sum_{i=1}^{\infty} Z ,$$

where $\chi_H(x)$ is the characteristic function of the set $H$. If $s$ is a positive integer or 0, choose a $y_s \in \sum_{i=1}^{\infty} Z$ so that $\sum_{i=1}^{\infty} (y_s)_i = s$; then for the measure $y_s(g)dm_D(g)$

$$\widehat{y_s m_D}(x) = \hat{m}_D(x - y_s) = \begin{cases} 1 & \text{if} \quad \Sigma(x - y_s) = 0 \\ & \text{i.e. } \Sigma(x)_i = s \\ 0 & \text{otherwise} \end{cases} .$$

Evidently for all $s$

$$\int_G |y_s(g)\, dm_D(g)| = 1 ,$$

so if $f \in H^p(T^{\omega})$ we can consider $f*(y_s dm_D)$ where $*$ denotes the usual convolution of a measure on $T^{\omega}$ with a function which is in $H^p(T^{\omega})$, hence in $L^1(T^{\omega})$. We have the following inequalities:

$$\text{(2)} \qquad \|f*(y_s dm_D)\|_p \leqq \|f\|_p \int_G |y_s(g) dm_D(g)| = \|f\|_p .$$

If we calculate the Fourier transform of $f*(y_s dm_D)$

$$\widehat{f*(y_s dm_D)}(x) = \hat{f}(x)\widehat{(y_s dm_D)}(x) = \widehat{P_s(f)}(x) .$$

Since $f*(y_s dm_D)$ and $P_s(f)$ have the same Fourier transform they are the same element of $H^p(T^{\omega})$, and so from equation (2)

$$\|P_s(f)\|_p = \|f*(y_s dm_D)\|_p \leqq \|f\|_p$$

and this completes the proof.

DEFINITION 1.7. If $f \in H^p(T^{\omega})$, then the *s homogeneous component of f is* $P_s(f)$.

If $f = P_s(f)$ for some $s$, we say *f is homogeneous of degree s*. This definition is justified by the fact that if $f$ is a homogeneous trigonometric polynomial of degree $s$ on $T^{\omega}$, then we have

$$\text{(3)} \qquad f(\lambda g) = \lambda^s f(g) \quad \text{for all} \quad g \in T^{\omega}$$

whenever $\lambda$ is a complex number of unit modulus and on the left we write $\lambda$ to mean $(\lambda, \lambda, \cdots)$.

Suppose that $f$ is a homogeneous function and that $f \in H^1(T^j)$, where $j$ is a positive integer or $\omega$. If $j$ is finite, then $f$ is necessarily a trigonometric polynomial and the following lemma and theorem are obvious. However if $j = \omega$, $f$ isn't necessarily a trigonometric polynomial, and the following lemma and theorem require proof.

LEMMA 1.8. *Suppose $f \in H^1(T^\omega)$ and that $f$ is homogeneous of degree $s$. Then equation* (3) *is satisfied for almost all $g \in T^\omega$ and almost all $\lambda$.*

*Proof.* If $f$ is a trigonometric polynomial there is nothing to prove. Otherwise by using an approximate identity we can find a sequence $\{f_n\}_{n=1}^\infty$ of homogeneous polynomials all of degree $s$ such that

$$\lim_{n\to\infty} f_n = f$$

in the norm of $H^1(T^\omega)$. There exists a subsequence of $\{f_n\}_{n=1}^\infty$ say $\{f_{n_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} f_{n_j}(g) = f(g) \text{ a.e.}$$

where a.e. means for almost all $g \in T^\omega$ with respect to Haar measure on $T^\omega$. $T^\omega \times T$ is the product of the measure spaces $T^\omega$ and $T$, and so $T^\omega \times T$ is a measure space with the product measure.

Let

$$W = \{(g, \lambda) \in T^\omega \times T \text{ such that } f(\lambda g) = \lambda^s f(g)\} .$$

Then $W$ is measurable and we wish to show that the measure of $W$ is 1. Now consider any fixed $\lambda \in T$; we have

$$\lim_{j\to\infty} f_{n_j}(g) = f(g)$$
$$\lim_{j\to\infty} f_{n_j}(\lambda g) = f(\lambda g)$$

except for a null set of $g$. But for each $j$

$$f_{n_j}(\lambda g) = \lambda^s f_{n_j}(g) ,$$
$$f(\lambda g) = \lim_{j\to\infty} f_{n_j}(\lambda g) = \lim_{j\to\infty} \lambda^s f_{n_j}(g) = \lambda^s f(g)$$

except for a null set of $g$. So $m(W) = 1$, which finishes the proof.

The next theorem is an application of a theorem about $\Lambda(p)$ sets. We digress for a moment to define $\Lambda(p)$ set.

DEFINITION 1.9. Let $G$ be a compact abelian group with dual group $\Gamma$. If $p > 1$ and $E \subset \Gamma$ we say $E$ is a $\Lambda(p)$ set if $L^1_E(G) = L^p_E(G)$.

DEFINITION 1.10. If $A$ is a subset of $\Gamma$ and $n$ is a positive integer we define $A^n = \{x \in \Gamma; x = a_1 + a_2 + \cdots + a_n$, where $a_i \in A$, $1 \leqq i \leqq n\}$.

THEOREM 1.11. *Suppose $G$ is a compact abelian group with torsion-free dual group $\Gamma$. If $E$ is an independent set in $\Gamma$, then $E^s$ is a $\Lambda(p)$ set for all $p < \infty$ and all positive integers $s$.*

*Proof.* See [3, p. 28, Theorem 4].

THEOREM 1.12. *Suppose $f \in H^1(T^\omega)$ and that $f$ is a homogeneous function of degree $s$ where $s$ is a positive integer or 0. Then $f \in H^p(T^\omega)$ for $1 \leqq p < \infty$.*

*Proof.* Let $E = \{z_i\}_{i=1}^\infty$. Then $E$ is independent as a set in $\sum_{i=1}^\infty Z$ and so $E^s$ is a $\Lambda(p)$ set for all $p < \infty$, by Theorem 1.11. But since $f \in H^1(T^\omega)$ and $f$ is homogeneous of degree $s, f$ is an $E^s$-function. By applying Theorem 1.11 we obtain that $f \in H^p(T^\omega)$ for all $p < \infty$, and this completes the proof.

COROLLARY 1.13. *Suppose $f \in H^1(T^\omega)$ and that $f$ is a finite sum of homogeneous functions; then $f \in H^p(T^\omega)$ for $1 \leqq p < \infty$.*

*Proof.* By assumption $f$ is a finite sum of homogeneous functions so we may write

$$f = \sum_{s=0}^{k} P_s(f) .$$

Since $f \in H^1(T^\omega)$ each $P_s(f) \in H^1(T^\omega)$ for $0 \leqq s \leqq k$. By Theorem 1.12 each $p_s(f) \in H^p(T^\omega)$ for $1 \leqq p < \infty$, so $f$ is a finite sum of functions in $H^p(T^\omega)$ hence $f \in H^p(T^\omega)$.

Theorem 1.12 is really a theorem about $H^1(T^\omega)$ rather than $L^1(T^\omega)$. In that context Theorem 1.12 is false. In fact Theorem 1.12 is false even for $L^1(T^2)$ and hence for $L^1(T^\omega)$.

If $j$ is a positive integer or $\infty$, we define homogeneity for arbitrary functions in $L^1(T^j)$ as follows: If $f \in L^1(T^j)$, we say $f$ is homogeneous of degree $s$ if

$$\hat{f}(x) = 0 \text{ if } x \in \sum_{i=1}^{j} Z \text{ and } \Sigma x_i \neq s .$$

To show that Theorem 1.12 can't be extended to $L^1(T^2)$, we shall construct for every $p > 1$ and for every positive integer $N$, a homogeneous polynomial $f$ of degree 0 on $T^2$ such that

$$||f||_1 = 1$$
$$||f||_p \geqq N .$$

For given $p > 1$, find a trigonometric polynomial $b$ defined on $T$ such that

$$||b||_1 = 1$$
$$||b||_p \geqq N$$

where $b(z_1)$ has Fourier series

$$b(z_1) = \sum_{k=0}^{t} a_k z_1^k .$$

Define the polynomial $f$ by

$$f(z_1, z_2) = \sum_{k=0}^{t} a_k z_1^k z_2^{-k} .$$

We wish to compute the norm of $f$ in $L^1(T^2)$ and in $L^p(T^2)$:

$$\begin{aligned} ||f||_1 &= \int_{T^2} |f(z_1, z_2)| dm_1(z_1) dm_2(z_2) \\ &= \int_{T^2} \left| \sum_{k=0}^{t} a_k (z_1 z_2^{-1})^k \right| dm_1(z_1) dm_2(z_2) \\ &= \int_{T^2} \left| \sum_{k=0}^{t} a_k (z_1)^k \right| dm_1(z_1) dm_2(z_2) = \int_T ||b||_1 dm_2(z_2) = \int_T 1\, dm_2(z_2) = 1 . \end{aligned}$$

The crucial equality in equation (4) is justified by the translation invariance of $dm_1(z_1)$. By a similar computation we have

$$||f||_p = ||b||_p \geqq N$$

and this provides the desired counterexample.

**2. A convergence theorem for $H^p(T^\omega)$.** By the M. Riesz theorem on conjugate functions [8], if $1 < p < \infty$ and $f \in H^p(T)$, then

$$f = \lim_{n \to \infty} \sum_{s=0}^{n} a_s z_1^s , \qquad a_s = \hat{f}(s)$$

in the norm of $H^p(T)$. In our terminology this can be written

$$f = \lim_{n \to \infty} \sum_{s=0}^{n} {}_1P_s(f) .$$

The next theorem gives an analogous result for $H^p(T^\omega)$. The proof uses a theorem about ordered groups so we digress for a moment to define the relevant terms.

Suppose $\Gamma$ is a discrete abelian group and $P$ is a subset of $\Gamma$ with the following properties:

1. If $\gamma_1 \in P$ and $\gamma_2 \in P$ then $\gamma_1 + \gamma_2 \in P$.

If $-P$ denotes the set whose elements are the inverses of the elements of $P$ then we have

2. $P \cap (-P) = \{0\}$
3. $P \cup (-P) = \Gamma$.

Under these conditions $P$ induces an order in $\Gamma$ as follows: For $\gamma_1$ and $\gamma_2$ elements of $\Gamma$, say $\gamma_1 \geqq \gamma_2$ if $\gamma_1 - \gamma_2 \in P$. It is easy to check that this is a linear order. A given group may have many different orders corresponding to different choices of $P$ with the three properties above.

DEFINITION 2.1. Suppose $G$ is a compact abelian group whose dual group $\Gamma$ is ordered. Let $f$ be a trigonometric polynomial on $G$ with Fourier series

$$f(g) \sim \sum_{\gamma \in \Gamma} a_\gamma(g, \gamma) .$$

Define $\Phi(f)$ by

$$\Phi(f)(g) \sim \sum_{\substack{\gamma \in \Gamma \\ \gamma \geqq 0}} a_\gamma(g, \gamma) .$$

We shall need the following generalization of the M. Riesz theorem on conjugate functions. It is due to Bochner [1].

THEOREM 2.2. *Suppose* $1 < p < \infty$. *Then there exists a constant* $A_p$, *independent of* $G$ *or the particular order in* $\Gamma$ *such that if* $f$ *is a trigonometric polynomial on* $G$, *then*

$$\|\Phi(f)\|_p \leqq A_p \|f\|_p .$$

THEOREM 2.3. *Let* $1 < p < \infty$. *Then if* $f \in H^p(T^\omega)$

$$\lim_{n\to\infty} \sum_{s=0}^{n} P_s(f) = f$$

*in the norm of* $H^p(T^\omega)$.

*Proof*. Fix $p$. Define $Y_n$ by

$$Y_n(f) = \sum_{s=0}^{n} P_s(f) \text{ if } f \in H^p(T^\omega) .$$

Clearly trigonometric polynomials are dense in $H^p(T^\omega)$ and

$$\lim_{n\to\infty} Y_n(f) = f$$

whenever $f$ is a trigonometric polynomial. It remains to show that the family $\{Y_n\}_{n=1}^{\infty}$ is uniformly bounded on trigonometric polynomials, i.e.

$$\| Y_n(f) \|_p \leqq K \| f \|_p$$

$f$ a trigonometric polynomial where $K$ is a positive constant independent of $n$ and $f$. Then by a standard argument in functional analysis, the proof is complete. I shall show that the norm of $Y_n$ is majorized by $A_p$, where $A_p$ is the constant of Theorem 2.2.

Our first task is to induce an order in $\sum_{i=1}^{\infty} Z$ so that we can apply Theorem 2.2. First choose a family $\{d_i\}_{i=1}^{\infty}$ of real numbers which satisfies the following properties:

1. $d_1 = -1$, $-1 < d_i < -n/(n+1)$ for $i \neq 1$.
2. The set $\{d_i\}$ is independent in the group sense as a subset of the reals.

We define a homomorphism from $\sum_{i=1}^{\infty} Z$ into the reals by

$$\pi\colon \sum_{i=1}^{\infty} \longrightarrow R$$

$$x \longmapsto \sum_{i=1}^{\infty} d_i x_i \ .$$

$\pi$ is clearly a homomorphism; since the $d_i$ are linearly independent, it has a trivial kernel, i.e. if $\pi(x) = 0$ then $x = 0$. Define

$$P = \left\{x\colon x \in \sum_{i=1}^{\infty} Z \text{ and } \pi(x) \geqq 0\right\} .$$

Then $P$ satisfies the necessary properties to induce an order in $\sum_{i=1}^{\infty} Z$. If $f(g)$ is an arbitrary trigonometric polynomial on $T^{\omega}$ define a trigonometric polynomial $f_1(g)$ as follows:

$$f_1(g) = z_1^{-n}(g) f(g) \ .$$

Let $f(g) = \Sigma a_x(g, x)$ . Then

$$f_1(g) = z_1^{-n}(g) f(g) = \Sigma a_x(g, -nz_1)(g, x) = \Sigma a_x(g, x - nz_1)$$

and

$$\phi(f_1) = \sum_{\pi(x - nz_1) \geqq 0} a_x(g, x - nz_1) \ .$$

If $\pi(x - nz_1) \geqq 0$, then

$$0 \leqq \pi(x - nz_1) = \pi(x) + \pi(-nz_1) = \pi(x) - n\pi(z_1) = \pi(x) + n$$

and $\pi(x) \geqq -n$. But $\pi(x) = \Sigma d_i x_i$, and by using property 1 of $\{d_i\}$ it is clear that $\pi(x) \geqq -n$ if and only if $\Sigma x_i \leqq n$. So $\phi(f_1) = \Sigma a_x(g, x - nz_1)$.

Then it is easy to compute that $\Sigma x_i \leqq n$

$$z_1^n \Phi(f_1) = \sum_{i=1}^{n} P_i(f) = Y_n(f) .$$

By Theorem 2.2 we have that

$$\|\Phi(f_1)\|_p \leqq A_p \|f_1\|_p .$$

So we have

$$\begin{aligned} \| Y_n(f) \|_p &= \| z_1^n \Phi(f_1) \|_p = \| \Phi f_1 \|_p \leqq A_p \| f_1 \|_p \\ &= A_p \| z_1^{-n} f \|_p = A_p \| f \|_p , \end{aligned}$$

so the norm of $Y_n$ is less than or equal to $A_p$ and the proof is complete.

**3. The complementation problem.** The next theorem shows that $H^p(T^\omega)$ is uncomplemented as a subspace of $L^p(T^\omega)$ if $p \neq 2$. This is in contrast to $H^p(T^n)$ which is complemented in $L^p(T^n)$ except when $p = 1$ or $p = \infty$. Although other examples of uncomplemented subspaces of an $L^p$ space are known, $H^p(T^\omega)$ has the advantage of being defined in a concrete way.

DEFINITION 3.1. Let $G$ be a compact abelian group. If $f \in L^1(G)$ let $f_{g_0}$ denote the *$g_0$-translate of $f$* where

$$f_{g_0}(g) = f(g_0 + g) .$$

LEMMA 3.2. *Let $G$ be a compact abelian group with dual group $\Gamma$. Suppose $1 \leqq p < \infty$ and that $T$ is a bounded projection from $L^p(G)$ onto $L^p_E(G)$. Then a linear operator $Q$ can be defined by*

$$Q(f) = \int_G [T(f_g)]_{-g} dm(g) \qquad f \in L^p(G) ,$$

*where the integral is the Bochner integral.*

$Q$ is the natural projection from $L^p(G)$ onto $L^p_E(G)$, i.e., if $f \in L^p(G)$ then $Q(f)$ is defined by its Fourier transform as follows:

$$\widehat{G(f)}(x) = \begin{Bmatrix} \hat{f}(x) & x \in E \\ 0 & \text{otherwise} \end{Bmatrix} .$$

*Proof.* The proof for the case $G = T$, $\Gamma = Z$, $E = Z^+$, $p = 1$ is given [4, page 154]. The proof in the general case is analogous.

THEOREM 3.3. *Suppose $p \neq 2$, then $H^p(T^\omega)$ is uncomplemented as subspace of $L^p(T^\omega)$.*

*Proof.* If $p = 1$ or $p = \infty$, there is really nothing to prove. There is a theorem in [4, pp. 154–155] which proves that $H^1(T)$ is uncomplemented in $L^1(T)$, and that $H^\infty(T)$ is uncomplemented in $L^\infty(T)$. Then since $H^i(T)$ and $L^i(T)$ can be isometrically embedded into $H^i(T^\omega)$ and $L^i(T^\omega)$ respectively for $i = 1, \infty$, the theorem is proved for $p = 1$ or $p = \infty$. In any case the argument which follows is valid for $p = 1$, and with slight modifications for $p = \infty$.

Let $S$ be the natural projection from $L^p(T^\omega)$ into $H^p(T^\omega)$ which is defined on trigonometric polynomials by

$$\begin{aligned} S\colon L^p(T^\omega) &\longrightarrow H^p(T^\omega) \\ f \quad &\longmapsto S(f) \end{aligned}$$

where

$$\widehat{S(f)}(x) = \begin{Bmatrix} \hat{f}(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{Bmatrix}.$$

We wish to show that $S$ can't be extended to a bounded operator defined on all of $L^p(T^\omega)$. To do this it is sufficient to find trigonometric polynomials $f_n$ on $T^\omega$ such that

$$\|f_n\|_p = 1 \tag{5}$$

$$\|S(f_n)\|_p = (1 + \varepsilon)^n \quad \text{where} \quad \varepsilon > 0 . \tag{6}$$

By [8, p. 295, Ex. 2] we can find a trigonometric polynomial $h$ defined on $T$ so that

$$h(z_1) = \sum_{k=-n}^{n} a_k z_1^k \qquad \|h\|_p = 1$$

and if

$$h_+(z_1) = \sum_{k=0}^{n} a_k z_1^k$$

then we have

$$\|h_+\|_p = 1 + \varepsilon$$

where $\varepsilon$ is some positive number which depends upon $p$. Consider the trigonometric polynomial $r$ defined on $T^2$ by

$$r(z_1, z_2) = h(z_1)h(z_2) = \left(\sum_{k=-n}^{n} a_k z_1^k\right)\left(\sum_{k=-n}^{n} a_k z_2^k\right).$$

Define $r_+$ by

$$r_+(z_1, z_2) = h_+(z_1)h_+(z_2) = \left(\sum_{k=0}^{n} a_k z_1^k\right)\left(\sum_{k=0}^{n} a_k z_2^k\right).$$

Then it is easy to compute that

$$\|r\|_p = \|h\|_p^2 = 1$$
$$\|r_+\|_p = (\|h_+\|_p)^2 = (1+\varepsilon)^2 .$$

We define trigonometric polynomials on $T^\omega$ by

$$f_1 = I_1(h) \qquad f_2 = I_2(r)$$

where $I_1$ and $I_2$ were defined in equation (1). It is easy to check that

$$S(f_1) = I_1(h_+) \qquad S(f_2) = I_2(r_+)$$

and since $I_1$ and $I_2$ are isometries we have

$$\|f_1\|_p = \|I_1(h)\|_p = \|h\|_p = 1$$
$$\|S(f_1)\|_p = \|I_1(h_+)\|_p = \|h_+\|_p = 1 + \varepsilon$$
$$\|f_2\|_p = \|I_2(r)\|_p = \|r\|_p = 1$$
$$\|S(f_2)\|_p = \|I_2(r_+)\|_p = \|r_+\|_p = (1+\varepsilon)^2 .$$

By a similar argument we can construct trigonometric polynomials $f_3, f_4, \cdots$ and hence $f_n$ for any $n$ and $f_n$ will satisfy equations (5) and (6). This shows that the natural projection from $L^p(T^\omega)$ into $H^p(T^\omega)$ isn't bounded. To finish the proof we must show there is no bounded projection of any kind from $L^p(T^\omega)$ into $H^p(T^\omega)$ which is the identity when restricted to $H^p(T^\omega)$.

Suppose there exists $\tilde{S}$ a linear transformation from $L^p(T^\omega)$ into $H^p(T^\omega)$ which is the identity when restricted to $H^p(T^\omega)$. Define a linear operator $Q$ by

$$Q(f) = \int_{T^\omega} [\tilde{S}(f_g)]_{-g} dm(g)$$

where the integral is the Bochner integral. Then $Q$ is a bounded linear operator from $L^p(T^\omega)$ into $H^p(T^\omega)$ and by Lemma 3.2 we have that $Q = S$, where $S$ is the natural projection from $L^p(T^\omega)$ into $H^p(T^\omega)$. But we know that $S$ isn't a bounded projection and this provides the contradiction which finishes the proof.

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WAYNE STATE UNIVERSITY

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# ON THE COMPLETION OF LOCALLY SOLID VECTOR LATTICES

D. H. FREMLIN

**Let $E$ be a Riesz space ($=$ vector lattice), with a locally solid Hausdorff linear space topology. Then its completion also has a Riesz space structure. In this paper it is shown how a pair of important properties which may be possessed by $E$ are inherited by its completion.**

In general this article will rest on the foundations of [4] and [5]. A linear space topology on a Riesz space $E$ is *locally solid* if 0 has a neighbourhood basis consisting of solid sets. In this case, the lattice operations are uniformly continuous; consequently (assuming that the topology is Hausdorff) they can be extended to the linear topological space completion $\hat{E}$ of $E$, and $\hat{E}$ will also be a locally solid topological Riesz space ([5, p. 235; 4, p. 108]). $E$ is now a *Riesz subspace* of $\hat{E}$, i.e. a linear subspace which is also a sublattice.

My object is to show how two important and common properties are preserved by the process of completion. Unfortunately, although these properties have been studied by various authors (see e.g. [3]), no satisfactory terminology has been devised. I hope that my use of the words "Fatou" (§1) and "Lebesgue" (§5), suggested by the famous convergence theorems, will prove acceptable.

## 1. Fatou topologies.

Let $E$ be a Riesz space and $\mathfrak{T}$ a topology on $E$. I will call $\mathfrak{T}$ *Fatou* if (i) it is a linear space topology (ii) 0 has a base consisting of sets $U$ which are solid and such that if $\emptyset \subset A \subseteqq U$ and $A \uparrow x$ in $E$ (i.e. if $A$ is nonempty, directed upwards, and has $x$ for its least upper bound), then $x \in U$.

This property is exceedingly common. Consider, for example, $C(X)$ for any compact space $X$; the basic neighbourhoods of 0 are of the form $\{x: \|x\|_\infty \leqq \varepsilon\}$, and these all have the property described above. Similarly, in all the $L^p$ spaces, for $0 \leqq p \leqq \infty$, the usual topologies are Fatou.

The most striking thing about Fatou topologies is Nakano's theorem (see [2]). For its full strength this requires a further concept. Let us call a linear space topology on a Riesz space $E$ a *Levi* topology if every topologically bounded set $A \subseteqq E$ which is directed upwards has an upper bound in $E$. (For example, all the spaces adduced above have Levi topologies. Also, the weak topology associated with a locally convex Hausdorff Levi topology will always be Levi). Then: A Levi Fatou Hausdorff topology on a Dedekind complete Riesz space is complete. For a proof of this theorem, see [4], Proposition IV. 1.5. ([4] uses the phrases "locally order complete" and "boundedly complete" for Fatou and Levi topologies respectively in Dedekind complete spaces).

2. **Extensions of Riesz spaces; the spaces $C_\infty(X)$.** Let $E$ be a Riesz space. I shall call a Riesz subspace $F$ of $E$ *orderdense* if, for every $x \geqq 0$ in $E$,

$$x = \sup \{y \colon y \in F,\ 0 \leqq y \leqq x\} .$$

An important consequence of this is that if $A$ is a nonempty subset of $F$ and $x = \sup A$ in $F$, that is, if $x$ is the least member of $F$ which is an upper bound of $A$, then $x = \sup A$ in $E$. It follows that if $F$ is orderdense in $E$, and $G$ is orderdense in $F$, then $G$ is orderdense in $E$.

Let $X$ be a compact extremally disconnected Hausdorff topological space. Let $C_\infty(X)$ be the set of all those continuous functions $x$ from $X$ to the extended real line $[-\infty, \infty]$ such that $\{t \colon -\infty < x(t) < \infty\}$ is dense in $X$. Because every continuous real-valued function defined on a dense open subset of $X$ has a unique extension to a member of $C_\infty(X)$ ([6, Lemma V. 2.1]), $C_\infty(X)$ has a natural Riesz space structure under which it is Dedekind complete ([6, Theorem V. 2.2]). The point is that every Archimedean Riesz space can be embedded as an orderdense Riesz subspace of some $C_\infty(X)$ ([6, Theorems IV. 11.1 and V. 4.2]).

[6] gives several properties of the space $C_\infty(X)$, but not the one we shall need; so I set it out here.

PROPOSITION 1. *Let $X$ and $C_\infty(X)$ be as above. Let $A \subseteqq C_\infty(X)^+$ be a nonempty set such that for every $x > 0$ in $C_\infty(X)$ there is an $n \in N$ such that*

$$nx \neq \sup_{y \in A} y \wedge nx .$$

*Then $A$ is bounded above in $C_\infty(X)$.*

*Proof.* Define $w \colon X \to [0, \infty]$ by

$$w(t) = \sup_{y \in A} y(t)\ \forall\, t \in X .$$

Then $w$ is lower semi-continuous. Define $v \colon X \to [0, \infty]$ by

$$v(t) = \inf \left\{ \sup_{u \in U} w(u) : U \text{ a } nhd \text{ of } t \right\}$$

for every $t \in X$. Then $v$ is continuous ([6, Theorem V. 1.1]). My aim is to prove that $v \in C_\infty(X)$, i.e. that $v$ is finite on a dense set.

Suppose that $G \subseteqq X$ is open and not empty. As $X$ is compact and Hausdorff, there is a continuous function $x$ on $X$ such that $x > 0$ but $x(t) = 0 \ \forall \ t \in X \backslash G$. Now $x \in C_\infty(X)$, so there is an $n \in N$ such that

$$nx \neq \sup_{y \in A} y \wedge nx ,$$

that is, there is a $z > 0$ in $C_\infty(X)$ such that

$$y \wedge nx \leqq nx - z \ \forall \ y \in A .$$

Of course $z \leqq nx$, so $z$ is finite everywhere and $z(t) = 0 \ \forall \ t \in X \backslash G$. Let $H = \{t: z(t) > 0\}$; then $H$ is not empty and $H \subseteqq G$.

But if $t \in H$, $y(t) \leqq nx(t) - z(t) \ \forall \ y \in A$, so $w(t) \leqq nx(t) - z(t)$; and as $nx - z$ is continuous, $v(t) \leqq nx(t) - z(t) < \infty \ \forall \ t \in H$.

Consequently, $\{t: v(t) < \infty\}$ meets $G$. As $G$ is arbitrary, $v \in C_\infty(X)$ and is the required upper bound for $A$.

3. THEOREM 1. *Let $E$ be an Archimedean Riesz space with a Hausdorff Fatou topology. Let $\hat{E}$ be its linear topological space completion with its natural Riesz space structure. Then* (i) *$E$ is an orderdense Riesz subspace of $\hat{E}$* (ii) *the topology on $\hat{E}$ is Fatou.*

*Proof.* My method is to find a complete Riesz space extending $E$ which has the required properties.

(a) Let $X$ be a compact extremally disconnected Hausdorff topological space such that $E$ can be embedded as an orderdense Riesz subspace of $C_\infty(X)$ (§2 above). Let $\mathscr{B}$ be the set of all neighbourhoods $U$ of 0 in $E$ satisfying the Fatou property in §1, i.e. such that $U$ is solid and if $\emptyset \subset A \subseteqq U$ and $A \uparrow x$ in $E$ then $x \in U$. Then $\mathscr{B}$ is a base of neighbourhoods of 0. For each $U \in \mathscr{B}$, set

$$\tilde{U} = \{w: w \in C_\infty(X), \forall \ x \in E, |x| \leqq |w| \Rightarrow x \in U\} .$$

Then $\tilde{U}$ is a solid subset of $C_\infty(X)$. Note that $\tilde{U} \cap E = U$.

(b) Suppose that $U$ and $V$ belong to $\mathscr{B}$ and that $U + U \subseteqq V$. Then $\tilde{U} + \tilde{U} \subseteqq \tilde{V}$. For suppose that $w_1, w_2 \in \tilde{U}$ and that $x \in E$ is such that $|x| \leqq |w_1 + w_2|$. Set $v_1 = |w_1| \wedge |x|$ and $v_2 = |x| - v_1 \leqq |w_2|$. Then $A_i = \{y: y \in E, 0 \leqq y \leqq v_i\} \uparrow v_i$ for $i = 1, 2$, so $A_1 + A_2 \uparrow v_1 + v_2 = |x|$ in $E$. But $A_1 + A_2 \subseteqq U + U \subseteqq V$, so $|x| \in V$ and $x \in V$. As $x$ is arbitrary, $w_1 + w_2 \in \tilde{V}$; as $w_1$ and $w_2$ are arbitrary, $\tilde{U} + \tilde{U} \subseteqq \tilde{V}$.

(c) It follows that if we set

$$H = \bigcap_{U \in \mathscr{B}} \bigcup_{\alpha \in \mathscr{R}} \alpha \tilde{U} ,$$

then $H$ is a solid linear subspace of $C_\infty(X)$, including $E$, and $\{\tilde{U} \cap H: U \in \mathscr{B}\}$ is a neighbourhood basis at 0 for a linear space topology $\mathfrak{T}$ on $H$. As every $\tilde{U} \cap H$ is solid, $\mathfrak{T}$ is locally solid; as $\tilde{U} \cap E = U$ for every $U \in \mathscr{B}$, $\mathfrak{T}$ induces the original topology on $E$. Also, $\mathfrak{T}$ is Hausdorff, for if $w \in H$ and $w \neq 0$, there is an $x \in E$ such that $0 < x \leqq |w|$; now if $U \in \mathscr{B}$ is such that $x \notin U$, $w \notin \tilde{U}$.

(d) If $U \in \mathscr{B}$, $\varnothing \subset A \subseteqq \tilde{U}$, and $A \uparrow w$ in $C_\infty(X)$, then $w \in \tilde{U}$. For suppose that $x \in E$ and that $|x| \leqq |w|$. Then

$$\{y^+ + w^-: y \in A\} \uparrow w^+ + w^- = |w| \geqq |x| ,$$

so

$$\{|x| \wedge (y^+ + w^-): y \in A\} \uparrow |x| .$$

Now set

$$B = \{z: z \in E, \exists\, y \in A, 0 \leqq z \leqq |x| \wedge (y^+ + w^-)\} .$$

Then $B \uparrow$, and as $E$ is orderdense in $C_\infty(X)$, $B \uparrow |x|$. But if $z \in B$ there is a $y \in A$ such that

$$z \leqq y^+ + w^- \leqq y^+ + y^- = |y| ,$$

so, as $y \in \tilde{U}$, $z \in U$. Because $U \in \mathscr{B}$, $x \in U$. As $x$ is arbitrary, $w \in \tilde{U}$.

(e) Consequently the sets $\tilde{U} \cap H$ all satisfy the Fatou condition, and $\mathfrak{T}$ is Fatou. (Here we have used the fact that $H$ is orderdense in $C_\infty(X)$, so that if $A \uparrow w$ in $H$, then $A \uparrow w$ in $C_\infty(X)$).

(f) It also follows that $\mathfrak{T}$ is Levi. For suppose that $A \subseteqq H$ is directed upwards, is not empty, and is bounded. Then of course $B = \{y^+: y \in A\}$ is directed upwards, and it is bounded because $\mathfrak{T}$ is locally solid. Now suppose that $x > 0$ in $C_\infty(X)$. Let $U \in \mathscr{B}$ be such that $x \notin \tilde{U}$. Let $n > 0$ be such that $A \subseteqq nU$. Now

$$\{n^{-1}y \wedge x: y \in B\}$$

is a subset of $\tilde{U}$, directed upwards; so its supremum belongs to $\tilde{U}$ and cannot be $x$. Thus $\sup_{y \in B} y \wedge nx$ is not $nx$, and $B$ satisfies the condition of Proposition 1; so $B$, and therefore $A$, is bounded above in $C_\infty(X)$. Let $z_0 = \sup A$ in $C_\infty(X)$; this exists as $C_\infty(X)$ is Dedekind complete. If $V \in \mathscr{B}$, there is an $m > 0$ such that $m^{-1}A \subseteqq \tilde{V}$, so by (d) again $m^{-1}z_0 \in \tilde{V}$ i.e. $z_0 \in m\tilde{V}$. As $V$ is arbitrary, $z_0 \in H$, and is the required upper bound for $A$ in $H$.

(g) Thus $\mathfrak{T}$ satisfies the conditions of Nakano's theorem, and $H$ is complete. So $\hat{E}$ may be regarded as the closure of $E$ in $H$. Because $E$ is orderdense in $H$, it is orderdense in $\hat{E}$. Finally, it is easy to see that the topology on $\hat{E}$ induced by $\mathfrak{T}$ is Fatou, because $\mathfrak{T}$ itself is Fatou and $\hat{E}$ is orderdense in $H$.

REMARK. Of course the condition "Archimedean" in the hypotheses of the theorem is redundant, because any Riesz space with a Hausdorff locally solid linear space topology must be Archimedean. The same applies to Theorem 2 below.

4. **Counter-example.** Suppose that $E = C([0, 1])$, the space of real-valued continuous functions on the unit interval. Give $E$ the topology induced by $\|\ \|_1$ where

$$\|x\|_1 = \int |x| d\mu_L \ \forall\ x \in E ,$$

$\mu_L$ being Lebesgue measure. Then $\|\ \|_1$ is a Riesz norm so the topology is locally solid. But it is not Fatou and $E$ is not orderdense in its completion $L^1(\mu_L)$.

5. **Lebesgue topologies.** I should now like to proceed to a stronger condition, also fulfilled by many examples. Because it is of great interest in many contexts, I give as general a definition as I can. Let $E$ be any partially ordered set. A topology $\mathfrak{T}$ on $E$ is *Lebesgue* if, whenever $A$ is a non-empty subset of $E$ and either $A \uparrow x$ or $A \downarrow x$ in $E$, then $x$ belongs to the closure $\bar{A}$ of $A$. We shall be interested, of course, in linear space topologies on Riesz spaces; in this case, $\mathfrak{T}$ is Lebesgue iff $0 \in \bar{A}$ whenever $\varnothing \subset A \downarrow 0$.

Now the ordinary topologies on the $L^p$ spaces, for $0 \leqq p < \infty$, are Lebesgue; so is the norm topology on $c_0(N)$. We note that the exceptions are the $L^\infty$ and $C(X)$ spaces. However, the weak topology $\mathfrak{T}_s(L^\infty, L^1)$ is Lebesgue; in fact it is the case that the Mackey topology $\mathfrak{T}_k(L^\infty, L^1)$ is Lebesgue. Of course, if $\mathfrak{T}$ is Lebesgue and $\mathfrak{S}$ is weaker than $\mathfrak{T}$, then $\mathfrak{S}$ is Lebesgue.

Lebesgue topologies have many remarkable properties. I give one of the simplest.

LEMMA 1. *A Lebesgue locally solid linear space topology on a Riesz space is Fatou.*

*Proof.* Let $U$ be any neighbourhood of 0; let $V$ be a closed neighbourhood of 0 included in $U$; let $W$ be a solid neighbourhood of 0 included in $V$. The point is that $\bar{W}$ is solid ([4, Proposition IV.

4.8]). But now $\bar{W} \subseteqq U$ and $\bar{W}$ satisfies the Fatou condition because the topology is Lebesgue.

6. THEOREM 2. *Let $E$ be an Archimedean Riesz space with a Lebesgue locally solid Hausdorff linear space topology. Then the completion $\hat{E}$ of $E$ also has a Lebesgue topology.*

*Proof.* We know by Lemma 1 and Theorem 1 that $E$ is orderdense in $\hat{E}$. Suppose, if possible, that $A \downarrow 0$ in $\hat{E}$, $A$ is not empty, but that $0 \notin \bar{A}$. Let $U$ be a solid neighbourhood of 0 in $\hat{E}$ such that $A$ does not meet $U$. Let $V$ be a solid neighbourhood of 0 in $\hat{E}$ such that $V + V + V \subseteqq U$. Fix $x_0 \in A$ and find a $y_0 \in E$ such that $x_0 - y_0 \in V$; without loss of generality, I may suppose that $y_0 \geqq 0$. Now

$$\{y_0 \wedge (x_0 - x)^+; x \in A\} \uparrow y_0 \wedge x_0 ,$$

so if

$$B = \{z: z \in E, \exists\, x \in A, 0 \leqq z \leqq y_0 \wedge (x_0 - x)^+\} ,$$

$B \uparrow x_0 \wedge y_0$ in $\hat{E}$. Similarly,

$$C = \{w: w \in E, 0 \leqq w \leqq (y_0 - x_0)^+\} \uparrow (y_0 - x_0)^+ ,$$

and so $B + C \uparrow y_0$ in $E$. As the topology on $E$ is Lebesgue, there exist $z \in B$ and $w \in C$ such that

$$y_0 - w - z \in V .$$

But as $V$ is solid, $w \in V$, so $y_0 - z \in V + V$, and

$$x_0 - z = y_0 - z + (x_0 - y_0) \in V + V + V \subseteqq U .$$

However, there is an $x \in A$ such that $0 \leqq z \leqq (x_0 - x)^+$, and there is an $x_1 \in A$ such that $x_1 \leqq x \wedge x_0 \leqq x_0 - z$. But $U$ is solid, so $x_1 \in U$; which is the contradiction we require.

7. **Conclusion.** I think that Theorem 1 is more surprising than Theorem 2. Both Fatou and Lebesgue topologies are frequently mysterious; but when we require a topology to be both locally solid and Lebesgue we are imposing such a powerful condition that we expect agreeable results to follow quickly. The Fatou property is harder to tackle. Its actual applications in Theorem 1, while certainly essential (see § 4), are buried too deep in the argument to be readily disentangled; so it's not clear just what it is about Fatou topologies that makes the theorem true.

Theorem 1 is reminiscent of the result in [1] that if $E$ is any Riesz space, then the canonical image of $E$ in $E^{\times\times}$ or $(E_n^\sim)_n^\sim$ is orderdense.

In fact this can be deduced from Theorem 1, though (as far as I know) only by an extremely involved route. But there may be some hope that the techniques of [1] could be adapted to give a simpler proof of Theorem 1.

Theorem 2 is more straightforward, and can be proved independently of Theorem 1 without much difficulty. If in Theorem 2 we know that $E$ is locally convex, there is a proof direct from the result in [1] quoted above. But the hypothesis of local convexity doesn't seem to help in Theorem 1.

Theorem 2 recalls the construction of the ordinary function spaces. If the spaces $L^1$, $L^2$ etc. are thought of as completions of the space $S$ of equivalence classes of simple functions under the appropriate norms, their properties can be deduced from the fact that each of these norms induces a Lebesgue locally solid topology on $S$.

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# ESSENTIAL CENTRAL SPECTRUM AND RANGE FOR ELEMENTS OF A VON NEUMANN ALGEBRA

HERBERT HALPERN

**A closed two-sided ideal $\mathscr{J}$ in a von Neumann algebra $\mathscr{A}$ is defined to be a central ideal if $\sum A_iP_i$ is in $\mathscr{J}$ for every set $\{P_i\}$ of orthogonal projections in the center $\mathscr{Z}$ of $\mathscr{A}$ and every bounded subset $\{A_i\}$ of $\mathscr{J}$. Central ideals are characterized in terms of the existence of continuous fields and their form is completely determined.**

**If $\mathscr{J}$ is a central ideal of $\mathscr{A}$ and $A \in \mathscr{A}$, then $A_0 \in \mathscr{Z}$ is said to be in the essential central spectrum of $A$ if $A_0 - A$ is not invertible in $\mathscr{A}$ modulo the smallest closed ideal containing $\mathscr{J}$ and $\zeta$ for every maximal ideal $\zeta$ of $\mathscr{Z}$. It is shown that the essential central spectrum is a nonvoid, strongly closed subset of $\mathscr{Z}$ and that it satisfies many of the relations of the essential spectrum of operators on Hilbert space. Let $\mathscr{A}^{\sim}$ be the space of all bounded $\mathscr{Z}$-module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$. The essential central numerical range of $A \in \mathscr{A}$ with respect to $\mathscr{J}$ is defined to be $\mathscr{K}_{\mathscr{J}}(A)=\{\phi(A) \mid \phi \in \mathscr{A}^{\sim}, \|\phi\| \leqq 1, \phi(1) = P_{\mathscr{J}}, \phi(\mathscr{J}) = (0)\}$. Here $P_{\mathscr{J}}$ is the orthogonal complement of the largest central projection in $\mathscr{J}$. The essential central numerical range is shown to be a weakly closed, bounded, $\mathscr{Z}$-convex subset of $\mathscr{Z}$. It possesses many of the properties of the essential numerical range but in a form more suited to the fact that $A$ is in $\mathscr{A}$ rather than a bounded operator. It is shown that if $\mathscr{A}$ is properly infinite and $\mathscr{J}$ is the ideal of finite elements (resp. the strong radical) of $\mathscr{A}$, then $\mathscr{K}_{\mathscr{J}}(A)$ is the intersection of $\mathscr{Z}$ with the weak (resp. uniform) closure of the convex hull of $\{UAU^{-1} \mid U$ unitary in $\mathscr{A}\}$.**

In a final section, we give some applications of these facts. We extend a result of J. G. Stampfli [19] to show that the range of a derivation on a von Neumann algebra is never uniformly dense. We also prove a theorem on self-adjoint commutators using a calculation of M. David [5].

2. Central ideals. Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. For any subset $\mathscr{B}$ of $\mathscr{A}$ let $(\mathscr{B})$ denote the set of all projections of $\mathscr{B}$. Throughout this paper *all* ideals will be assumed to be closed two-sided ideals. An ideal $\mathscr{J}$ in $\mathscr{A}$ is said to be a central ideal or a $\mathscr{Z}$-ideal if given a norm bounded set $\{A_i \mid i \in I\}$ of elements of $\mathscr{J}$ and a corresponding set $\{P_i \mid i \in I\}$ of mutually orthogonal projections in $\mathscr{Z}$, then the sum $\sum A_iP_i$, which exists in the strong topology, is also in $\mathscr{J}$. (Similar definitions were used by I. Kaplansky

[22, §1] and M. Goldman [13; §4] in the theory of $AW^*$-modules; however, here there is no canonical inner product.) Any ideal $\mathscr{J}$ in $\mathscr{A}$ is contained in a smallest central ideal $\langle\mathscr{J}\rangle$ given by $\langle\mathscr{J}\rangle = \{\sum\{A_iP_i \mid i \in I\} \mid \{A_i \mid i \in I\}$ is a bounded subset of $\mathscr{J}$ and $\{P_i \mid i \in I\}$ is a mutually orthogonal subset of $(\mathscr{Z})$ of sum 1$\}$ ([19], remarks preceding corollary to (a5) implies (a1)). If $\mathscr{J}$ a central ideal in $\mathscr{A}$ and if $A$ is an element of $\mathscr{A}$, then it is clear that there is an element $P$ in $(\mathscr{Z})$ such that $AP \in \mathscr{J}$ and $AQ \notin \mathscr{J}$ for every $Q$ in $(\mathscr{Z})$ with $0 < Q \leqq 1 - P$. The following definition is now possible.

DEFINITION 2.1. Let $\mathscr{A}$ be a von Neumann algebra and let $\mathscr{J}$ be a central ideal of $\mathscr{A}$. Then $P_{\mathscr{J}}$ will denote the orthogonal complement of the largest central projection in $\mathscr{J}$. We notice that $QP_{\mathscr{J}} \in \mathscr{J}$ for a central projection $Q$ implies $QP_{\mathscr{J}} = 0$.

We now describe central ideals with regard to finite element

PROPOSITION 2.2. *Let $\mathscr{A}$ be a semi-finite von Neumann algebra with center $\mathscr{Z}$, let $\mathscr{J}$ be a central ideal of $\mathscr{A}$, and let $\mathscr{A}P$ be the weak closure of $\mathscr{J}$, where $P \in (\mathscr{Z})$. Then $\mathscr{J}$ contains every finite projection of $\mathscr{A}$ majorized by $P$.*

*Proof.* Let $F$ be a finite projection of $\mathscr{A}$ majorized by $P$. Let $Q$ be an element of $(\mathscr{Z})$ such that $FQ \in \mathscr{J}$ and $FR \notin \mathscr{J}$ for every $R$ in $(\mathscr{Z})$ with $0 < R \leqq 1 - Q$ (preliminary remarks). We note that $Q' = 1 - Q \leqq P$. We obtain a contradiction by assuming that $Q' \neq 0$. Since the weak closure of $\mathscr{J}$ is $\mathscr{A}P$ and since linear combinations of projections are dense in $\mathscr{J}$, there is a projection $E$ in $\mathscr{J}$ with $EQ' \neq 0$. There is an $R$ in $(\mathscr{Z}Q')$ such that $ER \prec FR$ and $F(Q' - R) \prec E(Q' - R)$. Either $ER \neq 0$ or $E(Q' - R) \neq 0$. Now if $ER \neq 0$, there is nonzero $S$ in $(\mathscr{Z}R)$ and projections $E_1, \cdots, E_n$ in $\mathscr{A}$ such that $ES = E_1 \sim E_2 \sim \cdots \sim E_n$ and $FS - \sum E_i \prec E_1$. This means that $FS$ is in $\mathscr{J}$. This is contrary to the choice of $Q$, so we must assume that $E(Q' - R) \neq 0$. But this also implies that $F(Q' - R)$ is in $\mathscr{J}$. So we must conclude that $Q' = 0$. Hence, we have shown that every finite projection majorized by $P$ is in $\mathscr{J}$.

COROLLARY 2.3. *An ideal in a finite von Neumann algebra is a central ideal if and only if it is weakly closed.*

*Proof.* If the ideal $\mathscr{J}$ in the finite von Neumann algebra $\mathscr{A}$ is weakly closed, then there is a central projection $P$ in $\mathscr{A}$ such that $\mathscr{J} = \mathscr{A}P$ [9, I, 3, Theorem 2, Corollary 2]. Obviously the ideal

$\mathscr{A}P$ is a central ideal of $\mathscr{A}$.

Conversely, let $\mathscr{J}$ be a central ideal of $\mathscr{A}$. Let $P$ be the central projection of $\mathscr{A}$ such that the weak closure of $\mathscr{J}$ is $\mathscr{A}P$. Then $\mathscr{J}$ contains every finite projection majorized by $P$; in particular, it contains $P$ itself. So $\mathscr{J} = \mathscr{A}P$ and $\mathscr{J}$ is weakly closed.

We now describe central ideals for an arbitrary von Neumann algebra $\mathscr{A}$ with center $\mathscr{Z}$. Let $P$ be a projection in $\mathscr{Z}$ and let $E$ be a properly infinite projection in $\mathscr{A}$ majorized by $P$. (By convention we assume that 0 is a properly infinite projection in a finite algebra $\mathscr{Z}$.) Let $(\mathscr{J}_P(E))$ be the set of all projections in $\mathscr{A}$ given by $(\mathscr{J}_p(E)) = \{F \in (\mathscr{A}) \mid F \leqq P \text{ and } QE \prec QF \text{ for some } Q \in (\mathscr{Z}) \text{ implies } QE = 0\}$. Let $\mathscr{J}_P(E)$ be the ideal generated by $(\mathscr{J}_P(E))$.

We shall use the following lemma of F. B. Wright [32; §2].

LEMMA. *Suppose $\mathscr{P}$ is a set of projections on a von Neumann algebra $\mathscr{A}$ that satisfies the following properties:*

(1) *if $E \in (\mathscr{A})$, $F \in \mathscr{P}$ and $E \prec F$, then $E \in \mathscr{P}$; and*

(2) *if $E$ and $F$ are in $\mathscr{P}$, then the least upper bound* $\mathrm{lub}\{E, F\}$ *of $E$ and $F$ is in $\mathscr{P}$.*

*Then the set of projections of the ideal generated by $\mathscr{P}$ is exactly $\mathscr{P}$.*

THEOREM 2.4. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. In order for the ideal $\mathscr{J}$ in $\mathscr{A}$ to be a central ideal, it is a necessary and sufficient condition that there exist a projection $P$ in $\mathscr{Z}$ and a properly infinite projection $E$ majorized by $P$ with $\mathscr{J} = \mathscr{J}_P(E)$.*

REMARK. The sufficiency is an adaptation of the proof we gave for a special case in an earlier paper [18, Proposition 2.1].

*Proof.* Let $E$ be a properly infinite projection majorized by the central projection $P$. We show that $\mathscr{J}_P(E) = \mathscr{J}$ is a central ideal. Let $P_1$ and $P_2$ be orthogonal central projections of sum 1 such that $\mathscr{A}P_1$ is a finite algebra and $\mathscr{A}P_2$ is a properly infinite. It is sufficient to show that $\mathscr{J}P_i$ is a central ideal in $\mathscr{A}P_i$ $(i = 1, 2)$. However, we have that $\mathscr{J}P_i$ is generated by $(\mathscr{J}P_i) = \{F \in (AP_i) \mid F \leqq PP_i, EQP_i \prec FQ \text{ for some } Q \text{ in } (\mathscr{Z}P_i) \text{ implies } EP_iQ = 0\}$. Now setting $E_i = EP_i$, we obtain a properly infinite projection in $\mathscr{A}P_i$ so that $\mathscr{J}P_i = \mathscr{J}_{PP_i}(E_i)$. Hence, there is no loss of generality in assuming that $\mathscr{A}$ is either finite or properly infinite.

Let $\mathscr{A}$ be finite. Then $E = 0$ and $(\mathscr{J}_P(0)) = \{F \in (A) \mid F \leqq P\}$. Hence $\mathscr{J} = \mathscr{A}P$ and so $\mathscr{J}$ is a central ideal.

Now assume that $\mathscr{A}$ is properly infinite. There is no loss of generality in the assumption that $P = 1$. We show that $\mathscr{J}$ satisfies properties (1) and (2) of the lemma of F. B. Wright. By the definition of $(\mathscr{J})$ is clear that it satisfies property (1). Now let $E_1$ and $E_2$ be in $(\mathscr{J})$. Since $\text{lub}\{E_1, E_2\} - E_1 \prec E_2$ [21, Theorem 5.4], we have that $\text{lub}\{E_1, E_2\} - E_1$ is in $(\mathscr{J})$ by (1). So there is no loss of generality in the assumption that $E_1$ and $E_2$ are orthogonal. There is $Q \in (\mathscr{Z})$ such that $QE_1 \prec QE_2$ and $(1 - Q)E_2 \prec (1 - Q)E_1$. Since $Q(E_1 + E_2) \in (\mathscr{J})$ and $(1 - Q)(E_1 + E_2) \in (\mathscr{J})$ implies that $E_1 + E_2 \in (\mathscr{J})$, there is no loss of generality in the assumption that $E_1 \prec E_2$. There is a $Q \in (\mathscr{Z})$ such that $QE_2$ is finite and $(1 - Q)E_2$ is properly infinite. Hence, we may assume that either $E_2$ is finite or properly infinite. If $E_2$ is finite, then $E_1$ is finite since $E_1 \prec E_2$ and so $E_1 + E_2$ is finite. [9, III, 2]. If $Q$ is a central projection with $QE \prec Q(E_1 + E_2)$, then $QE$ is finite and so $QE = 0$. So we are left with the situation that $E_1 \prec E_2$, $E_1E_2 = 0$, and $E_2$ is properly infinite. Because $E_2$ is properly infinite, there are projections $F_1$, $F_2$ satisfying the relations: $F_1 \sim F_2 \sim E_2$, $F_1F_2 = 0$, and $F_1 + F_2 = E_2$. [9; III, 8, Corollary 2]. We have that $E_1 + E_2 \sim E_1 + F_2 < F_2 + F_2 = E_2$. By property (1) of the lemma, we conclude that $E_1 + E_2 \in (\mathscr{J})$. Hence $(\mathscr{J})$ satisfies properties (1) and (2) of the lemma and this means that the set of projections of the ideal $\mathscr{J}$ generated by $(\mathscr{J})$ is precisely $(\mathscr{J})$. Now we show $\mathscr{J}$ is a central ideal. Let $\{A_i \mid i \in I\}$ be a bounded set in $\mathscr{J}$ and let $\{Q_i \mid i \in I\}$ be an orthogonal subset of $(\mathscr{Z})$ of sum 1. For every $\varepsilon > 0$ and every $i \in I$ there is a projection $F_i$ in $(\mathscr{J})$ such that $\|A_i - A_iF_i\| \leqq \varepsilon$. Then $\sum F_iQ_i = F$ is in $(\mathscr{J})$. Indeed, if $EQ \prec FQ$ for some $Q$ in $(\mathscr{Z})$, then $E(Q_iQ) \prec F(Q_iQ) = F_i(Q_iQ)$ for every $i \in I$. Thus $(EQ)Q_i = 0$ for every $i \in I$ and $EQ = \sum (EQ)Q_i = 0$. This means that $F \in (\mathscr{J})$. However, we have that

$$\|\sum A_iQ_i - (\sum A_iQ_i)F\| \leqq \text{lub}\,\|A_i - A_iF_i\| \leqq \varepsilon\ .$$

Since $(\sum A_iQ_i)F$ is in $\mathscr{J}$ and since $\mathscr{J}$ is uniformly closed, we have that $\sum A_iQ_i \in \mathscr{J}$. This proves that $\mathscr{J}$ is a central ideal.

We now show that every central ideal $\mathscr{J}$ is of the form $\mathscr{J}_P(E)$. Given a nonzero $P \in (\mathscr{Z})$ it is sufficient to prove that there is a properly infinite projection $E$ in $\mathscr{A}$, a nonzero $Q$ in $(\mathscr{Z})$, and an $R \in (\mathscr{Z})$ with $R \leqq Q \leqq P$ such that $\mathscr{J}_R(ER)Q = \mathscr{J}Q$. Indeed, suppose we have verified this statement. Let $\{P_i \mid i \in I\}$ be a maximal set of mutually orthogonal nonzero central projections such that for each $P_i$ there is a properly infinite projection $E_i$, and a $Q_i \in (\mathscr{Z})$ majorized by $P_i$ such that $\mathscr{J}_{Q_i}(EQ_i)P_i = \mathscr{J}P_i$. By the maximality of $\{P_i\}$, we conclude that $\sum P_i = 1$. Setting $E = \sum E_iQ_i$ (resp. $Q = \sum Q_i$) we obtain a properly infinite (resp. central) projection $E$ majorized by $Q$ such that $\mathscr{J}_Q(E) = \mathscr{J}$. In fact, since $\mathscr{J}_Q(E)$ and $\mathscr{J}$ are generated by their respective projections, it is sufficient to show that $(\mathscr{J}_Q(E)) = (\mathscr{J})$, But we may verify immediately that $(\mathscr{J}_Q(E))P_i = (\mathscr{J}_{Q_i}(E_iQ_i))$, and so we have that $F \in (\mathscr{J}_Q(E))$ if and only if $FP_i \in (\mathscr{J}_{Q_i}(E_iQ_i)) = (\mathscr{J}P_i)$ for every $P_i$ since $\mathscr{J}_Q(E)$ is a central ideal by the first part of this theorem. However, the ideal $\mathscr{J}$ is also a central ideal and thus $F \in (\mathscr{J}_Q(E))$ if and only if $F \in (\mathscr{J})$. So it is sufficient to verify the required statement. We do this in the next paragraph.

Let $P$ be a nonzero element in $(\mathscr{Z})$. Since we are looking for a nonzero central projection $Q$ majorized by $P$, we may assume at the outset that $P = 1$ and that either $\mathscr{A}$ is finite or $\mathscr{A}$ is properly infinite. If $\mathscr{A}$ is finite there is a $Q$ in $(\mathscr{Z})$ with $\mathscr{J} = \mathscr{A}Q$ (corollary 2.3). Then we verify immediately that $\mathscr{J} = \mathscr{J}_Q(0)$. Hence, we may assume that $\mathscr{A}$ is properly infinite. Suppose that there is a projection $P \neq 1$ in $(\mathscr{Z})$ such that $AP = A$ for every $A$ in $\mathscr{J}$. Then we have that $\mathscr{J}_0(0)(1 - P) = 0 = \mathscr{J}(1 - P)$. So we may assume that $\mathscr{J}$ is weakly dense in $\mathscr{A}$. Now suppose that $P_{\mathscr{J}} \neq 1$. Then the nonzero central projection $Q = 1 - P_{\mathscr{J}}$ is in $\mathscr{J}$. This means $\mathscr{J}Q = \mathscr{A}Q = \mathscr{J}_Q(0)$. Hence, we may pass to the case that $P_{\mathscr{J}} = 1$. By making a further reduction if necessary, we may assume that 1 is the sum of an infinite set $\{E_i \mid i \in I\}$ of orthogonal, equivalent, $\sigma$-finite projections [9, III, 1, Lemma 1]. Let $\mathscr{S}(I)$ be the family of all subsets $s$ of $I$ such that there is a nonzero projection $P_s$ in $\mathscr{Z}$ with

$$\sum \{E_i \mid i \in s\} Q \notin \mathscr{J}$$

for every nonzero $Q \in (\mathscr{Z}P_s)$. The family $\mathscr{S}(I)$ is nonvoid since $I \in \mathscr{S}(I)$ with $P_I = 1$. There is an $s_0 \in \mathscr{S}(I)$ such that $\operatorname{Card} s_0 \leqq \operatorname{Card} s$ for every $s \in \mathscr{S}(I)$. We may assume that $P_{s_0} = 1$. Let $\sum \{E_i \mid i \in s_0\} = E$; we notice that $E$ is a properly infinite projection of central support 1. We show that $\mathscr{J}_1(E) = \mathscr{J}(E)$ is equal to $\mathscr{J}$. First we prove that $(\mathscr{J}) \subset (\mathscr{J}(E))$. Let $F \in (\mathscr{J})$. If $EP \prec FP$ for some $P \in (\mathscr{Z})$, then by choice of $s_0$ we have that $EP = 0$. So $F \in (\mathscr{J}(E))$ by definition and hence $(\mathscr{J}) \subset (\mathscr{J}(E))$. To show the converse relation $(\mathscr{J}(E)) \subset (\mathscr{J})$ we consider two cases: (i) $\operatorname{Card} s_0$ is finite, and (ii) $\operatorname{Card} s_0$ is infinite. For case (i) we have that $E$ is a $\sigma$-finite projection of central support 1. Then we have that $(\mathscr{J}(E))$ is exactly the set of finite projections of $\mathscr{A}$ [9; III, 8, Corollary 5]. But by our preliminary reduction $\mathscr{J}$ is weakly dense in $\mathscr{A}$ and therefore contains all finite projections of $\mathscr{A}$ (Proposition 2.2). So $(\mathscr{J}(E)) \subset (\mathscr{J})$. Now we consider case (ii). Let $F \in (\mathscr{J}(E))$. Since $\mathscr{J}$ is a central ideal, there is a $P \in (\mathscr{Z})$ such that $PF \in \mathscr{J}$ and $QF \notin \mathscr{J}$ for every nonzero $Q$ in $(\mathscr{Z}(1 - P))$. We obtain a contradiction by assuming $1 - P \neq 0$. Because $\mathscr{J}$ contains all finite projections (Proposition 2.2), we have that $F(1 - P)$ is properly infinite with central support $1 - P$. We may find a nonzero projection $Q$ in $(\mathscr{Z}(1 - P))$ such that $FQ$ is the sum of a set $\{F_i \mid i \in s\}$ of orthogonal, equivalent, properly infinite $\sigma$-finite projections [9; III, 1, Lemma 7]. We have that $F_1 \sim E_jQ$ for every $i \in s$ and $j \in s_0$. [9; III, 8, Corollary 5]. Since $\sum \{F_i \mid i \in s\} = FQ \prec EQ = \sum \{E_iQ \mid i \in s_0\}$, and since Card $s_0$ is infinite, we have that Card $s \leqq$ Card $s_0$ [9; III, 1, Lemma 6]. If Card $s_0 \leqq$ Card $s$, we would have a contradiction in that $EQ \prec FQ$ and $EQ \neq 0$. Thus Card $s \neq$ Card $s_0$. But if $s'$ is a subset of $\mathscr{S}(I)$ with Card $s' =$ Card $s$, then $\sum \{E_i \mid i \in s'\}Q' \sim FQ'$ for every $Q'$ in $(\mathscr{Z}Q)$ and so $\sum \{E_i \mid i \in s'\}Q' \notin \mathscr{J}$ for every nonzero $Q'$ in $(\mathscr{Z}Q)$. This contradicts the choice of $s_0$. Hence, $1 - P = 0$ and $F \in (\mathscr{J})$. So in case (ii) we have $(\mathscr{J}(E)) \subset (\mathscr{J})$. Therefore, we have completed the crucial step, and so there is an $E \in (\mathscr{A})$ and a $Q \in (\mathscr{Z})$ with $\mathscr{J} = \mathscr{J}_Q(E)$.

Now let $E$ be a properly infinite projection majorized by the central projection $P$ in the von Neumann algebra $\mathscr{A}$. Let $Q$ be the central projection of $\mathscr{A}$ such that $\mathscr{A}Q$ is equal to the weak closure of $\mathscr{J}_P(E) = \mathscr{J}$. Then it is clear that $\mathscr{J}_Q(EQ) = \mathscr{J}$. We say a representation $\mathscr{J}_P(E)$ for a central ideal $\mathscr{J}$ is in *canonical form* if $\mathscr{A}P$ is the weak closure of $\mathscr{J}_P(E)$.

PROPOSITION 2.5. *Let $\mathscr{J}_P(E)$ and $\mathscr{J}_Q(F)$ be two central ideals of a von Neumann algebra $\mathscr{A}$ that are represented in canonical form. Then $\mathscr{J}_P(E) = \mathscr{J}_Q(F)$ if and only if $P = Q$ and $E \sim F$.*

*Proof.* If $P = Q$ and $E \sim F$, then it is clear that $\mathscr{J}_P(E) = \mathscr{J}_Q(F)$.

Now let $\mathscr{J}_P(E) = \mathscr{J}_Q(F) = \mathscr{J}$. Since $\mathscr{A}P =$ weak closure $\mathscr{J} = \mathscr{A}Q$, we have that $P = Q$. Now let $R$ be the largest central projection majorized by $P$ such that $RE \sim RF$. Suppose $R' = P - R \neq 0$. There is a central projection $R''$ majorized by $R'$ such that $R''E \prec R''F$ and $(R' - R'')F \prec (R' - R'')E$. If $R'' \neq 0$, then $SR''F \prec SR''E$ for some central projection $S$ implies that $SR''F = 0$. Otherwise, we would have that $SR''E \sim SR''F$ and so $R$ would not be the largest central projection with $RE \sim RF$. This means that $R''E \in \mathscr{J}$. Hence $R''E = 0$ and so $\mathscr{J}R'' = \mathscr{A}R''$. This means that $FR'' = 0$ and consequently that $ER'' \sim FR''$. This is a contradiction. A similar contradiction arises if $R' - R'' \neq 0$. So we must have that $R = P$, i.e., $E \sim F$.

REMARK 2.6. In the sequel we assume *all* representations of central ideals are in *canonical* form.

COROLLARY 2.7. *Let $\mathscr{A}$ be a von Neumann algebra and let $\mathscr{J}$ be a central ideal of $\mathscr{A}$ given by $\mathscr{J} = \mathscr{J}_P(E)$ in canonical form.*

*Then in order that* $P_{\mathscr{J}} = 1$, *a necessary and sufficient condition is that* $P$ *be the central support of* $E$.

*Proof*. If the central support of $E$ is $Q$, then from the definition of $\mathscr{J}_P(E) = \mathscr{J}$, it is clear that $P - Q \in \mathscr{J}$. This means that $P_{\mathscr{J}} \neq 1$ if $P - Q \neq 0$. Conversely, if $1 - P_{\mathscr{J}} \neq 0$, then $(1 - P_{\mathscr{J}})E = 0$. But $(1 - P_{\mathscr{J}}) \leqq P$ and thus $E$ cannot have central support $P$.

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. Let $Z$ be the spectrum of $\mathscr{Z}$. Let $C_C(Z)$ be the set of order-continuous functions of $Z$ into the set of cardinal numbers. J. Tomiyama [30] showed that there is a dimension function $D$ of $\mathscr{A}$ into $C_C(Z)$ such that $D(E) \leqq D(F)$ if and only if $E \prec F$. W. Wils [31] described the range of $D$ in $C_C(Z)$ as being a certain subset $\varDelta$ of functions in $C_C(Z)$. Although it is not important in the sequel, one may see that the set of projections of a central ideal $\mathscr{J}_P(E)$ satisfies a certain dimension relation relative to $P$ and $E$. We therefore feel justified in introducing a name for the following relation.

DEFINITION 2.8. Let $\mathscr{J}$ be a central ideal in a von Neumann algebra $\mathscr{A}$. Let $P$ be a central projection and let $E$ be a properly infinite projection majorized by $P$ with $\mathscr{J} = \mathscr{J}_P(E)$. A projection $F$ in $\mathscr{A}$ is said to have *dimension* greater than that of $\mathscr{J}$ if $F$ has central support $P_{\mathscr{J}}$ and if $F \succ EP_{\mathscr{J}}$ (in symbols, $\dim F > \dim \mathscr{J}$).

The following proposition characterizes the projections whose dimension is greater than the dimension of $\mathscr{J}$.

PROPOSITION 2.9. *Let* $\mathscr{A}$ *be a von Neumann algebra and let* $\mathscr{J}$ *be a central ideal of* $\mathscr{A}$. *Then a projection* $F$ *of* $\mathscr{A}$ *has dimension greater than that of* $\mathscr{J}$ *if and only if* $F$ *has central support* $P_{\mathscr{J}}$ *and* $FQ \in \mathscr{J}$ *for some central projection* $Q$ *implies* $FQ = 0$.

*Proof*. Let $\mathscr{Z}$ be the center of $\mathscr{J}$. Let $E \in (\mathscr{A})$ and let $P \in (\mathscr{Z})$ so that $\mathscr{J}_P(E)$ represents $\mathscr{J}$ in canonical form. First let $F \in (\mathscr{A})$ with central support $P_{\mathscr{J}}$ such that $QF \in \mathscr{J}$ for some $Q \in (\mathscr{Z})$ implies $QF = 0$. There is an $R \in (\mathscr{Z})$ such that $RE \prec RF$ and such that $R'E \prec R'F$ for $R' \in (Z(1 - R))$ implies $R' = 0$. Then $FP(1 - R) \in \mathscr{J}_P(E)$ by definition and so $FP(1 - R) = 0$. Thus we obtain that $FPR = FP$. So $EP_{\mathscr{J}} = EPP_{\mathscr{J}} \prec FP \leqq F$, i.e. $\dim F > \dim \mathscr{J}$.

Conversely, let $\dim F > \dim \mathscr{J}$. Then by definition we have that $F$ has central support $P_{\mathscr{J}}$. Let $Q \in (\mathscr{Z})$ and let $QF \in \mathscr{J}$. We have that $EP_{\mathscr{J}} \prec F$ implies that $EQP_{\mathscr{J}} \in \mathscr{J}$ (lemma of F. B. Wright). Since $EQP_{\mathscr{J}} \prec EQP_{\mathscr{J}}$, we have that $EQP_{\mathscr{J}} = 0$ and thus $QPP_{\mathscr{J}} \in \mathscr{J}$.

By definition of $P_{\mathscr{I}}$ we find that $QPP_{\mathscr{I}} = 0$. Also $\mathscr{I}(1 - P) = (0)$ and so $QF = QPF + Q(1 - P)F = 0$.

Now we can give some examples.

EXAMPLE 2.10. In a factor algebra, every ideal is a central ideal.

EXAMPLE 2.11. In a semi-finite algebra $\mathscr{A}$, the ideal $\mathscr{I}$ generated by all finite projections of $\mathscr{A}$ is a central ideal. If $\mathscr{A}$ is finite, then $\mathscr{I} = \mathscr{A}$; if $\mathscr{A}$ is properly infinite, then $\mathscr{I} = \mathscr{I}_1(E)$, where $E$ is a properly infinite projection of central support 1 for which there is a set $\{P_i\}$ of mutually orthogonal central projections of sum 1 such that $EP_i$ is $\sigma$-finite for every $P_i$ [8; III, 1, Lemma 7].

EXAMPLE 2.12. If $\mathscr{A}$ is a properly infinite von Neumann algebra, then the strong radical $\mathscr{J}$ (i.e. the intersection of all maximal ideals) is a central ideal with $\mathscr{J} = \mathscr{I}_1(1)$.

**3. The essential central spectrum.** Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. If $\mathscr{I}$ is an ideal in $\mathscr{A}$, let $\mathscr{A}(\mathscr{I})$ denote the algebra $\mathscr{A}$ reduced modulo $\mathscr{I}$, and let $A(\mathscr{I})$ denote the image of an element $A$ under the canonical homomorphism of $\mathscr{A}$ into $\mathscr{A}(\mathscr{I})$. The algebra $\mathscr{A}(\mathscr{I})$ is a $C^*$-algebra under the norm $||A(\mathscr{I})|| = \text{glb}\{||A + B|| \mid B \in \mathscr{I}\}$. If $\zeta$ is an element in the spectrum $Z$ of $\mathscr{Z}$, let $[\zeta]$ denote the smallest ideal in $\mathscr{A}$ containing $\zeta$. For simplicity we let $\mathscr{A}([\zeta])$ and $A([\zeta])$ be denoted by the symbols $\mathscr{A}(\zeta)$ and $A(\zeta)$, respectively. Then J. Glimm [12; Lemma 10] has shown that for fixed $A \in \mathscr{A}$ the map $\zeta \rightarrow ||A(\zeta)||$ is continuous on the spectrum $Z$. For every $A$ in $\mathscr{A}$ and $\zeta$ in $Z$, the norm $||A(\zeta)||$ is equal to $||A(\zeta)|| = \text{glb}\{||AP|| \mid P \in (\mathscr{Z})$ and $P^\wedge(\zeta) = 1\}$. Here $P^\wedge$ denotes the Gelfand transform of $P$. If $\mathscr{I}_1$ and $\mathscr{I}_2$ are ideals in $\mathscr{A}$, then the algebraic sum $\mathscr{I}_1 + \mathscr{I}_2$ is also an ideal of $\mathscr{A}$. In the sequel we denote the sum $\mathscr{I} + [\zeta]$ of an ideal $\mathscr{I}$ and the special ideal $[\zeta]$ formed from $\zeta \in Z$ by $\mathscr{I}(\zeta)$. For an element $A$ in $\mathscr{A}$, we denote the spectrum of $A(\mathscr{I}(\zeta))$ in $\mathscr{A}(\mathscr{I}(\zeta))$ by $\text{Sp}\,A(\mathscr{I}(\zeta))$.

The next lemma is used repeatedly.

LEMMA 3.1. *Let $\mathscr{A}$ be a von Neumann algebra, let $\mathscr{Z}$ be the center of $\mathscr{A}$, let $P \in (\mathscr{Z})$, let $Z$ be the spectrum of $\mathscr{Z}$, and let $\mathscr{I}$ be a central ideal of $\mathscr{A}$. If $A$ is an element of $\mathscr{A}$ such that $f_A(\zeta) = ||A(\mathscr{I}(\zeta))||$ vanishes for every $\zeta$ in the support of $P$ given by supp $P = \{\zeta \in Z \mid P^\wedge(\zeta) = 1\}$, then the element $AP$ is in $\mathscr{I}$.*

*Proof.* For every $\zeta$ in supp $P$ and $\varepsilon > 0$ there is a $B_\zeta$ in $\mathscr{I}$ such that $||(A - B_\zeta)(\zeta)|| < \varepsilon$. Hence there is a $P_\zeta$ in $(\mathscr{Z})$ with $P_\zeta^\wedge(\zeta) = 1$

such that $||(A - B_\zeta)P_\zeta|| < \varepsilon$. Using the fact that supp $P$ is compact, we may find a set $P_1, \cdots, P_n$ of orthogonal projections in $\mathscr{Z}$ of sum $P$ and a corresponding set $B_1, \cdots, B_n$ in $\mathscr{J}$ such that

$$||AP - \sum B_i P_i|| = \text{lub}\, ||(A - B_i)P_i|| < \varepsilon .$$

Since $\mathscr{J}$ is closed, the element $AP$ is in $\mathscr{J}$.

We characterize those ideals $\mathscr{J}$ for which $\zeta \to ||A(\mathscr{J}(\zeta))||$ is continuous on $Z$ for every $A$ in $\mathscr{A}$.

THEOREM 3.2. *Let $\mathscr{A}$ be a von Neumann algebra, let $\mathscr{Z}$ be the center of $\mathscr{A}$, and let $Z$ be the spectrum of $\mathscr{Z}$. Let $\mathscr{J}$ be an ideal of $\mathscr{A}$. In order that $f_A(\zeta) = ||A(\mathscr{J}(\zeta))||$ be a continuous function on $Z$ for every $A$ in $\mathscr{A}$, a necessary and sufficient condition is that $\mathscr{J}$ be a central ideal of $\mathscr{A}$.*

*Proof.* The sufficiency follows by a proof that is virtually the same as the one we gave in the corollary of (a5) implies (a1) of [19].

Conversely, let $f_A$ be continuous on $Z$ for every $A$ in $\mathscr{A}$. We show that $\mathscr{A}$ is a central ideal. If $\{A_i | i \in I\}$ is a bounded subset of $\mathscr{J}$ and if $\{P_i | i \in I\}$ is an orthogonal set in $(\mathscr{Z})$ of sum 1, then we prove that $A = \sum A_i P_i$ is in $\mathscr{A}$. Indeed, the set $\cup_i \{\zeta \in Z | P_i^\wedge(\zeta) = 1\}$ is a dense set of $Z$ on which $f_A(\zeta)$ vanishes since $f_A(\zeta) = ||A_i(\mathscr{J}(\zeta))|| = 0$ whenever $P_i^\wedge(\zeta) = 1$. By the continuity of $f_A$, we see that $f_A$ vanishes on $Z$. Hence, the element $A$ is in $\mathscr{J}$ by Lemma 3.1.

REMARK 3.3. If $\mathscr{J}$ is the strong radical of a properly infinite von Neumann algebra, then $\mathscr{J}(\zeta) = \mathscr{J} + [\zeta]$ is the unique maximal ideal which contains $\zeta$ [24 and 15, Proposition 2.3].

Now we prove the main result of this section. It is convenient to separate the following lemma.

LEMMA 3.4. *Let $\mathscr{Z}$ be a commutative von Neumann algebra and let $X_1, \cdots, X_n$ be closed sets which cover the spectrum $Z$ of $\mathscr{Z}$. Then there are orthogonal projections $R_1, \cdots, R_n$ in $(\mathscr{Z})$ of sum 1 such that $\{\zeta \in Z | R_i^\wedge(\zeta) = 1\} \subset X_i$ for $1 \leqq i \leqq n$.*

*Proof.* Let $\{P_i | i \in I\}$ be a maximal set of nonzero mutually orthogonal projections such that for each $i \in I$ there is an $i(j)$ with $1 \leqq i(j) \leqq n$ so that $Y_i = \{\zeta \in Z | P_i^\wedge(\zeta) = 1\} \subset X_{i(j)}$. We obtain a contradiction if $P = 1 - \sum P_i \neq 0$. Indeed, the set $Y = \{\zeta \in Z | P^\wedge(\zeta) = 1\}$ is nonvoid and is covered by the closed sets $Y \cap X_1, \cdots, Y \cap X_n$. By the Baire category theorem one of the set $Y \cap X_m$ has a nonvoid interior in $Y$. This means that there is nonzero projection $Q$ in $\mathscr{Z}$ such that $\{\zeta \in Z \mid Q^\wedge(\zeta) = 1\} \subset Y \cap X_m$. This contradicts the maximality of $\{P_i\}$. We must have that $\sum P_i = 1$. The remainder of the proof consists in adding the projections $P_i$. Let $I_j = \{i \in I \mid Y_i \subset X_j\}$ for $1 \leqq j \leqq n$ and let $R_j = \sum \{P_i \mid i \in I_j - \cup \{I_k \mid 0 \leqq k \leqq j-1\}\}$ for $1 \leqq j \leqq n$. Here $I_0 = \emptyset$. Then it is clear that $R_1, R_2, \cdots, R_n$ satisfy the requirements of the lemma.

THEOREM 3.5. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$, let $\mathscr{J}$ be a central ideal of $\mathscr{A}$, and let $A$ be an element of $\mathscr{A}$. Let $X_0$ be a closed subset of the complex plane $\boldsymbol{C}$ such that the intersection $S(\zeta)$ of $X_0$ with the spectrum (resp. left-spectrum, right-spectrum, the intersection of the left-spectrum and the right-spectrum) of $A(\mathscr{J}(\zeta))$ is nonvoid for every $\zeta$ in the spectrum $Z$ of $\mathscr{Z}$. Then there is an element $A_0$ in the center of $\mathscr{A}$ such that $A_0^\wedge(\zeta) \in S(\zeta)$ for every $\zeta$ in $Z$.*

*Proof*. We first prove that there exists $A_0$ in $\mathscr{Z}$ such that $A_0^\wedge(\zeta)$ is in the intersection $S(\zeta)$ of $X_0$ with the spectrum $\operatorname{Sp} A(\mathscr{J}(\zeta))$ of $A(\mathscr{J}(\zeta))$ for every $\zeta$ in $Z$. Since $\operatorname{Sp} A(\mathscr{J}(\zeta))$ is contained in $\operatorname{Sp} A$, there is no loss of generality in assuming $X_0 \subset \operatorname{Sp}(A)$. We prove the theorem by an approximation argument that involves decomposing the space $Z$.

For every compact set $X$ in the complex plane, let $X(Z) = \{\zeta \in Z \mid X \cap S(\zeta) \neq \emptyset\}$. We show that $X(Z)$ is closed in $\boldsymbol{C}$. Let $\{\zeta_i\}$ be a net in $X(Z)$ converging to $\zeta$. Let $\alpha_i \in S(\zeta_i) \cap X$; by passing to a subnet, we may assume that $\{\alpha_i\}$ converges to $\alpha \in X \cap X_0$. Arguing by contradiction we show that $\alpha \in \operatorname{Sp} A(\mathscr{J}(\zeta))$. If $\alpha \notin \operatorname{Sp} A(\mathscr{J}(\zeta))$, then there is a $B \in \mathscr{A}$ with

$$\begin{aligned} &\|(B(\alpha - A) - 1)(\mathscr{J}(\zeta))\| = \\ &\|((\alpha - A)B - 1)(\mathscr{J}(\alpha))\| = 0 \,. \end{aligned}$$

By Theorem 3.2, we see that there is a $\zeta_i$ and $\alpha_i$ such that

$$\|(B(\alpha_i - A) - 1)(\mathscr{J}(\zeta_i))\| < 1$$

and $\|((\alpha_i - A)B - 1)(\mathscr{J}(\zeta_i))\| < 1$. This means that $\alpha_i \notin \operatorname{Sp} A(\mathscr{J}(\zeta_i))$ and this is contrary to assumption. So $\alpha \in X(Z)$ and $X(Z)$ is closed.

We now begin the approximation argument by decomposing $Z$ into subsets on which we shall approximate $A_0$. Suppose we have, for every $m$ less than or equal to the natural number $n$, constructed sets of integers $I_m = \{1, 2, \cdots, p_m\}$ such that for every $s$ in $I_1 \times \cdots \times I_m = I(m)$ there is a compact subset $X(s)$ of $\boldsymbol{C}$ of diameter $\leqq 2^{-m}$ and a $P(s)$ in $(\mathscr{Z})$ which satisfies the following properties:

(1) For $s \in I(m)$, $\cup \{X(s; j) \mid j \in I_{m+1}\} = X(s)$ whenever $1 \leqq m < n$ and

$$\cup \{X(j) \mid j \in I(1)\} = X_0 ;$$

(2) Supp $P(s) = \{\zeta \in Z \mid P(s)^\wedge(\zeta) = 1\} \subset X(s)(Z)$ for every $s \in I(m)$ $(1 \leqq m \leqq n)$; and
(3) for $s \in I(m)$, $\{P(s; j) \mid j \in I_{m+1}\}$ is a set of orthogonal projections of sum $P(s)$ whenever $1 < m < n$ and $\{P(j) \mid j \in I(1)\}$ is a set of orthogonal projections of sum 1.
We shall construct a set $I_{n+1} = \{1, \cdots, p_{n+1}\}$, compact sets $X(s)(s \in I(n+1) = I_1 \times \cdots \times I_{n+1})$ of diameter $\leqq 2^{-(n+1)}$ in the complex plane, and projections $P(s)(s \in I(n+1))$ in $\mathscr{Z}$ which satisfy (1), (2), (3). Indeed, let $\{Y_j \mid j \in I_{n+1}\}$ be compact sets of diameter $\leqq 2^{-(n+1)}$ which cover $X_0$. Let $X(s, j) = X(s) \cap Y_j$ for $s \in I(n)$ and $j \in I_{n+1}$. Then $\{X(s) \mid s \in I(m), m = 1, 2, \cdots, n+1\}$ satisfies property (1). Now let $s$ be fixed in $I(n)$; we have that $\cup \{X(s; j)(Z) \mid j \in I_{n+1}\} = X(s)(Z)$. Since supp $P(s)$ is contained in $X(s)(Z)$, the sets $X(s; j)(Z) \cap \operatorname{supp} P(s)(j \in I_{n+1})$ form a closed cover of supp $P(s)$. By the Lemma 3.4, there are orthogonal central projections $P(s; j)$ $(j \in I_{n+1})$ of sum $P(s)$ such that

$$\operatorname{supp} P(s; j) \subset X(s; j)(Z)$$

for every $j \in I_{n+1}$. Thus $P(s)(s \in I(n+1))$ satisfies (2) and (3).

We continue by induction to construct $I(n)$, compact sets $X(s)$ $(s \in I(n))$ of diameter $\leqq 2^{-n}$, and central projections $P(s)$ $(s \in I(n))$ satisfying (1), (2), and (3) for every $n = 1, 2, \cdots$. We notice that if $X(s)$ is void then $P(s) = 0$.

We now construct the approximating elements. Let $n = 1, 2, \cdots$ be fixed. If $s \in I(n)$, let $\alpha(s) \in X(s)$ if $X(s)$ is non-void, and $\alpha(s) = 0$ if $X(s)$ is void. Let $A_n = \sum \{\alpha(s)P(s) \mid s \in I(n)\}$. Then $A_n$ is an element in the center of $\mathscr{A}$.

We show that $\{A_n\}$ is a Cauchy sequence. Indeed, we have that

$$\| A_n - A_{n+1} \| = \operatorname{lub} \{\| (A_n - A_{n+1})P(s) \| \mid s \in I(n+1)\}$$

since $\sum \{P(s) \mid s \in I(n+1)\} = \sum \{P(s) \mid s \in I(n)\} = \cdots = 1$. However, if $s \in I(n+1)$ is of the form $s = (s'; j)$ with $s' \in I(n)$ and $j \in I_{n+1}$, then

$$\| (A_n - A_{n+1})P(s) \| = \| (\alpha(s') - \alpha(s))P(s) \| \leqq 2^{-n}$$

since $\alpha(s) \in X(s')$ whenever $P(s) \neq 0$. Hence, we obtain that

$$\| A_n - A_{n+1} \| \leqq 2^{-n}$$

for every $n = 1, 2, \cdots$ and so $\{A_n\}$ is a Cauchy sequence in $\mathscr{Z}$.

We show that the limit $A_0$ of $\{A_n\}$ satisfies the requirements of the Theorem 3.5. Let $\zeta$ be an arbitrary point in $Z$. Given $\varepsilon > 0$ we show that there is $\alpha \in S(\zeta)$ such that $|A_0^\wedge(\zeta) - \alpha| \leqq \varepsilon$. Since $S(\zeta)$ is closed and since $\varepsilon > 0$ is arbitrary, this will mean that $A_0^\wedge(\zeta) \in S(\zeta)$. Let $m$ be a natural number with $2^{-m+2} < \varepsilon$. Then $|A_0^\wedge(\zeta) - A_m^\wedge(\zeta)| \leqq \|A_0 - A_m\| < 2^{-1}\varepsilon$. There is an $s \in I(m)$ such that $P(s)^\wedge(\zeta) = 1$ since $\sum \{P(s) \mid s \in I(m)\} = 1$. By property (2), we have that $\zeta$ is in $X(s)(Z)$. So there is an element $\alpha$ in $X(s)$ such that $\alpha \in S(\zeta)$. However we have that $A_m^\wedge(\zeta) = \alpha(s) \in X(s)$, and so $|\alpha(s) - \alpha| < 2^{-m}$ since the diameter of $X(s)$ is less than $2^{-m}$. Now we obtain that $|A_0^\wedge(\zeta) - \alpha| \leqq \varepsilon$, and by the preceding remarks that $A_0^\wedge(\zeta) \in S(\zeta)$. This completes the proof for the case of $X_0 \cap \operatorname{Sp} A(\mathscr{I}(\zeta)) \neq \emptyset$.

We may prove the existence of an element $A_0$ in $\mathscr{Z}$ such that $(A_0 - A)(\mathscr{I}(\zeta))$ is not left (resp. right, left nor right) invertible in $\mathscr{A}(\mathscr{I}(\zeta))$ and $A_0^\wedge(\zeta) \in X_0$ by the same proof we just gave for an invertible element by using the additional fact that, for any element $B$ in a Banach algebra $\mathscr{B}$ with identity, the set of all complex $\alpha$ such that $\alpha - B$ is not left (resp. right, left nor right) invertible is a non-void compact set ([26; 1.5.4 and 1.4.6]; also cf. [11; Theorem 3.1]).

The following definition is now meaningful.

Definition 3.6. Let $\mathscr{A}$ be a von Neumann algebra, let $\mathscr{Z}$ be the center of $\mathscr{A}$ and let $Z$ be the spectrum of $\mathscr{Z}$. Then the *essential central spectrum* $\mathscr{Z} - \operatorname{Sp}_{\mathscr{I}} A$ of an element $A$ in $\mathscr{A}$ with respect to the central ideal $\mathscr{I}$ is the set of all $A_0$ in $\mathscr{Z}$ such that $A_0^\wedge(\zeta) \in \operatorname{Sp} A(\mathscr{I}(\zeta))$ for every $\zeta \in Z$. The left-essential (resp. right-essential) central spectral $\mathscr{Z} - \operatorname{Sp}_{\mathscr{I}}^e A$ (resp. $\mathscr{Z} - \operatorname{Sp}_{\mathscr{I}}^r A$) of $A$ with respect to $\mathscr{I}$ is defined in a similar manner. The intersection $\mathscr{Z} - \operatorname{Sp}_{\mathscr{I}}^b A = (\mathscr{Z} - \operatorname{Sp}_{\mathscr{I}}^e A) \cap (\mathscr{Z} - \operatorname{Sp}_{\mathscr{I}}^r A)$ is called the two-sided essential central spectrum of $A$ with respect to $\mathscr{I}$.

Remark 3.7. All sets defined in Definition 3.6 are non-void (Theorem 3.5).

Remark 3.8. For every $A_0 \in \mathscr{Z} - \operatorname{Sp}_{\mathscr{I}} A$, we have that $A_0(1 - P_{\mathscr{I}}) = 0$. Since $(\mathscr{Z} - \operatorname{Sp}_{\mathscr{I}}^e A) \cup (\mathscr{Z} - \operatorname{Sp}_{\mathscr{I}}^r A) \subset \mathscr{Z} - \operatorname{Sp}_{\mathscr{I}} A$, the projection $1 - P_{\mathscr{I}}$ annihilates the other essential central spectrums.

We note that these definitions correspond to the usual ones if $\mathscr{A}$ is the algebra of all bounded operators on a Hilbert space and $\mathscr{I}$ is the ideal of compact operators.

Proposition 3.9. *Let $\mathscr{A}$ be a von Neumann algebra. Then the essential (resp. left-, right-essential) central spectrum of an element $A$ in $\mathscr{A}$ with respect to a central ideal $\mathscr{I}$ is closed in the strong operator topology.*

*Proof.* Let $\{A_i\}$ be a net in the essential central spectrum of $A$ with respect to $\mathscr{J}$ which converges strongly to $A_0$ in the center $\mathscr{Z}$ of $\mathscr{A}$. There is a net $\{P_n\}$ of mutually orthogonal central projections of sum 1 such that for each $P_n$ there is a sequence $\{A_{i(n)}\}$ in $\cup_i\{A_i\}$ with $\lim A_{i(n)}P_n = A_0P_n$ (uniformly) [28; Corollary 13.1]. Since $A^{\wedge}_{i(n)}(\zeta) \in \operatorname{Sp} A(\mathscr{J}(\zeta))$ for every $\zeta$ in the spectrum $Z$ of $\mathscr{Z}$ and since $\operatorname{Sp} A(\mathscr{J}(\zeta))$ is closed, we have that $A_0^{\wedge}(\zeta) \in \operatorname{Sp} A(\mathscr{J}(\zeta))$ for every $\zeta$ in the dense subset $X = \cup_n\{\zeta \in Z \mid P_n^{\wedge}(\zeta) = 1\}$ of $Z$ [7]. Let $\{\zeta_i\}$ be a net in $X$ which converges to $\zeta$ in $Z$. If $A_0^{\wedge}(\zeta) \notin \operatorname{Sp} A(\mathscr{J}(\zeta))$, then there is a $B$ in $\mathscr{A}$ with

$$\|(B(A_0 - A) - 1)(\mathscr{J}(\zeta))\| = \|((A_0 - A)B - 1)(\mathscr{J}(\zeta))\| = 0 .$$

This means that there is a $\zeta_i$ with

$$\|(B(A_0 - A) - 1)(\mathscr{J}(\zeta_i))\| < 1 \quad \text{and} \quad \|(A_0 - A)B - 1)(\mathscr{J}(\zeta_i))\| < 1$$

and thus that $A_0^{\wedge}(\zeta_i)$ is not in $\operatorname{Sp} A(\mathscr{J}(\zeta_i))$. Hence, we must have that $A_0^{\wedge}(\zeta)$ is in $\operatorname{Sp} A(\mathscr{J}(\zeta))$ for every $\zeta$ in the closure $Z$ of $X$. This proves that $\mathscr{Z} - \operatorname{Sp}_{\mathscr{J}} A$ is strongly closed.

The statements concerning the left- and right-essential central spectra are proved in an analogous fashion.

For future reference we note some simple facts in the following proposition.

PROPOSITION 3.10. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ on the Hilbert space $H$, let $\mathscr{J}$ be a central ideal in $\mathscr{A}$, let $P_1$ and $P_2$ be orthogonal projections of sum 1 in $\mathscr{Z}$, and let $A$ be an element of $\mathscr{A}$. Let $\mathscr{A}_i$ be the von Neumann algebra $\mathscr{A}P_i$ with center $\mathscr{Z}_i = \mathscr{Z}P_i$ on the Hilbert space $P_iH$, let $\mathscr{J}_i$ be the central ideal $\mathscr{J}P_i$ in $\mathscr{A}_i$, and let $A_i$ be the element $AP_i$ in $\mathscr{A}_i$ for $i = 1, 2$. Then $\mathscr{Z} - \operatorname{Sp}_{\mathscr{J}} A = \{B_1 + B_2 \mid B_i \in \mathscr{Z}_i - \operatorname{Sp}_{\mathscr{J}_i} A_i, i = 1, 2\}$.*

REMARK. A similar statement holds for the left- and right-essential central spectrums.

*Proof.* This follows from the fact that the spectrum of $\mathscr{Z}_i$ is $\{\zeta P_i \mid \zeta \in Z, P_i^{\wedge}(\zeta) = 1\}$, where $Z$ is the spectrum of $\mathscr{Z}$, and thus that $[\zeta P_i]$ in $\mathscr{A}P_i$ is equal $[\zeta]P_i$.

We now restrict our attention to self-adjoint elements. We note that the essential central spectrum of a self-adjoint element consists of self-adjoint elements.

PROPOSITION 3.11. *Let $\mathscr{A}$ be a von Neumann algebra, let $\mathscr{Z}$ be* *the center of* $\mathscr{A}$, *and let* $A$ *be a self-adjoint element of* $\mathscr{A}$. *Then there are elements* $C_u$ *and* $C_l$ *in the essential central spectrum* $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$ *of* $A$ *with respect to the central ideal* $\mathscr{J}$ *such that* $C_l \leqq C \leqq C_u$ *for every* $C$ *in* $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$.

*Proof.* The set $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$ is a monotonely increasing net in $\mathscr{Z}$. Indeed, if $C$ and $C'$ are in $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$, then there is a $P\in(\mathscr{Z})$ such that $\mathrm{lub}\,\{C, C'\} = PC + (1-P)C'$. Since $PC+(1-P)C'$ is in $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$ (by 3.10), the set $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$ is monotonely increasing. Then the least upper bound $C_u$ of $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$ is the strong limit of elements in $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$ and so $C_u$ is in the essential central spectrum of $A$ with respect to $\mathscr{J}$ (Proposition 3.8).

In an analogous manner, we may show that $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$ is monotonely decreasing and thus we may find a greatest lower bound $C_l$ for $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$ in $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$.

PROPOSITION 3.12. *Let* $\mathscr{A}$ *be a von Neumann algebra with center* $\mathscr{Z}$, *let* $\mathscr{J}$ *be a central ideal of* $\mathscr{A}$, *and let* $A$ *be a self-adjoint element of* $\mathscr{A}$. *Let* $C_u$ *and* $C_l$ *be the least upper bound and the greatest lower bound of the essential central spectrum of* $A$ *with respect to* $\mathscr{J}$, *respectively. Then* $C_u^\wedge(\zeta) = \mathrm{lub}\,\mathrm{Sp}\,A(\mathscr{J}(\zeta))$ *and* $C_l^\wedge(\zeta) = \mathrm{glb}\,\mathrm{Sp}\,A(\mathscr{J}(\zeta))$ *for every* $\zeta$ *in the spectrum* $Z$ *of* $\mathscr{Z}$.

*Proof.* Since $C_u^\wedge(\zeta)\in\mathrm{Sp}\,A(\mathscr{J}(\zeta))$ for every $\zeta\in Z$, we have that $C_u^\wedge(\zeta)\leqq\alpha_\zeta=\mathrm{lub}\,\mathrm{Sp}\,A(\mathscr{J}(\zeta))$, for every $\zeta\in Z$. Conversely, we obtain a contradiction if we assume that $\alpha_\zeta - C_u^\wedge(\zeta) = 2\varepsilon > 0$ for some $\zeta\in Z$. Indeed, let $E$ be the spectral projection of $A-C_u$ corresponding to the interval $[\varepsilon, +\infty)$. Because $(A-C_u)(1-E)\leqq\varepsilon(1-E)$, we have that $E(\mathscr{J}(\zeta))\neq 0$. Hence, there is a $P\in(\mathscr{Z})$ such that $P^\wedge(\zeta)=1$ and $E(\mathscr{J}(\zeta'))\neq 0$ for all $\zeta'$ in $\mathrm{supp}\,P=\{\zeta'\in Z\,|\,P^\wedge(\zeta')=1\}$ (Theorem 3.2). Since $\varepsilon E\leqq(A-C_u)E$, we have that $\mathrm{Sp}\,(A-C_u)(\mathscr{J}(\zeta'))\cap[\varepsilon,+\infty)\neq\varnothing$ for all $\zeta'\in\mathrm{supp}\,P$. Reducing to the algebra $\mathscr{A}P$ with center $\mathscr{Z}P$, we see that $S(\zeta')=\mathrm{Sp}\,(A-C_u)P((\mathscr{J}P)(\zeta'))\cap[\varepsilon,+\infty)$ is non-void for every $\zeta'$ in the spectrum $X$ of $\mathscr{Z}P$. Because $\mathscr{J}P$ is a central ideal in $\mathscr{A}P$, we may find a $B$ in $\mathscr{Z}$ such that $(BP)^\wedge(\zeta')\in S(\zeta')$ for every $\zeta'\in X$ (Theorem 3.5). If $D$ is an arbitrary element in $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}(A-C_u)$, then $PB+(1-P)D=B'$ is in $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}(A-C_u)$ (Proposition 3.10), and consequently, the element $B''=B'+C_u$ is in $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$. But we have that $B''P+C_u(1-P)$ is in $\mathscr{Z}-\mathrm{Sp}_{\mathscr{J}}\,A$ (Proposition 3.10) and that $B''P+C_u(1-P)\geqq C_u+\varepsilon P$. This contradicts the definition of $C_u$. Thus we must have that $C_u^\wedge(\zeta)=\mathrm{lub}\,\mathrm{Sp}\,A(\mathscr{J}(\zeta))$ for every $\zeta\in Z$.

A similar proof holds for $C_l$.

The following proposition shows that if $A_0$ is in the essential central spectrum of $A$ with respect to $\mathscr{J}$, then $A_0 - A$ is small on a large subspace with respect to $\mathscr{J}$.

PROPOSITION 3.13. *Let $\mathscr{A}$ be a von Neumann algebra, let $\mathscr{J}$ be a central ideal of $\mathscr{A}$, let $A$ be a self-adjoint element of $\mathscr{A}$, let $A_0$ be an element of the essential central spectrum of $A$ with respect to $\mathscr{J}$, and let $\varepsilon > 0$. If $F$ is the spectral projection of $A_0 - A$ corresponding to the interval $[-\varepsilon, \varepsilon]$, then $FP \in \mathscr{J}$ for some central projection $P$ implies $P \in \mathscr{J}$* (i.e. $P \leqq 1 - P_{\mathscr{J}}$).

*Proof.* Let $P$ be a central projection with $PF \in \mathscr{J}$. We show $P \in \mathscr{J}$. We may assume that $P \neq 0$. Let $\zeta$ be a point in the spectrum of the center of $\mathscr{A}$ such that $P^{\wedge}(\zeta) = 1$. We have that

$$(A_0 - A)(\mathscr{J}(\zeta)) = (A_0 - A)(1 - F)(\mathscr{J}(\zeta)).$$

If $(1 - F)(\mathscr{J}(\zeta)) \neq 0$, then $(A_0 - A)(\mathscr{J}(\zeta))$ is invertible in $\mathscr{A}(\mathscr{J}(\zeta))$. Since this is not possible, we have that $1(\mathscr{J}(\zeta)) = 0$. This means that $P \in \mathscr{J}(\zeta)$. Since $\zeta$ with the property $P^{\wedge}(\zeta) = 1$ is arbitrary in the last relation, we have that $P \in \mathscr{J}$ by Lemma 3.1.

We now characterize the essential central spectrum of a self-adjoint element in terms of the canonical form of a central ideal (cf. Remark 2.6ff. and Definition 2.8).

PROPOSITION 3.14. *Let $\mathscr{A}$ be a von Neumann algebra with no finite type I direct summand, let $\mathscr{J}$ be a central ideal of $\mathscr{A}$, and let $A$ be a self-adjoint element in $\mathscr{A}$. An element $A_0$ is in the essential central spectrum of $A$ with respect to $\mathscr{J}$ if and only if there is an orthogonal sequence $\{E_n\}$ of projections in $\mathscr{A}$ of dimension greater than* $\dim \mathscr{J}$ *such that $AE_n(\mathscr{J}) = E_nA(\mathscr{J})$ and $\|(A_0 - A)E_n(\mathscr{J})\| \leqq n^{-1}$ for every $n = 1, 2, \cdots$ and $A_0 = A_0P_{\mathscr{J}}$.*

*Proof.* Let $A_0$ be in the essential central spectrum of $A$ with respect to $\mathscr{J}$. There is no loss of generality in the assumption that $P_{\mathscr{J}} = 1$ and that $A_0 = 0$. [9; III, 5, Problem 7]. Let $F_n$ be the spectral projection of $A$ corresponding to the interval $[-n^{-1}, n^{-1}]$ for $n = 1, 2, \cdots$; then we have that $\{F_n\}$ is a monotonely decreasing sequence of projections such that $\dim F_n > \dim \mathscr{J}$ (Propositions 3.13 and 2.9).

Let $\mathscr{J}$ be represented in the form $\mathscr{J} = \mathscr{J}_S(E)$ (2.4–2.6). Now let $\{P_i\}$ be a maximal set of mutually orthogonal central projections such that for each $i$ there is a natural number $j(i)$ with $(F_k - F_{k+1})P_i \in \mathscr{J}$ whenever $k \geqq j(i)$. This means that $AF_{j(i)}P_i \in \mathscr{J}$ since

$$||AF_{j(i)}P_i(\mathscr{J})|| = ||AF_kP_i(\mathscr{J})|| \leqq k^{-1}$$

for arbitrary $k \geqq j(i)$. Hence, setting $F = \sum F_{j(i)}P_i$ and $P = \sum P_i$, we obtain a projection $F$ of central support $P$ such that $AF \in \mathscr{J}$ and $EP \prec F$ (Proposition 2.9). Since $\mathscr{A}$ has no finite type I direct summands, we may find a sequence $\{G'_n\}$ of orthogonal projections of sum $FP$ such that the central support of $G'_n$ is $P$ and such that $EP \prec G'_n$. Indeed, there is a central projection $R$ majorized by $P$ such that $FR$ is properly infinite and $F(P - R)$ is finite. In the first instance $FR$ is the sum of a sequence of mutually orthogonal projections each equivalent to $FR$ [9; III, 8, Corollary 2]. In the second instance, we have that $E(P - R) = 0$. Indeed, $E$ is a properly infinite projection and $E(P - R)$ is finite since $E(P - R) \prec F(P - R)$. Now $F(P - R)$ may be written as the sum of a sequence of orthogonal projections of central support $P - R$ [9; III, 1, Theorem 1, Corollary 3].

Now, for every nonzero central projection $Q$ majorized by $P' = 1 - P$ and for every $n = 1, 2, \cdots$, there is a nonzero central projection $Q'$ with $Q' \leqq Q$ and a natural number $m \geqq n$ such that $(F_m - F_{m+1})Q'$ has central support $Q'$ and $EQ' \prec (F_m - F_{m+1})Q'$ (Proposition 2.9). By induction we may find sets $\{G_{ni} | i \in I_n\}$ $(1 \leqq n < \infty)$ of projections with the following properties:

(1) if $Q_{ni}$ denotes the central support of $G_{ni}$, then $EQ_{ni} \prec G_{ni}Q_{ni}$ $(i \in I_n;\ n = 1, 2, \cdots)$;

(2) $\{Q_{ni} | i \in I_n\}$ is a mutually orthogonal set of sum $P'$;

(3) for each $i \in I_n$ there is a natural number $s = s(i) \geqq n$ with $G_{ni} = (F_s - F_{s+1})Q_{ni}$; and

(4) if $i \in I_m$, $j \in I_n$, and $Q_{mi}Q_{nj} \neq 0$ then $s(i) < s(j)$ whenever $m < n$.

Here $I_n$ is a countable indexing set with $I_m \cap I_n \neq \varnothing$ for $m \neq n$.

Indeed, at the $(n + 1) -$ st stage of the induction we work in algebras of the form $\mathscr{A}Q_{1i_1} \cdots Q_{ni_n}(i_j \in I_j)$ and then sum the appropriate pieces together by summing over those pieces corresponding to the same $s(i)$. Setting $G''_n = \sum \{G_{ni} | i \in I_n\}$, we obtain sequence of mutually orthogonal projections of central support $P'$ such that $EP' \prec G'_nP'$, $AG''_n = G''_nA$, and $||AG''_n|| \leqq n^{-1}$ for every $n = 1, 2, \cdots$. Setting $E_n = G'_n + G''_n$ for $n = 1, 2, \cdots$, we obtain a sequence $\{E_n\}$ of mutually orthogonal projections of central support 1 such that

$$E \prec E_n,\ AE_n(\mathscr{J}) = E_nA(\mathscr{J})\ , \quad \text{and} \quad ||AE_n(\mathscr{J})|| \leqq n^{-1}$$

for every $n$.

Conversely, let $\{E_n\}$ be a sequence of (not necessarily orthogonal) projections which satisfy the conditions of the proposition for the central element $A_0$. Suppose there is a $B$ in $\mathscr{A}$ with $B(A_0 - A)(\mathscr{J}(\zeta)) = 1 \neq 0$ for some $\zeta$ in the spectrum of the center. Then we have that

$$\|E_n(\mathscr{J}(\zeta))\| = \|B(A_0 - A)E_n(\mathscr{J}(\zeta))\| \leqq n^{-1}\|B\|,$$

for every $n = 1, 2, \cdots$ implies $\|E_n(\mathscr{J}(\zeta))\| = 0$ for all sufficiently large $n$. However, this means that $\|E_n(\mathscr{J}(\zeta'))\| = 0$ for all $\zeta'$ in a neighborhood of $\zeta$ since $\zeta' \to \|E_n(\mathscr{J}(\zeta'))\|$ is a continuous function of the spectrum of the center into $\{0, 1\}$ (Theorem 3.2). So there is a projection $P$ in the center with $P^\wedge(\zeta) = 1$ such that $E_nP \in \mathscr{J}$ (Lemma 3.1). But this contradicts the hypothesis that $\dim E_n > \dim \mathscr{J}$ (Proposition 2.9). Consequently, the element $A_0$ is in the essential spectrum of $A$ with respect to $\mathscr{J}$.

COROLLARY 3.15. *Let $\mathscr{A}$ be a von Neumann algebra with no finite type I direct summands* **and** *let $\mathscr{J}$ be a central ideal of $\mathscr{A}$. Then the essential central spectrum with respect to $\mathscr{J}$ of a self-adjoint element $A$ contains $A_0$ if and only if there is a sequence $\{E_n\}$ of mutually orthogonal projections of dimension greater than* $\dim \mathscr{J}$ *such that $\|(A_0 - A)E_n\| \leqq n^{-1}$ for every $n = 1, 2, \cdots$ and $A_0 = A_0P_{\mathscr{J}}$.*

*Proof.* There is no loss of generality in the assumption that $A_0 = 0$ since every element in the essential central spectrum of $A$ is self-adjoint. Then there are orthogonal projections $\{F_n\}$ such that $\dim F_n > \dim \mathscr{J}$, $AF_n(\mathscr{J}) = F_nA(\mathscr{J})$ and $\|AF_n(\mathscr{J})\| < (2n)^{-1}$ for every $n = 1, 2, \cdots$ (Proposition 3.14). For every $n$ there is a $B_n \in \mathscr{J}$ with $\|AF_n - B_nF_n\| < (2n)^{-1}$. There is a projection $G_n \in \mathscr{J}$ such that $G_n \leqq F_n$ and $\|B_nF_n(1 - G_n)\| \leqq (2n)^{-1}$. Let $E_n = F_n - G_n$. If $Q$ is a central projection with $QE_n \in \mathscr{J}$, then $QF_n \in \mathscr{J}$ and $QP_{\mathscr{J}} = 0$ and so $\dim E_n > \dim \mathscr{J}$ (Proposition 2.9.). But we have that

$$\|AE_n\| \leqq \|(A - B_n)E_n\| + \|B_nE_n\| \leqq n^{-1}.$$

Thus $\{E_n\}$ is the required sequence.

The converse is derived from Proposition 3.14 since $\|B(\mathscr{J})\| \leqq \|B\|$ for every $B \in \mathscr{A}$.

COROLLARY 3.16. *Let $\mathscr{A}$ be a von Neumann algebra with no finite type I direct summand and let $\mathscr{J}$ be a central ideal in $\mathscr{A}$. If the left-essential* (*resp. right-essential*) *central spectrum of an element $A$ in $\mathscr{A}$ contains $A_0$, then there is a sequence $\{E_n\}$ of orthogonal projections in $\mathscr{A}$ such that* $\dim E_n > \dim \mathscr{J}$ *and* $\|(A_0 - A)E_n\| \leqq n^{-1}$ (*resp.* $\|(A_0 - A)^*E_n\| \leqq n^{-1}$) *for every $n = 1, 2, \cdots$.*

*Proof.* Since the essential central spectrum of $(A_0 - A)^*(A_0 - A)$ contains 0, the Corollary 3.15 can be applied.

REMARK 3.17. If $\mathscr{A}$ is a finite type I algebra, $\mathscr{J}$ is a central ideal, $A \in \mathscr{A}$, and $A_0$ in the essential central spectrum of $A$ with respect to $\mathscr{J}$, then D. Deckard and C. Pearcy [6] showed that there is an abelian projection $E$ of central support $P_{\mathscr{J}}$ in $\mathscr{A}$ with $(A_0 - A)E = 0$.

## 4. The essential central range.

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. Then $\mathscr{A}$ may be considered as a module over $\mathscr{Z}$. Let $\mathscr{A}^{\sim}$ be the $\mathscr{Z}$-module of all bounded module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$ and let $\mathscr{A}^{\sim +}$ be the set of all elements of $\mathscr{A}^{\sim}$ which map $\mathscr{A}^{+}$ into $\mathscr{Z}^{+}$. For a central ideal $\mathscr{J}$ of $\mathscr{A}$, let $E_a(\mathscr{J}) = \{\phi \in \mathscr{A}^{\sim +} \mid \phi(\mathscr{J}) = (0) \text{ and } \phi(P_{\mathscr{J}}) = P_{\mathscr{J}}\}$. Here $P_{\mathscr{J}}$ is the orthogonal complement of the largest central projection in $\mathscr{J}$. We notice that $E_a(\mathscr{J})$ is the set of all *states* (i.e. elements $\phi$ of $\mathscr{A}^{\sim +}$ with $\phi(1) = 1\}$ of $\mathscr{A}^{\sim}$ which vanish on $\mathscr{J}$ whenever $P_{\mathscr{J}} = 1$, or equivalently, if $\mathscr{J} = \mathscr{J}_P(E)$ (Remark 2.6), whenever the central support of $E$ is equal to $P$ (Corollary 2.7). In particular, the set $E_a(\mathscr{J})$ is equal to the set of all states which vanish on $\mathscr{J}$ whenever $\mathscr{J}$ is the ideal generated by the set of all finite projections or $\mathscr{J}$ is the strong radical of a properly infinite von Neumann algebra (Examples 2.11 and 2.12). It is clear that $E_a(\mathscr{J})$ is compact in the topology of pointwise convergence on $\mathscr{A}$ where is $\mathscr{Z}$ taken with the weak topology, i.e., in the $\sigma_W(\mathscr{A}^{\sim}, \mathscr{A})$-topology of $\mathscr{A}^{\sim}$. If $\{\phi_i \mid i \in I\}$ is any subset of $E_a(\mathscr{J}\}$ and $\{P_i \mid i \in I\}$ is a set of orthogonal central projections of sum 1, then $\phi(A) = \sum P_i\phi_i(A)$ defines an element $\phi$ in $E_a(\mathscr{J})$. Furthermore, we see that $E_a(\mathscr{J})$ is *central-convex* in the sense that $C\phi_1 + (1 - C)\phi_2$ is in $E_a(\mathscr{J})$ for every $\phi_1$ and $\phi_2$ in $E_a(\mathscr{J})$ and $C$ in $\mathscr{Z}$ with $0 \leqq C \leqq 1$.

DEFINITION 4.1. *Let $\mathscr{A}$ be a von Neumann algebra, let $\mathscr{J}$ be a central ideal of $\mathscr{A}$, and let $A$ be an element of $\mathscr{A}$. The set $\mathscr{K}_{\mathscr{J}}(A) = \{\phi(A) \mid \phi \in E_a(\mathscr{J})\}$ will be called the* essential central range *of $A$ with respect to $\mathscr{J}$. We notice that $\mathscr{K}_{\mathscr{J}}(A)$ is a central-convex, weakly compact (and consequently uniformly closed) subset of the sphere in the center of $\mathscr{A}$ of radius $\|A\|$ about the origin.*

PROPOSITION 4.2. *Let $\mathscr{A}$ be a von Neumann algebra, let $\mathscr{J}$ be a central ideal of $\mathscr{A}$, and let $A$ be an element of $\mathscr{A}$. Then for every $\zeta$ in the spectrum of the center of $\mathscr{A}$, the set $\mathscr{K}_{\mathscr{J}}(A)(\zeta) = \{B^{\wedge}(\zeta) \mid B \in \mathscr{K}_{\mathscr{J}}(A)\}$ is a compact set of complex numbers.*

*Proof*. Since $\mathscr{K}_{\mathscr{J}}(A)(\zeta)$ is bounded, it is sufficient to show that $\mathscr{K}_{\mathscr{J}}(A)(\zeta)$ is closed. If $\alpha$ is the limit of a sequence $\{\phi_n(A)^{\wedge}(\zeta)\}$ where $\phi_n \in E_a(\mathscr{J})$ for every $n = 1, 2, \cdots$, we show that $\alpha \in \mathscr{K}_{\mathscr{J}}(A)(\zeta)$. There is no loss of generality in assuming that $\alpha = 0$. We may assume that $|\phi_n(A)^\wedge(\zeta)| < n^{-1}$ for every $n = 1, 2, \cdots$. There is a sequence $\{P_n\}$ of central projections with $\|\phi_n(A)P_n\| \leqq n^{-1}$, and $P_n^\wedge(\zeta) = 1$ for every $n = 1, 2, \cdots$. Let $Q_0 = \text{glb}\{P_1 \cdots P_n | n = 1, 2, \cdots\}$ and let $Q_1 = P_1(1 - P_2)$, $Q_2 = P_1P_2(1 - P_3)$, $\cdots$; then $\{Q_i | i = 0, 1, 2, \cdots\}$ is a sequence of orthogonal central projections of sum $P_1$. The homomorphism

$$\psi_n = (1 - P_1)\phi_1 + Q_0\phi_n + \sum\{Q_i\phi_i | i = 1, 2, \cdots\}$$

is an element of $E_a(\mathscr{J})$ and so

$$A_0 = \lim \psi_n(A) = (1 - P_1)\phi_1(A) + \sum\{Q_i\phi_i(A) | i = 1, 2, \cdots\}$$

is in $\mathscr{K}_{\mathscr{J}}(A)$. Since $(1 - P_1)^\wedge(\zeta) = 0$ and $Q_i^\wedge(\zeta) = 0$ for all $i \geqq 1$, either $Q_0^\wedge(\zeta) = 1$ or $\sum\{Q_i | i \geqq n\}^\wedge(\zeta) = 1$ for all $n = 1, 2, \cdots$. In either case $A_0^\wedge(\zeta) = 0$ since $\|\sum\{Q_i | i = 0, n, n + 1, \cdots\}A_0\| \leqq n^{-1}$. This means that $0 \in \mathscr{K}_{\mathscr{J}}(A)(\zeta)$.

We need the following lemma. Its proof is a simple reworking of [19; proof of corollary to (a5) implies (a1)].

Lemma 4.3. *Let $\mathscr{A}$ be a von Neumann algebra, let $\mathscr{J}$ be a central ideal of $\mathscr{A}$, and let $E$ be a projection in $\mathscr{A}$. There is positive module homomorphism of the module $\mathscr{A}$ into its center which vanishes on $\mathscr{J}$ and satisfies the relation $\phi(1) = \phi(E) = 1 - Q$ where $Q$ is the largest central projection of $\mathscr{A}$ such that $EQ \in \mathscr{J}$.*

THEOREM 4.4. *Let $\mathscr{A}$ be a von Neumann algebra. The essential central range of a self-adjoint element $A$ of $\mathscr{A}$ with respect to a central ideal $\mathscr{J}$ is the smallest central-convex subset of $\mathscr{A}$ which contains the essential central spectrum of $A$ with respect to $\mathscr{J}$.*

*Proof.* Let $\mathscr{Z}$ be the center of $\mathscr{A}$, let $\zeta$ be in the spectrum of $\mathscr{Z}$, and let $\phi$ be an element of $E_a(\mathscr{J})$. Let $\phi_\zeta$ be the bounded linear functional on $\mathscr{A}$ defined by $\phi_\zeta(B) = \phi(B)^\wedge(\zeta)$ for all $B \in \mathscr{A}$. If $B_1, \cdots, B_n$ are in $\mathscr{A}$ and $C_1, \cdots C_n$ are in $\zeta$, then

$$\phi_\zeta(\sum B_iC_i) = \sum C_i^\wedge(\zeta)\phi_\zeta(B_i) = 0 .$$

This proves that $\phi_\zeta$ vanishes on a dense subset of $[\zeta]$ and so vanishes on $[\zeta]$. Hence $\phi_\zeta$ vanishes on $\mathscr{J}(\zeta)$. Now let $C_l = \text{glb}\, \mathscr{Z} - \text{Sp}_{\mathscr{J}} A$ and $C_u = \text{lub}\, \mathscr{Z} - \text{Sp}_{\mathscr{J}} A$. We have that $C_l^\wedge(\zeta) \leqq A(\mathscr{J}(\zeta)) \leqq C_u^\wedge(\zeta)$ in $\mathscr{A}(\mathscr{J}(\zeta))$. (Proposition 3.12). This means that

$$\phi(C_l)^\wedge(\zeta) = C_l^\wedge(\zeta) \leqq \phi(A)^\wedge(\zeta) \leqq C_u^\wedge(\zeta) = \phi(C_u)^\wedge(\zeta)$$

for all $\zeta$ with $P_{\mathscr{J}}^\wedge(\zeta) = 1$. Consequently, we have that $C_l = C_lP_{\mathscr{J}} \leqq \phi(A) \leqq C_uP_{\mathscr{J}} = C_u$. So we may find a $C$ in $\mathscr{Z}$ with $0 \leqq C \leqq 1$ such that $CC_l + (1 - C)C_u = \phi(A)$. Hence, the smallest central convex set containing $\mathscr{Z} - \mathrm{Sp}_{\mathscr{J}}A$ contains $\mathscr{K}_{\mathscr{J}}(A)$.

Conversely, to show the opposite relation we simply must show that $C_l$ and $C_u$ are in $\mathscr{K}_{\mathscr{J}}(A)$. We work with $C_u$. Given $\varepsilon > 0$, there is a projection $E$ in $\mathscr{A}$ such that $E$ commutes with $A$, $\|(C_u - A)E\| < \varepsilon$, and if $EP$ is in $\mathscr{J}$ for a central projection $P$ then $P$ is in $\mathscr{J}$ (Proposition 3.13). There is a $\phi$ in $E_a(\mathscr{J})$ such that $\phi(E) = P_{\mathscr{J}}$ (Lemma 4.3). From the Cauchy-Schwarz inequality for elements of $A^{\sim +}$, we obtain

$$\begin{aligned}\|\phi(A) - C_u\| = \|\phi(A - C_u)\| = \\ \|\phi((A - C_u)E)\| + \|\phi((A - C_u)(1 - E))\| \\ \leqq \varepsilon + \|A - C_u\|\,\|\phi(1 - E)\| = \varepsilon\,.\end{aligned}$$

Because $\mathscr{K}_{\mathscr{J}}(A)$ is uniformly closed and because $\varepsilon > 0$ is arbitrary, we have that $C_u \in \mathscr{K}_{\mathscr{J}}(A)$. By a similar argument $C_l \in \mathscr{K}_{\mathscr{J}}(A)$.

COROLLARY 4.5. *Let $\mathscr{A}$ be a von Neumann algebra. The essential central range of an element $A$ in $\mathscr{A}$ with respect to a central ideal $\mathscr{J}$ is equal to a set $\{A_0\}$ if and only if $A_0P_{\mathscr{J}} = A_0$ and $A - A_0 \in \mathscr{J}$.*

*Proof.* First let the essential central range $\mathscr{K}_{\mathscr{J}}(A)$ of $A$ be equal to $A_0$. Then $\phi(A) = A_0$ for every $\phi \in E_a(\mathscr{J})$. Hence $\phi(A + A^*) = A_0 + A_0^*$ for every $\phi \in E_a(\mathscr{J})$. This means that the essential central spectrum of $A + A^* - (A_0 + A_0^*)$ with respect to the ideal $\mathscr{J}$ is equal to $\{0\}$ (Theorem 4.4). Hence $A + A^* - (A_0 + A_0^*) \in \mathscr{J}$ (Proposition 3.12 and Lemma 3.1). Similarly we find that $(A - A^*) - (A_0 - A_0^*) \in \mathscr{J}$. Consequently, we have that $A - A_0 \in \mathscr{J}$.

The converse is obvious.

The following remarks lead to a characterization of the essential central range. This reduces to the known characterization of the essential numerical range of the algebra of all bounded operators on a separable Hilbert space [11; 5.1]. Let $\mathscr{A}$ be a von Neumann algebra on the Hilbert space $H$ and let $\mathscr{Z}$ be the center of $\mathscr{A}$. Let $E$ be an abelian projection with central support $P$ in the commutant $\mathscr{Z}'$ of $\mathscr{Z}$ [cf. 9; I, §7]. For every $A \in \mathscr{A}$, there is a unique $\tau_E(A)$ in $\mathscr{Z}P$ with $EAE = \tau_E(A)E$. Then $A \to \tau_E(A)$ defines an element in $\mathscr{A}^{\sim +}$ with $\tau_E(1) = P$. For every projection $P$ in $\mathscr{Z}$ let $V_P(\mathscr{A}) = \{\tau_E \in \mathscr{A}^{\sim} \mid E$ is an abelian projection in $\mathscr{Z}'$ of central support $P\}$; for every $A \in \mathscr{A}$ let $W_P(A) =$ uniform closure $\{\phi(A) \mid \phi \in V_P(\mathscr{A})\}$.

We now need a version of the Toeplitz-Hausdorff Theorem.

LEMMA 4.6. *Let $\mathscr{A}$ be a von Neumann algebra. Then, for* *every* $A \in \mathscr{A}$ *and central projection* $P$, *the set* $\{\phi(A) \mid \phi \in V_P(\mathscr{A})\}$ *is central-convex.*

*Proof.* There is no loss of generality in the assumption that $P = 1$. Let $E_1$ and $E_2$ be maximal abelian projections (i.e. abelian projections with central support 1) in the commutant $\mathscr{Z}'$ of the center $\mathscr{Z}$ of $\mathscr{A}$ and let $C \in \mathscr{Z}$ with $0 \leqq C \leqq 1$. Setting $E = \text{lub}\{E_1, E_2\}$, we obtain a projection $E$ such that the reduced algebra $\mathscr{Z}'_E$ is the product of homogeneous algebras of type $I_n$ where $n \leqq 2$. Indeed, we have that $\text{lub}\{E_1, E_2\} - E_1 \prec E_1$ and so $\text{lub}\{E_1, E_2\} - E_1$ is abelian. So there is no loss of generality in the assumption that $\mathscr{A} = \mathscr{Z}'$ is homogeneous of degree 2 since the degee 1 case requires no further proof. Now we may write $\phi(B) = C\tau_{E_1}(B) + (1 - C)\tau_{E_2}(B)$ as $\phi(B) = A_1\tau_{F_1}(B) + A_2\tau_{F_2}(B)$ where $F_1$, $F_2$ are orthogonal maximal abelian projections of sum 1 and $A_1$, $A_2$ are elements in $\mathscr{Z}^+$ with $A_1 + A_2 = 1$ [14; §4]. So we may assume that $E_1$ and $E_2$ are orthogonal of sum 1. Let $\tau_i = \tau_{E_i}$ $(i = 1, 2)$. Since it is sufficient to find a maximal abelian projection $E$ with $\tau_E(A - \tau_2(A)) = C\tau_1(A - \tau_2(A))$, we may assume that $\tau_2(A) = 0$. Now there is a sequence $\{P_n\}$ of orthogonal projections in $\mathscr{Z}$ such that $\tau_1(A)P_n$ is invertible in $\mathscr{Z}P_n$ and $\tau_1(A)(1 - \sum P_n) = 0$. Because the sum of abelian projections with orthogonal central supports is again abelian, there is no loss of generality in the assumption that $\tau_1(A) = 1$.

The rest of this lemma is the classical Toeplitz-Hausdorff theorem. Let $U$ be a partial isometry of $\mathscr{A}$ with $U^*U = E_1$ and $UU^* = E_2$ and let $A = E_1 + A_1U + A_2U^*$, where $A_1, A_2 \in \mathscr{Z}$. There is a unitary operator $V$ in $\mathscr{Z}$ with $V|A_1 - A_2^*| = A_1 - A_2^*$. Let $T = V^*A_1 + VA_2$. There is a $D \in \mathscr{Z}$ with $-1 \leqq D \leqq 1$ such that

$$D^2 + D(1 - D^2)^{1/2}T = C$$

[6]. Now, by direct calculation, we find that

$$E = D^2E_1 + VD(1 - D^2)^{1/2}U + V^*D(1 - D^2)^{1/2}U^* + (1 - D^2)E_2$$

is a projection in $\mathscr{A}$ of central support 1 that vanishes on the range of $(1 - C^2)^{1/2}E_1 - VCE_2$. So $E$ must be a maximal abelian projection. Finally, by another calculation, we obtain that $EAE = CE$.

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$. Let $\mathscr{A}$ be considered as a $\mathscr{Z}$-module and let $\mathscr{A}_\sim$ be the $\mathscr{Z}$-module of all $\sigma$-weakly continuous module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$. Let $\mathscr{A}_\sim^+ = \mathscr{A}^{\sim+} \cap \mathscr{A}_\sim$ be the set of all *normal* (i.e. positive $\sigma$-weakly continuous) module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$.

Now we can extend Lemma 4.6.

LEMMA 4.7. *Let $\mathscr{A}$ be a von Neumann algebra, let $P$ be a central projection of $\mathscr{A}$, and let $A \in \mathscr{A}$; then*

$$W_P(A) = \{\phi(A) \mid \phi \in \mathscr{A}_{\sim}^{+}, \phi(1) = P\} .$$

*Proof.* First let $\phi \in \mathscr{A}_{\sim}^{+}$ with $\phi(1) = P$. We show that $\phi(A) \in W_P(A)$. There is a monotonely decreasing sequence $\{A_n\}$ of positive elements in the center $\mathscr{Z}$ of $\mathscr{A}$, and a sequence $\{E_n\}$ of orthogonal abelian projections in the commutant $\mathscr{Z}'$ of $\mathscr{Z}$ with central supports $\{P_n\}$ respectively such that $\lim A_n = 0$ (uniformly), $\sum A_n = P$ (strongly), $E_{n+1} \prec E_n$, $\operatorname{supp} P_n^{\wedge} = \operatorname{supp} A_n^{\wedge}$ $(n = 1, 2, \cdots)$, and $\phi(B) = \sum A_n \tau_{E_n}(B)$ (strongly) for all $B \in \mathscr{A}$ ([16; Theorem 2] and [14; §4]). There is a mutually orthogonal set $\{Q_i\}$ in $(\mathscr{Z})$ of sum $P$ such that

$$\lim_m \sum \{A_n Q_i \mid 1 \leqq n \leqq m\} = PQ_i$$

uniformly (cf. [14, Theorem 4.1]). For each $Q_i$ we may therefore find an $m_i$ with $\|B_i Q_i\| \leqq \varepsilon$, where $B_i = \sum \{A_n \mid n \geqq m_i\}$ and where $\varepsilon > 0$ is a preassigned constant. Now there are abelian projections $F_k (1 \leqq k \leqq m_i = m)$ of central support $PQ_i$ such that $E_k Q_i \leqq F_k$. Since $\operatorname{supp} P_k^{\wedge} = \operatorname{supp} A_k^{\wedge}$, we have that $\phi_i = \sum \{A_k \tau_{F_k} \mid 1 \leqq k \leqq m-1\} + B_i \tau_{F_m}$ is equal to $\sum \{A_k Q_i \tau_{E_k} \mid 1 \leqq k \leqq m-1\} + B_i \tau_{F_m}$. Since $\sum \{A_k Q_i \mid 1 \leqq k \leqq m-1\} + B_i Q_i = PQ_i$, there is an abelian projection $G_i$ in $\mathscr{Z}'$ of central support $PQ_i$ such that $\tau_{G_i}(A) = \phi_i(A)$ (Lemma 4.6). Notice that

$$\|(\phi_i(A) - \phi(A))Q_i\| \leqq \|B_i \tau_{F_m}(A)\| + \|\sum \{A_n \tau_{E_n}(A) \mid n \geqq m\} Q_i\| \leqq 2\varepsilon \|A\| .$$

Now $\sum G_i = G$ is an abelian projection of central support $P$ and

$$\|\phi(A) - \tau_G(A)\| \leqq \operatorname{lub} \|(\phi(A) - \tau_{G_i}(A))Q_i\| \leqq 2\varepsilon \|A\| .$$

So $\phi(A) \in W_P(A)$ since $\varepsilon > 0$ is arbitrary and $W_P(A)$ is closed.

The converse relation is obvious since $\tau_E$ is a normal module homomorphism.

PROPOSITION 4.8. *Let $\mathscr{A}$ be a von Neumann algebra. Then the essential central range of an element $A$ in $\mathscr{A}$ with respect to the central ideal $\mathscr{J}$ is equal to $\cap \{W_P(A+B) \mid B \in \mathscr{J}\}$. Here $P = P_{\mathscr{J}}$.*

*Proof.* Let $\phi \in E_a(\mathscr{J})$. Let $Q$ be the central projection in $\mathscr{A}$ such that $\mathscr{A}Q$ is a discrete algebra and $\mathscr{A}(1-Q)$ is a continuous algebra. There is a net $\{\phi_n'\}$ (resp. $\{\phi_m''\}$) of elements of $(\mathscr{A}Q)_{\sim}^{+}$ (resp. $(\mathscr{A}(1-Q))_{\sim}^{+}$) with $\phi_n'(Q) = P_{\mathscr{J}} Q$ (resp. $\phi_m''(1-Q) = P_{\mathscr{J}}(1-Q)$) such that $\lim \phi_n(BQ) = \phi(BQ)$ (resp. $\lim \phi_m(B(1-Q)) = \phi(B(1-Q))$) uniformly for every $B \in \mathscr{A}$. This follows from Theorem 5.4 (resp. Theorem 5.1) of [17]. Then setting $\phi_{nm}(B) = \phi_n'(BQ) + \phi_m''(B(1-Q))$, we obtain a net $\{\phi_{nm}\}$ in $\mathscr{A}_{\sim}^{+}$ with $\phi_{nm}(1) = P_{\mathscr{J}}$ for all $m$, $n$ and $\lim \phi_{nm}(B) = \phi(B)$ (uniformly) for all $B \in \mathscr{A}$. Let $B \in \mathscr{J}$ and let $\varepsilon > 0$; then there is a $\phi_{mn}$ with $\|\phi_{mn}(B))\| \leqq \varepsilon$ and $\|\phi_{mn}(A) - \phi(A)\| \leqq \varepsilon$ since $\phi(B) = 0$. Since $\varepsilon > 0$ is arbitrary and since $W_P(A + B)$ is closed, we have that $\phi(A) \in W_P(A + B)$ by Lemma 4.7. Since $B \in \mathscr{J}$ is arbitrary $\phi(A) \in \cap \{W_P(A + B) | B \in \mathscr{J}\}$. So $\mathscr{K}_{\mathscr{J}}(A) \subset \cap \{W_P(A + B) | B \in \mathscr{J}\}$.

We now prove that the opposite inclusion relation is true. First let $A$ be self-adjoint. We show that $0 \in \cap \{W_P(A + B) | B = B^* \in \mathscr{J}\}$ implies that $0 \in \mathscr{K}_{\mathscr{J}}(A)$. Let $\mathscr{Z}$ be the center of $\mathscr{A}$ and let $C_u = \text{lub}\,\mathscr{Z} - \text{Sp}_{\mathscr{J}}A$. Suppose there is an $\alpha > 0$ and a nonzero projection $Q$ in $\mathscr{Z}$ with $Q \leqq P$ and $C_u Q \leqq -2\alpha Q$. We have that $(C_u - A)(\mathscr{J}(\zeta)) \geqq 0$ for every $\zeta$ in the spectrum of $\mathscr{Z}$ (Proposition 3.12). If $f_+$ (resp. $-f_-$) is the function that is identity on the real interval $[0, \infty)$ (resp. $(-\infty, 0]$) and 0 on the complement, we have that $f_-(C_u - A)$ is a self-adjoint element in $\mathscr{J}$ (Lemma 3.1). However, by hypothesis there is an abelian projection $E$ in $\mathscr{Z}'$ of central support $P$ with $\|\tau_E(f_-(C_u - A) - A)\| \leqq \alpha$. On the other hand, we have that

$$Q\tau_E(f_-(C_u - A) - A) = Q\tau_E(f_+(C_u - A) - C_u) \geqq 2\alpha Q\ .$$

This is a contradiction. Hence, we find that $C_u P \geqq 0$. Since $1 - P \in \mathscr{J}$, we have that $C_u(1-P) = 0$ and so $C_u \geqq 0$ (cf. Remark 3.8). Similarly, we obtain $C_l = \text{glb}\,\mathscr{Z} - \text{Sp}_{\mathscr{J}}(A) \leqq 0$ and finally that $0 \in \mathscr{K}_{\mathscr{J}}(A)$ (Theorem 4.4).

Now let $A$ be an arbitrary element of $\mathscr{A}$ with $0 \in \cap \{W_P\{A + B) | B \in \mathscr{J}\}$. Let $\mathscr{S} = \{|B| = (B^*B)^{1/2} | B \in \mathscr{K}_{\mathscr{J}}(A)\}$. We note that $\mathscr{S}$ is a monotonely decreasing net in $\mathscr{Z}^+$. Indeed, let $B$ and $C$ be in $\mathscr{K}_{\mathscr{J}}(A)$. There is a central projection $Q$ with $Q|B| + (1 - Q)|C| = \text{glb}\{|B|, |C|\}$. But the set $\mathscr{K}_{\mathscr{J}}(A)$ is central-convex and so $\mathscr{K}_{\mathscr{J}}(A)$ contains $D = QB + (1 - Q)C$. Thus, we have that $|D| = Q|B| + (1 - Q)|C|$ is in $\mathscr{S}$. Thus $\mathscr{S}$ has a greatest lower bound $B_0$ in $\mathscr{Z}^+$. We show $B_0 = 0$ by arguing by contradiction. Suppose there is a point $\zeta$ in the spectrum $Z$ of $\mathscr{Z}$ with $B_0^\wedge(\zeta) > 0$. Then we may assume that $B_0^\wedge(\zeta) = \text{glb}\{C^\wedge(\zeta) | C \in \mathscr{S}\}$ since $B_0^\wedge(\zeta) = \text{glb}\{C^\wedge(\zeta) | C \in \mathscr{S}\}$ holds on a dense open set of $Z$ [7]. There is a $C \in \mathscr{K}_{\mathscr{J}}(A)$ such that $|C^\wedge(\zeta)| = B_0^\wedge(\zeta)$ (Proposition 4.2). Then we may find a unitary $U$ in $\mathscr{Z}$ such that $UC = |C|$. We have that $0 \in \cap \{W_P(UA + B) | B \in \mathscr{J}\}$ since $U(W_P(A + U^*B)) = W_P(UA + B)$ and that $B_0 = \text{glb}\{|B| \,| B \in \mathscr{K}_{\mathscr{J}}(UA)\}$ since $\mathscr{K}_{\mathscr{J}}(UA) = U\mathscr{K}_{\mathscr{J}}(A)$. Furthermore, we have that $|C| \in \mathscr{K}_{\mathscr{J}}(UA)$. Hence, there is no loss in generality in assuming that there is a $C \in \mathscr{K}_{\mathscr{J}}(A)$ with $C^\wedge(\zeta) = B_0^\wedge(\zeta)$. Now let $A_1 = (A + A^*)/2$ and $A_2 = (A - A^*)/2i$. We show that $0 \in \cap \{W_P(A_j + B) | B = B^* \in \mathscr{J}\}$ $(1 \leqq j \leqq 2)$. In fact, given $\varepsilon > 0$ and $B = B^* \in \mathscr{J}$, there is an abelian projection $E$

with central support $P$ in the commutant of $\mathscr{Z}$ such that $\|\tau_E(A+B)\| \leqq \varepsilon$. Hence, we have that

$$\|\tau_E(A + A^* + 2B)\| = \|\tau_E(A + B) + \tau_E(A + B)^*\| \leqq 2\varepsilon .$$

Similarly, we may find an abelian projection $F$ of central support $P$ such that $\|\tau_F(A - A^* + 2iB)\| \leqq 2\varepsilon$. Now by the preceding paragraph we conclude that $0 \in \mathscr{K}_{\mathscr{J}}(A_j)$ $(1 \leqq j \leqq 2)$. Let $\phi$ be an element of $\mathscr{A}^{\sim+}$ with $\phi(1) = P$, $\phi(\mathscr{J}) = 0$, and $\phi(A_1) = 0$. However, every element of the form $\alpha\phi(A) + (1 - \alpha)C$ $(0 \leqq \alpha \leqq 1)$ is in $\mathscr{K}_{\mathscr{J}}(A)$ and so there is at least one $\alpha$ with $0 \leqq \alpha \leqq 1$ such that

$$|\alpha\phi(A)^\wedge(\zeta) + (1 - \alpha)C^\wedge(\zeta)| < C^\wedge(\zeta) = B_0^\wedge(\zeta) .$$

Indeed $\phi(A)^\wedge(\zeta)$ is pure imaginary. This contradicts the choice of $B_0$. Hence, we must have that $0 \in \mathscr{K}_{\mathscr{J}}(A)$.

PROPOSITION 4.9. *Let $\mathscr{A}$ be a von Neumann algebra; then $A_0$ is in the essential central range of $A \in \mathscr{A}$ with respect to the central ideal $\mathscr{J}$ if $A_0P_{\mathscr{J}} = A_0$ and if, given $\varepsilon > 0$, there is a projection $E$ with* $\dim E > \dim \mathscr{J}$ *such that $\|E(A_0 - A)E\| \leqq \varepsilon$. Conversely, if $A \in \mathscr{A}$ is self-adjoint and if $A_0$ is in the essential central range of $A$ with respect to $\mathscr{J}$, then there is a projection $E$ in $\mathscr{A}$ with* $\dim E >$ $\dim \mathscr{J}$ *such that $\|E(A_0 - A)E\| \leqq \varepsilon$.*

*Proof.* The first statement follows from Lemma 4.3 and Proposition 2.9 since the essential central range $\mathscr{K}_{\mathscr{J}}(A)$ of $A$ with respect to $\mathscr{J}$ is uniformly closed.

Now let $A$ be self-adjoint and let $A_0 \in \mathscr{K}_{\mathscr{J}}(A)$. There is no loss of generality in assuming at the outset that $A_0 = 0$ and that $P_{\mathscr{J}} = 1$. Let $\mathscr{J}$ have the canonical form $\mathscr{J} = \mathscr{J}_P(F)$ (Remark 2.6). Let $\varepsilon > 0$ be given. Let $C_l = \operatorname{glb} \mathscr{Z} - \mathrm{Sp}_{\mathscr{J}} A$ and let $C_u = \operatorname{lub} \mathscr{Z} - \mathrm{Sp}_{\mathscr{J}} A$ where $\mathscr{Z}$ is the center of $\mathscr{A}$. Since $0 \in \mathscr{K}_{\mathscr{J}}(A)$, we have that $C_l \leqq 0 \leqq C_u$ (Theorem 4.4).

Now let $R$ be the largest central projection such that $\mathscr{A}R$ is of type $I$ and $\mathscr{J}R = 0$. Consequently, if $G$ is a finite type $I$ projection majorized by $1 - R$, then $G \in \mathscr{J}(1 - R)$ (Proposition 2.2). By Proposition 3.10 we may assume that either $R = 1$ or $1 - R = 1$.

First suppose theat $R = 1$. We may assume that $\mathscr{A}$ is equal the commutant of its center [9; I, 8, Theorem 1]. Then there are abelian projections $E_1$ and $E_2$ of central support 1 in $\mathscr{A}$ such that

$$\|\tau_{E_1}(A) - C_l\| + \|\tau_{E_2}(A) - C_u\| \leqq \varepsilon .$$

(Theorem 4.4 and Proposition 4.8). There is a $C$ in $\mathscr{Z}$ with $0 \leqq C \leqq 1$ such that $CC_l + (1 - C)C_u = 0$, and there is an abelian projection $E$

of central support 1 in $\mathscr{A}$ such that $\tau_E(A) = C\tau_{E_1}(A) + (1 - C)\tau_{E_2}(A)$ (Lemma 4.6). Thus, we obtain

$$||EAE|| \leqq ||C||\,||\tau_{E_1}(A) - C_l|| + ||1 - C||\,||\tau_{E_2}(A) - C_u|| \leqq \varepsilon .$$

So we may assume that $1 - R = 1$. Because the closure of every open subset of the spectrum $Z$ of $\mathscr{Z}$ is open, we may find a sequence $\{P_n \mid n = 0, 1, 2, \cdots\}$ of mutually orthogonal central projections of sum 1 such that

$$C_l P_n \leqq -n^{-1}P_n < 0 < n^{-1}P_n \leqq C_u P_n$$

for $n = 1, 2, \cdots$, and $C_l\ C_u\ P_0 = 0$. We shall find projections $E_n$ of central support $P_n$ such that $FP_n \prec E_n$ and $||E_nAE_n|| \leqq 4\varepsilon$. Then we shall have that $E = \sum E_n$ has central support 1, $F \prec E$, and $||EAE|| = \text{lub}_n ||E_nAE_nP_n|| \leqq 4\varepsilon$ (cf. [9, III, §1]). Now, we have that $\mathscr{I}_n = \mathscr{I} P_n = \mathscr{I}_{PP_n}(EP_n)$ is a representation of the central ideal $\mathscr{I}_n$ of $\mathscr{A} P_n$ in canonical form. Since $C_lC_uP_0 = 0$, there is a $P_0'$ in $(\mathscr{Z} P_0)$ with $P_0'C_l + (P_0 - P_0')C_u = 0$ (Lemma 3.4). Thus, we see that $0 \in \mathscr{Z} P_0 - \text{Sp}_{\mathscr{I}_0}(AP_n)$ (Proposition 3.10) and so we may find the projection $E_0$ (Proposition 3.13). By reducing to an algebra $\mathscr{A} P_n$, we may assume that $C_l \leqq -\alpha < 0 < \alpha \leqq C_u$ (Proposition 3.10).

It is sufficient to show that every nonzero $Q \in (\mathscr{Z})$ majorizes a nonzero $R \in (\mathscr{Z})$ such that there is a $G \in (\mathscr{A})$ of central support $R$ with $FR \prec G$ and $||GAG|| \leqq 4\varepsilon$. Then the usual maximality argument for the projections $R$ may be employed to find the projection $E_n$. By making yet another reduction to a direct summand of $\mathscr{A}$, we may assume, without loss of generality, that there are natural numbers $m$, $n$, and $p$ such that

$$||mp^{-1} + C_l|| \leqq p^{-1} \leqq \varepsilon \quad \text{and} \quad ||np^{-1} - C_u|| \leqq \varepsilon .$$

We now find $n$ (resp. $m$) orthogonal projections $F_i$ of dimension greater than $\dim \mathscr{I}$ such that $||(C_l - A)F_i|| \leqq \varepsilon$ (resp. $||(C_u - A)F_i|| \leqq \varepsilon$). We normally would apply Proposition 3.14, however it is necessary for the combined set of $m + n$ projections to be orthogonal and so the following additional argument is required. Let $A_1$ and $A_2$ be elements of $\mathscr{A}^+$ such that $A_1 - A_2 = A$ and $A_1A_2 = 0$. For every $\zeta \in Z$, we have that $-C_l^\wedge(\zeta) = ||A_2(\mathscr{I}(\zeta))||$ and $C_u^\wedge(\zeta) = ||A_1(\mathscr{I}(\zeta))||$ (Proposition 3.12). Let $G_1$ and $G_2$ be the domain projections of $A_1$ and $A_2$, respectively. For definiteness, let $G = G_1$. If $Q$ is a central projection with $GQ \in \mathscr{I}$, then $Q = 0$; otherwise, there is a $\zeta \in Z$ with $G(\mathscr{I}(\zeta)) = GQ(\mathscr{I}(\zeta)) = 0$ and consequently with $||A_1(\mathscr{I}(\zeta))|| = 0$. This implies that $\dim G > \dim \mathscr{I}$. So there is a projection $G'$ with $F \sim G' \leqq G$. We now restrict $\mathscr{A}$ to the subspace of the Hilbert space determined by $G$ to obtain the von Neumann algebra $\mathscr{A}_G =$

$G\mathscr{A}G$ (cf. [9; I, § 2]). The set $G\mathscr{J}G = \mathscr{J}_G$ is easily seen to be a central ideal of $\mathscr{A}_G$ since the center of $\mathscr{A}_G$ is $\mathscr{Z}G = \mathscr{Z}_G$ [9; III, 5, Problem 7]. We have that $A_1 \in \mathscr{A}_G^+$. Since the spectrum of $\mathscr{Z}_G$ is the set $X = \{\zeta G \mid \zeta \in Z\}$, we have that the smallest ideal $[\zeta G]$ of $\mathscr{A}_G$ which contains $\zeta G$ is $G[\zeta]G$. We may now easily show that

$$||A_1(\mathscr{J}_G + [\zeta G])|| = \inf\{||A + B + C|| \mid B \in \mathscr{J}_G, C \in [\zeta G]\}$$

is equal to $||A_1(\mathscr{J}(\zeta))||$ for every $\zeta \in Z$. This means that $C_u G =$ lub $\mathscr{Z}_G - \mathrm{Sp}_{\mathscr{J}_G} A_1$ (Proposition 3.12). Therefore, we may find a set $F_1, F_2, \cdots, F_m$ of mutually orthogonal projections in $\mathscr{A}_G$ of dimension greater than $\dim \mathscr{J}_G$ such that

$$A_1F_i(\mathscr{J}_G) = F_iA_1(\mathscr{J}_G) \quad \text{and} \quad ||(C_uG - A_1)F_i(\mathscr{J}_G)|| \leqq \varepsilon$$

for every $i = 1, 2, \cdots, m$ (Proposition 3.14). Indeed, the algebra $\mathscr{A}_G$ has no finite type I direct summands. Thus, we may find orthogonal projections $F_1, F_2, \cdots, F_m$ majorized in $\mathscr{A}$ by $G = G_1$ such that

$$F \sim G' \prec F_i,\ AF_i(\mathscr{J}) = F_iA(\mathscr{J})\ , \quad \text{and} \quad ||(C_u - A)F_i(\mathscr{J})|| \leqq \varepsilon$$

for every $i = 1, 2, \cdots, m$. Likewise, we may find orthogonal projections $F_{m+1}, \cdots, F_{m+n}$ majorized by $G_2$ such that $F \prec F_i$, $AF_i(\mathscr{J}) = F_iA(\mathscr{J})$, and $||(C_l - A)F_i(\mathscr{J})|| \leqq \varepsilon$ for every $i = m + 1, \cdots, m + n$. Since $G_1$ and $G_2$ are orthogonal, the projections $F_1, \cdots, F_{m+n}$ are mutually orthogonal. There are partial isometries $U_{ij} (1 \leqq i, j \leqq m+n)$ of $\mathscr{A}$ which satisfy the following properties:

(1) $U_{ij}U_{kl} = \delta_{il}U_{kj}$ ($\delta =$ Kronecker delta);

(2) $U_{ij} = U_{ji}^*$; and

(3) $U_{ii}$ is a projection with $F \sim U_{ii} \prec F_i$, for all $i, j, k, l$.

The element $E' = (m + n)^{-1} \sum U_{ij}$ is a projection in $\mathscr{A}$ with $E' \sim F$, i.e. $\dim E' > \dim \mathscr{J}$. Here, indeed, a calculation using $(m + n) \times (m + n)$ complex matrices suffices. Furthermore, using the fact that $AF_i(\mathscr{J}) = F_iA(\mathscr{J})$ for every $i$, we have that

$$\begin{aligned}
&||E'AE'(\mathscr{J})|| \\
&\quad = ||E'(A - (C_u \textstyle\sum \{U_{ii} \mid i \leqq m\} + C_l \sum \{U_{ii} \mid i > m\}))E'(\mathscr{J})|| \\
&\qquad + ||E'(\textstyle\sum \{C_uU_{ii} \mid i \leqq m\} + \sum \{C_lU_{ii} \mid i > m\})E'(\mathscr{J})|| \\
&\quad \leqq ||(m + n)^{-2} \textstyle\sum_{ij} (\sum \{U_{ki}(A - C_u)U_{jk} \mid k \leqq m\} \\
&\qquad + \textstyle\sum \{U_{ki}(A - C_l)U_{jk} \mid k > m\})(\mathscr{J})|| \\
&\qquad + ||(m + n)^{-1}(mC_u + nC_l)E'(\mathscr{J})|| \leqq 2\varepsilon\ .
\end{aligned}$$

Now there is a $B \in \mathscr{J}$ with $||E'AE' - E'BE'|| \leqq 3\varepsilon$. In the ideal $\mathscr{J}$, we may find a spectral projection $E''$ for $E'B^*BE'$ majorized by $E'$ so that $||BE'(1 - E'')|| \leqq \varepsilon$. If $Q(E' - E'') \in \mathscr{J}$ for some $Q \in (\mathscr{Z})$, then $QE' \in \mathscr{J}$ and consequently $Q = 0$. This means that

$$\dim (E' - E'') > \dim \mathscr{J}$$

(Proposition 2.9). Setting $E = E' - E''$, we obtain the relation

$$\|EAE\| \leqq \|E(A - B)E\| + \|EBE\| \leqq 4\varepsilon .$$

REMARK 4.10. If $\mathscr{A}$ is the algebra of all bounded operators on a separable Hilbert space $H$ and $\mathscr{J}$ is the ideal of completely continuous operators, then Fillmore, Stampfli, and Williams [11, Theorem 5.1, Corollary] have obtained Proposition 4.8 without the added restriction that $A$ is self-adjoint. The theorem of Fillmore, et al., depends on properties of Hilbert-Schmidt operators on separable $H$; however, it is likely that the restriction can also be removed here.

Let $\mathscr{A}$ be a von Neumann algebra. Let $U(\mathscr{A})$ be the group of unitary operators of $\mathscr{A}$ and let $\mathscr{E}$ be the set of positive real-valued functions $f$ of finite support such that $\sum \{f(U) \mid U \in U(\mathscr{A})\} = 1$. For each $f \in \mathscr{E}$ and $A$ in $\mathscr{A}$, let $f \cdot A = \sum \{f(U)U^*AU \mid U \in U(\mathscr{A})\}$ and let $\mathscr{K}'(A)$ be the uniform closure of $\{f \cdot A \mid f \in \mathscr{E}\}$. If $B \in \mathscr{K}'(A)$, then $\mathscr{K}'(B) \subset \mathscr{K}'(A)$. Then the intersection $\mathscr{K}(A)$ of $\mathscr{K}'(A)$ with the center is a nonvoid closed convex subset of the center ([8]; cf. also [9; III, §5]). Furthermore the set $\mathscr{K}'(A)$ (resp. $\mathscr{K}(A)$) is central-convex in the sense that $CC_1 + (1 - C)C_2$ is in $\mathscr{K}'(A)$ (resp. $\mathscr{K}(A)$) for every $C_1$ and $C_2$ in $\mathscr{K}'(A)$ (resp. $\mathscr{K}(A)$) and $C$ in the center with $0 \leqq C \leqq 1$ [19; proof, Lemma 6].

The following forms the basis for our analysis of $\mathscr{K}(A)$.

PROPOSITION 4.11. *Let $\mathscr{A}$ be a von Neuman algebra and let $A$ be an element in $\mathscr{A}$. Let $\zeta$ be a point in the spectrum of the center of $\mathscr{A}$. Then the set $\mathscr{K}(A)(\zeta) = \{B^\wedge(\zeta) \mid B \in \mathscr{K}(A)\}$ is a compact subset of the complex plane.*

*Proof.* Because $\mathscr{K}(A)(\zeta)$ is bounded, it is sufficient to show that $\mathscr{K}(A)(\zeta)$ contains an arbitrary limit point $\alpha$. Due to the fact that $\mathscr{K}(A - \alpha)(\zeta) = \mathscr{K}(A)(\zeta) - \alpha$, there is no loss in generality in proving that $0 \in \mathscr{K}(A)(\zeta)$ whenever 0 is a limit point of $\mathscr{K}(A)(\zeta)$. We proceed to do this. For every $n = 1, 2, \cdots$, there is a function $f_n$ in the subset $\mathscr{E}$ of real-valued functions on the unitary operators of $\mathscr{A}$ and a central projection $P_n$ of $\mathscr{A}$ with $P_n^\wedge(\zeta) = 1$ and $\|(f_n \cdot A)P_n\| \leqq 2^{-n}$. Let $\{Q_i\}$ be the sequence of orthogonal projections defined by $Q_1 = P_1 - P_1P_2$, $Q_2 = P_1P_2 - P_1P_2P_3, \cdots$, and let $B \in \mathscr{K}'(A)$. Then let $C_n = B(1 - P_1) + \sum \{(f_i \cdot A)Q_i \mid 1 \leqq i \leqq n\} + (f_{n+1} \cdot A)Q'_n$ $(n = 1, 2, \cdots)$. Here $Q'_n = P_1 \cdots P_{n+1}$ is the orthogonal complement of $(1 - P_1) + \sum \{Q_i \mid 1 \leqq i \leqq n\}$. We notice that $C_n \in \mathscr{K}'(A)$ for every $n$ since $\mathscr{K}'(A)$ is central-convex. However, the sequence $\{C_n\}$ is Cauchy since $\|C_n - C_{n+1}\| \leqq \max\{\|(f_{n+1} \cdot A)Q'_{n+1}\|, \|(f_{n+2} \cdot A)Q'_{n+1}\|\} \leqq 2^{-n-1}$. This means that $\{C_n\}$ converges to an element $C$ in $\mathscr{K}'(A)$. We have that

$$||C(\zeta)|| = \lim ||C_n(\zeta)|| = \lim ||((f_{n+1}\cdot A)Q'_n)(\zeta)|| \leqq \limsup 2^{-n} = 0$$

and thus $C$ is in the ideal $[\zeta]$. This means that $\mathscr{K}'(C) \subset [\zeta]$. However, we have that $\mathscr{K}'(C) \subset \mathscr{K}'(A)$ because $C \in \mathscr{K}'(A)$. This means that $\mathscr{K}(A) \cap [\zeta] \neq \varnothing$, or equivalently, that $0 \in \mathscr{K}(A)(\zeta)$.

THEOREM 4.12. *Let $\mathscr{A}$ be a properly infinite von Neumann algebra, let $\mathscr{J}$ be the strong radical of $\mathscr{A}$, and let $A$ be an element of $\mathscr{A}$. Then the set $\mathscr{K}(A)$ is equal to the set $\mathscr{K}_{\mathscr{J}}(A) = \{\phi(A) \mid \phi$ is a state of $\mathscr{A}^\sim$ with $\phi(\mathscr{J}) = (0)\}$.*

REMARK 4.13. Here notice $E_a(\mathscr{J})$ is the set of all states of $\mathscr{A}^\sim$ which vanish on $\mathscr{J}$.

*Proof.* First let $A$ be self-adjoint. We show that every element $C$ in the essential central spectrum of $A$ with respect to $\mathscr{J}$ is in $\mathscr{K}(A)$. There is no loss of generality in assuming for this that $C = 0$. Then for every $\varepsilon > 0$, there is a projection $E$ in $\mathscr{A}$ such that

$$||AE|| \leqq \varepsilon \quad \text{and} \quad E \sim 1$$

(Example 2.12 and Corollary 3.15).

There are orthogonal projections $E'$ and $E''$ of sum $E$ such that $E' \sim E'' \sim E$ [9; III, 8, Corollary 2]. By replacing $E$ by $E'$, we may assume that $||AE|| \leqq \varepsilon$ and $E \sim 1 - E \sim 1$. Then the element

$$2^{-1}((E - (1 - E))A(E - (1 - E)) + A) = EAE + (1 - E)A(1 - E)$$

is in $\mathscr{K}'(A)$. Now let $E_1, \cdots, E_n$ be orthogonal projections of sum $E$ with $E_1 \sim \cdots \sim E_n \sim E$, and let $U_1, \cdots, U_n$ be unitary operators in $\mathscr{A}$ so that the domain support of $(1 - E)U_i$ equals $E_i$. For every unit vector $x$ in the Hilbert space, we have

$$\begin{aligned}
&||\sum\{n^{-1}U_i^{-1}(EAE + (1 - E)A(1 - E))U_i x \mid 1 \leqq i \leqq n\}|| \\
&\quad ||\sum n^{-1}U_i^{-1}EAEU_i x|| + ||\sum n^{-1}U_i^{-1}(1 - E)A(1 - E)U_i x|| \\
&\quad\quad \leqq n(n^{-1}\varepsilon) + ||\sum n^{-1}E_i U_i^{-1}(1 - E)A(1 - E)U_i E_i x|| \\
&\quad\quad \leqq \varepsilon + n^{-1}||A|| .
\end{aligned}$$

This proves that $\mathscr{K}'(A)$ contains an element of norm less than or equal to $\varepsilon + n^{-1}||A||$. Because $\varepsilon > 0$ and $n$ are arbitrary, the set $\mathscr{K}(A)$ contains 0. This means that the essential central spectrum of $A$ with respect to $\mathscr{J}$ is contained in $\mathscr{K}(A)$. Hence, the least upper bound $C_u$ and the greatest lower bound $C_l$ of the essential central spectrum are in $\mathscr{K}(A)$. Since $\mathscr{K}_{\mathscr{J}}(A)$ is the smallest central-convex set containing $C_l$ and $C_u$ (Theorem 4.4) and since $\mathscr{K}(A)$ is central convex, we have that $\mathscr{K}_{\mathscr{J}}(A) \subset \mathscr{K}(A)$.

Now let $A$ be an arbitrary element of $\mathscr{A}$ and let $\phi \in E_a(\mathscr{J})$. We may assume that $\phi(A) = 0$. We show that 0 is in $\mathscr{K}(A)(\zeta) = \{B^\wedge(\zeta) \mid B \in \mathscr{K}(A)\}$ for every $\zeta$ in the spectrum of the center. Since $\mathscr{K}(A)(\zeta)$ is compact (Proposition 4.11), there is a $C$ in $\mathscr{K}(A)$ with $|C^\wedge(\zeta)| = \mathrm{glb}\{|\alpha| \mid \alpha \in \mathscr{K}(A)(\zeta)\}$. There is no loss of generality in assuming $C^\wedge(\zeta) \geqq 0$. We obtain a contradiction by assuming $C^\wedge(\zeta) > 0$. Indeed, we have that $\phi(A+A^*) = \phi(A) + \phi(A)^* = 0$. By the preceding paragraph we conclude that $0 \in \mathscr{K}(2^{-1}(A + A^*))$ and so there is a sequence $\{f_n\}$ in the subset $\mathscr{E}$ of functions on the unitary operators of $\mathscr{A}$ with $\lim f_n \cdot (2^{-1}(A + A^*)) = 0$. We may also assume that $\{f_n \cdot ((2i)^{-1}(A - A^*))\}$ converges to a self-adjoint element $B$ in the center [9; III, §5, Problem 2]. Hence, the element $iB$ is in $\mathscr{K}(A)$. However, we must have that $B^\wedge(\zeta) = 0$. Indeed, if $B^\wedge(\zeta) \neq 0$, then the distance to the origin of the line segment $L$ in the complex plane with end-points $C^\wedge(\zeta)$ and $iB^\wedge(\zeta)$ is less than $C^\wedge(\zeta)$. However, this contradicts the definition of $C$ since $L \subset \mathscr{K}(A)(\zeta)$. So we must have that $C^\wedge(\zeta) = 0$, and hence $0 \in \mathscr{K}(A)(\zeta)$. The proof is now completed by a compactness argument. Let $\varepsilon > 0$ be given. For every $\zeta$ in the spectrum of the center, there a $C_\zeta$ in $\mathscr{K}(A)$ and a central projection $P_\zeta$ with $P_\zeta^\wedge(\zeta) = 1$ such that $\|C_\zeta P_\zeta\| \leqq \varepsilon$. Due to the compactness of the spectrum of the center, we may find $C_1, \cdots, C_n$ in $\mathscr{K}(A)$ and orthogonal central projections $P_1, \cdots, P_n$ of sum 1 such that

$$\| \sum C_i P_i \| \leqq \varepsilon \ .$$

However, $\mathscr{K}(A)$ is central-convex and so $\sum C_i P_i \in \mathscr{K}(A)$. Since $\varepsilon > 0$ is arbitrary and since $\mathscr{K}(\mathscr{A})$ is closed, we have that $0 \in \mathscr{K}(A)$. This completes the first part of the proof.

Conversely, let $C \in \mathscr{K}(A)$. There is no loss of generality in assuming $C = 0$. We find $\phi$ in $E_a(\mathscr{J})$ with $\phi(A) = 0$: Let $\phi_0$ be a state of $\mathscr{A}^\sim$ that vanishes on $\mathscr{J}$ (Lemma 4.3). Let $\{f_n\}$ be a sequence of functions in $\mathscr{E}$ such that $\lim f_n \cdot A = 0$. Let $\phi_n$ be the state of $E_a(\mathscr{J})$ given by $\phi_n(B) = \phi_0(f_n \cdot B)$ for every $B$ in $\mathscr{A}$. Due to the compactness of the state space of $\mathscr{A}^\sim$ in the $\sigma_W(\mathscr{A}^\sim, \mathscr{A})$-topology, there is a subnet $\{\phi_{n_j}\}$ of $\{\phi_n\}$ and a state $\phi$ of $\mathscr{A}^\sim$ such that $\{\phi_{n_j}(B)\}$ converges weakly to $\phi(B)$ for every $B$ in $\mathscr{A}$. Clearly, the state $\phi$ vanishes on $\mathscr{J}$. However, for every $x$ and $y$ in the Hilbert space, we have that

$$|(\phi(A)x, y)| = \lim_j |(\phi_{n_j}(A)x, y)| \leqq \lim\sup \|\phi_0\| \, \|f_{n_j} \cdot A\| \, \|x\| \, \|y\| = 0 \ .$$

This proves that $\phi(A) = 0$, and so $0 \in \mathscr{K}_{\mathscr{J}}(A)$.

COROLLARY 4.14. *Let $\mathscr{A}$ be a properly infinite von Neumann* *algebra and let $A$ be an element of $\mathscr{A}$. Then the convex subset $\mathscr{K}(A)$ of the center is weakly compact.*

*Proof.* For any central ideal $\mathscr{J}$, the set $\mathscr{K}_{\mathscr{J}}(A)$ is weakly compact (Introduction, § 4).

Let $A$ be an element in the von Neumann algebra $\mathscr{A}$. Define $\mathscr{C}(A)$ to be the intersection of the weak closure of $\mathscr{K}'(A)$ with the center of $\mathscr{A}$. Using the tools we developed here, we can extend the theorem of J. Conway [4] from the case of properly infinite factors to pro perly infinite algebras with arbitrary centers. For this extension the following lemma is needed.

LEMMA 4.15. *Let $\mathscr{A}$ be a von Neumann algebra on the Hilbert space $H$. Let $f$ be a $\sigma_W(\mathscr{A}^\sim, \mathscr{A})$-continuous hermitian functional on $\mathscr{A}^\sim$* (i.e. *$f(\phi)$ is real for every $\phi$ in $\mathscr{A}^\sim$ which takes hermitian elements of $\mathscr{A}$ into hermitian elements of the center). Then there is an $x \in H$ and a self-adjoint $A \in \mathscr{A}$ such that $f(\phi) = (\phi(A)x, x)$ for every $\phi \in \mathscr{A}^\sim$.*

*Proof.* There are $x_1, \cdots, x_n, y_1, \cdots, y_n$ in $H$ and $A_1, \cdots, A_n$ in $\mathscr{A}$ such that $f(\phi) = \sum (\phi(A_i)x_i, y_i)$ for all $\phi$ in $\mathscr{A}^\sim$ [17; § 2, Introduction]. For each $i$ there are $z_{ij}$ $(1 \leqq j \leqq 4)$ such that

$$w_{x_i, y_i} = w_{z_{i1}} + w_{z_{i2}} + i(w_{z_{i3}} - w_{z_{i4}})$$

where $w_{x_i, y_i}(B) = (Bx_i, y_i)$ and $w_z = w_{z,z}$ on the center of $\mathscr{A}$ [9; I, 4, Theorem 6 and III, 1, Theorem 4, Corollary]. Then there is an $x$ in $H$ with $w_x = \sum_{i,j} w_{z_{ij}}$ [9; III, 1, Theorem 4, Corollary]. For each $ij$, there is a positive element $C_{ij}$ in the center with $(BC_{ij}x, x) = (Bz_{ij}, z_{ij})$ for all $B$ in the center (Radon-Nikodym theorem). Thus there is an element $B = \sum A_i(C_{i1} - C_{i2} + i(C_{i3} - C_{i4}))$ in $\mathscr{A}$ with $f(\phi) = (\phi(B)x, x)$ for every $\phi$ in $\mathscr{A}^\sim$. If $\phi^*(B) = \phi(B^*)^*$ for $\phi \in \mathscr{A}^\sim$, then

$$(\phi(B^*)^*x, x) = f(\phi^*) = f(\phi)^- = (\phi(B)x, x)^-$$

for every $\phi$ in $\mathscr{A}^\sim$ implies that $f(\phi) = (\phi(B^*)x, x) = (\phi(B)x, x)$ for every $\phi$ in $\mathscr{A}$. Hence, $f(\phi) = (\phi(A)x, x)$ for every $\phi$ in $\mathscr{A}^\sim$. Here $A = 2^{-1}(B + B^*)$.

THEOREM 4.16. *Let $\mathscr{A}$ be a properly infinite von Neumann algebra, and let $\mathscr{J}$ be the ideal of finite elements of $\mathscr{A}$; then $\mathscr{C}(A) = \mathscr{K}_{\mathscr{J}}(A)$ for every $A$ in $\mathscr{A}$.*

*Proof.* One may prove the theorem using the same steps (with appropriate modifications) that Conway [4] employed in his proof for factor algebras. We content ourselves with pointing out the appropriate steps. Let $\mathscr{C}$ be the set of all states of $\mathscr{A}^\sim$ such that $\phi(A) \in \mathscr{C}(A)$ for all $A$ in $\mathscr{A}$. For every $A \in \mathscr{A}$ and $A_0 \in \mathscr{C}(A)$, there is a $\phi \in \mathscr{C}$ such that $\phi(A) = A_0$. This uses the $\sigma_W(\mathscr{A}^\sim, \mathscr{A})$-topology instead of the weak *-topology of the dual of $\mathscr{A}$ [4; Lemma 5]. The set $\mathscr{C}(A)$ is equal to $\{0\}$ for every $A \in \mathscr{J}$ [4; Lemma 6]. Hence, the set $\mathscr{C}$ is a subset of $E_a(\mathscr{J})$. But if $A$ is self-adjoint and $\phi \in E_a(\mathscr{J})$, then $\phi(A) \in \mathscr{C}(A)$ since the least upper bound and the greatest lower bound of the essential central spectrum of $A$ with respect to $\mathscr{J}$ are in $\mathscr{C}(A)$ (argue as in [4; Lemma 4] based on Proposition 3.13) and since $\mathscr{C}(A)$ is central-convex (use the fact that $\mathscr{K}'(A)$ is central-convex). If there is $\phi_0$ in $E_a(\mathscr{J})$ but not in the $\sigma_W(\mathscr{A}^\sim, \mathscr{A})$-compact convex set $\mathscr{C}$, then there is a $\sigma_W(\mathscr{A}^\sim, \mathscr{A})$-continuous hermitian functional on $\mathscr{A}^\sim$ which strongly separates $\phi_0$ from $\mathscr{C}$. However, every $\sigma_W(\mathscr{A}^\sim, \mathscr{A})$-continuous hermitian functional $f$ of $\mathscr{A}^\sim$ is of the form $f(\phi) = (\phi(A)x, x)$ for some fixed self-adjoint $A$ in $\mathscr{A}$ and some vector $x$ in the Hilbert space. This contradicts the fact that $\phi_0(A) \in \mathscr{C}(A)$ and so that $\phi_0(A) = \phi(A)$ for some $\phi \in \mathscr{C}$. Hence, $\mathscr{C} = E_a(\mathscr{J})$ and $\mathscr{K}_{\mathscr{J}}(A) = \mathscr{C}(A)$.

COROLLARY 4.17. *Let $\mathscr{A}$ be a $\sigma$-finite properly infinite von Neumann algebra; then $\mathscr{K}(A) = \mathscr{C}(A)$ for every $A$ in $\mathscr{A}$.*

*Proof.* The ideal generated by the finite elements of $\mathscr{A}$ is the strong radical of $\mathscr{A}$. The corollary then follows from Theorems 4.12 and 4.16.

5. **Applications.** Using the notions of essential central spectrum and essential numerical range, we can extend some theorems on commutators and derivations to arbitrary properly infinite von Neumann algebras. These theorems are known for the algebra of all bounded linear operators on a Hilbert space, which is generally assumed to be separable, but the techniques employed there also suffice here.

A linear map $\delta$ of an algebra is said to be a *derivation* if $\delta(AB) = A\delta(B) + \delta(A)B$ for every $A$ and $B$ in the algebra. S. Sakai [27] proved that every derivation $\delta$ of a von Neumann algebra $\mathscr{A}$ is inner in the sense that there is an $A$ in $\mathscr{A}$ such that $\delta(B) = AB - BA$ for every $B$ in $\mathscr{A}$. The next proposition is due to J. G. Stampfli [29] for the algebra of bounded linear operators on a Hilbert space. His technique suffices here.

PROPOSITION [*Stampfli*] 5.1. *The range of a derivation on a von*

*Neumann algebra is not uniformly dense in the algebra.*

*Proof.* Since every von Neumann algebra may be written as a product of a finite and a properly infinite von Neumann algebra, it is sufficient to consider these two cases separately. If the algebra is finite, then the range of the derivation is contained in the set of elements whose canonical operator-valued trace vanishes. So the range of a derivation cannot be dense. If the von Neumann algebra $\mathscr{A}$ is properly infinite and the derivation $\delta$ on $\mathscr{A}$ is given by $\delta(B) = AB - BA$, then we construct an operator that is not in the closure of the range of $\delta$. Let $A_0$ be a central element such that $(A - A_0)(\mathscr{J}(\zeta))$ is neither left nor right invertible for all $\zeta$ in the spectrum of the center. Here $\mathscr{J}$ is the strong radical of $\mathscr{A}$ (Theorem 3.5). Because $\delta(B) = (A - A_0)B - B(A - A_0)$ for all $B \in \mathscr{A}$, we may assume $A_0 = 0$. There are sequences $\{E_n\}$ and $\{F_n\}$ of mutually orthogonal projections in $\mathscr{A}$ such that $E_n \sim 1 \sim F_n$, $||AE_n|| \leqq n^{-1}$, and $||F_nA|| \leqq n^{-1}$ for for every $n = 1, 2, \cdots$ (Example 2.12 and Corollary 3.16). Then there is a partial isometry $U$ in $\mathscr{A}$ with domain support $E = \sum E_i$ and range support $F = \sum F_i$ such that $UE_i = F_iU$. We show that $\alpha = ||U - \delta(B)|| \geqq 1$ for every $B \in \mathscr{A}$. Indeed, for every $n = 1, 2, \cdots$, we have that

$$1 = ||F_nUE_n|| \leqq ||F_n(U - \delta(B))E_n|| + ||F_n\delta(B)E_n|| \leqq \alpha + 2n^{-1}||B|| .$$

Hence the open ball of radius 1 about $U$ does not meet the range of $\delta$.

In [18], we showed that an element $A$ in a properly infinite von Neumann algebra $\mathscr{A}$ is a *commutator* in $\mathscr{A}$ (i.e. there are elements $B$ and $C$ with $A = BC - CB$) provided $0 \in \mathscr{K}(A)$. We can also prove that $0 \in \mathscr{K}(A)$ provided $A = BC - CB$ and $\pm(B^*B - BB^*)$ is a positive operator in $\mathscr{A}$. Now an element $A$ is said to be a self-adjoint commutator if $A = BC - CB$ with $B = B^*$. H. Radjavi [25] characterized those self-adjoint elements in the algebra $B(H)$ of all bounded linear operators on a separable Hilbert space $H$ which are self-adjoint commutators and J. Anderson [1] recently announced that he has completely characterized self-adjoint commutators in $B(H)$. We prove a proposition in this direction for properly infinite von Neumann algebras using a matrix calculation of M. David [5].

PROPOSITION 5.2. *Let $\mathscr{A}$ be a properly infinite von Neumann algebra and let $A$ be a self-adjoint element in $\mathscr{A}$. If 0 is in the essential central spectrum of $A$ with respect to the strong radical of $\mathscr{A}$, then $A$ is a self-adjoint commutator in $\mathscr{A}$.*

*Proof.* There is a sequence $\{E_n\}$ of orthogonal projections with $E_n \sim 1$ and $\|AE_n\| \leqq 1/n!$ for all $n = 1, 2, \cdots$ (Lemma 3.16 and Example 2.12). Thus, $\|E_m A E_n\| \leqq \min\{1/m!, 1/n!\}$. Then the matrix calculation of M. David [5; Theorem 3] is applicable.

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*Added in Proof, August 24, 1972.* We have obtained a better version of Proposition 5.2 by showing that $A$ is a self-adjoint commutator whenever 0 is in the essential central range of $A$.

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UNIVERSITY OF CINCINNATI

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# SUPERADDITIVITY INTERVALS AND BOAS' TEST

G. D. JOHNSON

**A test is given for determining maximal intervals of superadditivity for convexo-concave functions. The test is then applied to several families of ogive-shaped functions.**

1. Superadditive functions have been widely studied [8, 11] for their own sake but have also found important applications in reliability theory, e.g. [6]. However, tests for superadditivity were non existent in the literature until Bruckner's work [3] in 1962. A more constructive (hence more readily applicable) test due to Boas was given in 1964 in a paper by Beckenbach [2] on analytic inequalities, an area where superadditivity is of use (see [2] for a derivation of Whittaker's inequality [12]). Boas' test is here viewed in the light of Bruckner's result, strengthened, and applied to some families of convexo-concave functions as suggested in [2].

2. Consider a continuous, real-valued function, $f$, of a real variable, $x \in \boldsymbol{R}$. Then $f$ is called "superadditive" on $[\beta, b] \subset \boldsymbol{R}$ if

$$f(x) + f(y) \leqq f(x + y)$$

for every $x, y, x + y$ in $[\beta, b]$. We normalize to the cases $\beta = 0$, $b > 0$. In this event, superadditivity implies $f(0) \leqq 0$. The following sufficient condition for superadditivity is due to Boas [2]:

THEOREM (Boas' Test). *Assume $f$ is nonnegative on $[0, b]$ with $f(0) = 0$ and $f$ has a continuous derivative on $[0, b]$. If there are numbers $a \leqq b/2$ and $c \leqq a$ such that*

(0) *$f$ is star-shaped*[1] *on $[0, 2a]$,*

(i) *$f$ is concave*[2] *and satisfies $f(x/2) \leqq f(x)/2$ on $[c, b]$,*

(ii) $f'(0) < f'(b)$,

(iii) *$f'(x) - f'(b - x)$ has at most one zero in $(0, a)$.*

*Then $f$ is superadditive on $[0, b]$.*

A proof of the theorem can be made by considering separately the cases:

[1] $f$ is "star-shaped" on $[0, A]$ means for every $x \in [0, A]$, and every $\alpha \in [0, 1]$ it is true that $f(\alpha x) \leqq \alpha f(x)$. For $f \in C^1[0, A]$ it is necessary and sufficient [4] that $f'(x) \geqq f(x)/x$ for all $x \in (0, A]$.

[2] The function $f$ is called "convex" on $[a, b]$ if for every $x, y \in [a, b]$ it is true that $f((x+y)/2) \leqq (f(x)+f(y))/2$; $f$ is called "concave" if $-f$ is convex.

(I) $0 \leqq x \leqq a$, $0 \leqq y \leqq a$;

(II) $x \geqq a$, $y \geqq a$, $x + y \leqq b$;

(III) $x < a < y < b$, $x + y \leqq b$.

It was conjectured that this test could be applied to finding superadditivity intervals of such ogive-shaped functions as $\exp(-1/\alpha x)$ $(0 < \alpha \leqq 1)$; $\ln(1 + x^\lambda)$ and $\arctan x^\lambda (\lambda > 1)$. But it is easy to show that for some of these functions, Boas' test does not apply: consider $\ln(1 + x^2)$. A simple calculation shows that $1 \leqq c \leqq 2\sqrt{2}$ whereas $2a < 2$ and hence $a < c$. It is our primary goal to modify Boas' test so that it can be used to determine intervals of superadditivity for a larger class of functions. Along the way we shall be able to determine conditions giving maximal intervals of superadditivity, and finally a tabulation of intervals of superadditivity is given for some of the functions previously mentioned.

3. We are interested in determining intervals, $[0, b]$, of superadditivity for a special class of functions, the "convexo-concave" functions [1]: $f$ is called convexo-concave on $[0, B]$ if it is convex on $[0, c]$ and concave on $[c, B]$, $0 \leqq c \leqq B$. Already, $f$ is superadditive on $[0, c]$ [4]; that is, $b \geqq c$. Bruckner has characterized superadditivity of such functions in the following way:

THEOREM [3]. *The convexo-concave function, $f$, with $f(0) \leqq 0$, is superadditive on $[0, b]$ if and only if* $\max_{0 \leqq x \leqq b} [f(x) + f(b - x)] \leqq f(b)$.

The main difficulties in applying Bruckner's test are first in obtaining the quantity "$b$", and second in taking the maximum on the lefthand side. By requiring $f \in C^1[0, b]$ we can ameliorate the second objection and turning to Boas' test we obtain a candidate for $b$: namely, let $b$ be the smallest positive root of $f(x) = 2f(x/2)$.

THEOREM. *Let $f \in C^1[0, b]$ be convexo-concave on $[0, b]$ $(0 < b < \infty)$ with $f(0) \leqq 0$ and*[3]

( i ) $f(b) \geqq 2f(b/2)$,

( ii ) $f'(0) < f'(b)$,

(iii-a) $f'(x) = f'(b - x)$ *no more than once on* $(0, b/2)$. *Then $f$ is superadditive on* $[0, b]$.

*Proof.* Consider the function $g(x) \equiv f(x) + f(b - x) - f(b)$. Then $f(0) \leqq 0$ implies $g(0) \leqq 0$. By (i) and (ii), $g(b/2) \leqq 0$ and $g'(0) < 0$, respectively. Suppose $g$ is positive on $(0, b/2)$. Then it has a positive

[3] It is important for generalizing to higher dimensions that condition (0) in Boas' test has been deleted. See [6].

maximum on $(0, b/2)$. Therefore $g'(x) = f'(x) - f'(b - x)$ has at least *two* zeros on $(0, b/2)$, contrary to (iii-a). Finally, then, $g(x) \leqq 0$ on $[0, b/2]$ and—by symmetry of $g$ about $x = b/2$,

$$\max_{0 \leqq x \leqq b} [f(x) + f(b - x)] \leqq f(b)$$

which, by Bruckner's theorem, shows $f$ superadditive on $[0, b]$.

For the function $f(x) \equiv \ln(1 + x^2)$ it is easy to check that (i), (ii) are satisfied for $b = 2\sqrt{2}$. Condition (iii-a) is also straight forward: it is true by Descartes' rule of signs.

Notice that for $f(0) < 0$, $f$ is superadditive at least as long as it is merely nondecreasing and nonpositive. This relatively arbitrary state of affairs will be avoided by assuming $f(0) = 0$ in what follows. For a further appreciation of (iii) we give a corollary to Bruckner's theorem.

COROLLARY. *Suppose convexo-concave $f$, with $f(0) = 0$, is continuously differentiable. Then $f$ is superadditive on $[0, b]$ if and only if for every $x_0$ in $[0, b]$ such that $f'(x_0) = f'(b - x_0)$, it is true that $f(x_0) + f(b - x_0) \leqq f(b)$.*

Thus we see how the maximizing duties in Bruckner's theorem have been replaced by a zero-*counting* operation in the other two theorems. The fourth condition in Boas' test is less restrictive than (iii-a) above since $b$ is not less than $2a$. But it is not hard to see that (iii-a) can be replaced by

(iii-b) $f'(x) = f'(b - x)$ no more than once on the *smaller* of the two intervals $(0, c)$, $(c, b)$,

which is a less restrictive condition than even Boas' fourth condition. (Here "$c$" is the inflection point of $f$.)

Perhaps a computational note is in order here. If we refer generically to conditions (iii), (iii-a), (iii-b) as "root conditions", then in applications the root condition can often be tested by Sturm's theorem [7]. For example, the functions $\ln(1 + x^n)$ $(n = 2, 3, 4, \cdots)$ have as derivatives rational functions with denominators not vanishing for positive arguments. Verifying a root condition is then a matter of counting the number of zeros of *polynomials* in a finite interval. Sturm sequences can also be readily computed for rational functions [10], and Sturm's idea can be extended to counting real zeros of even more general functions [5]. Finally, upon observing that $f'$ is unimodal[4], an optimum strategy for localizing the inflection point $c$ (as used in (iii-b)) is well-known [9].

4. Now it is quite striking that the choice of $b$ as the smallest positive root, $\sigma$, of $2f(x/2) = f(x)$ often turns out to be maximal. Certainly $\sigma$ is an upper bound on the interval of superadditivity. Consider the quantity $\min\{\sigma, \tau\}$ where $\sigma, \tau$ are the smallest positive, odd zeros of $2f(x/2) - f(x)$, $f'(0) - f'(x)$, respectively. Then we may be assured of a maximal interval of superadditivity.

THEOREM. *Suppose $f \in C^1[0, b]$ is superadditive on $[0, b]$ where $b \equiv \min\{\sigma, \tau\} < \infty$. Then $f$ is not superadditive on any larger interval, $[0, B]$, $B > b$.*

The proof is immediate by failure of superadditivity near $x = 0$ ($b = \tau$ case) and $x = B/2$ ($b = \sigma$ case) where $B = b + \varepsilon$, $\varepsilon > 0$ arbitrary. In our example, $2\sqrt{2}$ is the largest value of $b$ so that $\ln(1 + x^2)$ is superadditive on $[0, b]$. With this optimality result, then, we turn to computing intervals of superadditivity in the next section.

5. Tables of $\hat{b}$ are now given where $\hat{b}$ is the largest 7D approximation smaller or equal to $b$ and $[0, b]$ is the maximum interval of superadditivity for the function indicated.

| $\lambda$ | $\arctan x^\lambda$ | $\ln(1 + x^\lambda)$ | $\exp(-\lambda/x)$ | $\lambda$ |
|---|---|---|---|---|
| 1.1 | .5852351 | .3425001 | 1.586964 | 1.1 |
| 1.2 | .8532410 | .7280202 | 1.731234 | 1.2 |
| 1.3 | 1.051079 | 1.104767 | 1.875503 | 1.3 |
| 1.4 | 1.205188 | 1.452478 | 2.019773 | 1.4 |
| 1.5 | 1.328208 | 1.764139 | 2.164042 | 1.5 |
| 1.6 | 1.427957 | 2.039063 | 2.308312 | 1.6 |
| 1.7 | 1.509790 | 2.279467 | 2.452581 | 1.7 |
| 1.8 | 1.577572 | 2.488734 | 2.596851 | 1.8 |
| 1.9 | 1.634178 | 2.670539 | 2.741120 | 1.9 |
| 2 | 1.681792 | 2.828427 | 2.885390 | 2 |
| 3 | 1.906368 | 3.634241 | 4.328085 | 3 |
| 4 | 1.966894 | 3.868672 | 5.770780 | 4 |
| 5 | 1.987133 | 3.948700 | 7.213475 | 5 |
| 6 | 1.994715 | 3.978890 | 8.656170 | 6 |
| 7 | 1.997751 | 3.991011 | 10.09886 | 7 |
| 8 | 1.999019 | 3.996080 | 11.54156 | 8 |
| 9 | 1.999565 | 3.998260 | 12.98425 | 9 |
| 10 | 1.999804 | 3.999218 | 14.42695 | 10 |

[4] A function $f(x)$ is "unimodal" if there is a $\xi$ so that $f$ is either strictly increasing for $x \leqq \xi$ and strictly decreasing for $x > \xi$, or else strictly increasing for $x < \xi$ and strictly decreasing for $x \geqq \xi$.

Entries above or to the left of the stepped line were unattainable by Boas' original test.

For $\exp(-\lambda/x)$ $(\lambda \geqq 1)$ it is easy to verify (in this case, Boas' test is sufficient) that the intervals of superadditivity $[0, b(\lambda)]$ are determined by $b(\lambda) = \lambda/\ln 2$.

In [2] it is suggested that maximum intervals of superadditivity be computed not only for $f = f_\lambda$ but also for the "average function of $f$", $F = F_\lambda$, and for the "inverse average function," $\phi = \phi_\lambda$, where

$$F_\lambda(x) \equiv \begin{cases} 0 & x = 0\,, \\ \dfrac{1}{x}\displaystyle\int_0^x f_\lambda(t)dt & x > 0\,; \end{cases}$$

$$\phi_\lambda(x) \equiv f_\lambda(x) + x f'_\lambda(x) \qquad x \geqq 0\,.$$

For the case $f_\lambda(x) \equiv \exp(-\lambda/x)$ we can give the following maximum intervals of superadditivity:

| Function | $\hat{b}(\lambda)$-end point |
|---|---|
| $\phi_\lambda$ | $\lambda/1.116845$ |
| $f_\lambda$ | $\lambda/.6931472$ |
| $F_\lambda$ | $\lambda/.4243251$ |

where Boas' test was inapplicable to the $\phi_\lambda$-case.

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UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER 8, B. C.
CANADA

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# DERIVATION IN INFINITE PLANES

N. L. JOHNSON

**The purpose of this article is to study "derivation" in arbitrary affine planes. It is shown that the derivation process extends to arbitrary planes which possess a suitable set of Baer subplanes.**

**1. Introduction.** A basic problem of interest is developing Ostrom's finite net replacement theory in the infinite case. Some expected premiums could be that the procedures are valid in infinite planes which have no finite analogue. For example, the Moufang planes, *non*-Pappian Desarguesian planes, and certain Bol planes may permit net replacement (see §4).

The present article will be restricted to studying derivation in infinite planes. Concerning infinite planes, Rosati [18] found a class of infinite Hughes planes and Swift [21] remarked that derivation is probably valid in infinite Pappian planes. This statement was essentially confirmed by Pickert [17] who also gave an algebraic construction of the Ostrom-Rosati planes (see Panella [15]).

Sabharwal [20] constructed a class of infinite André nearfield planes and showed that derivation is valid in these planes and also considered the analogous infinite "derivable chains" of Fryxell [6].

Barlotti and Bose [3] have studied the derivation of dual translation planes of dimension 2 by means of linear representations in projective spaces of projective planes (see [3], [4], [5]). The Bose-Barlotti derivation theory is valid in all dual translation planes of dimension 2 whose associated spread of the corresponding translation plane is also a dual spread. However, this condition is not valid in every *infinite* dual translation plane of dimension 2 (see [7]).

This article will be devoted to derivation in arbitrary planes. The treatment is in the spirit of Ostrom's original construction (see [13], section III, and [14]). Section 2 is devoted to showing that the derivation process extends to arbitrary planes which possess a suitable set of Baer (see (2.1)) subplanes. Section 3 is concerned with certain conditions sufficient for a given subplane to be a Baer subplane and develops some theory related to the derivation of translation planes and their duals. Finally, applications of the theory to certain infinite planes are considered in §4.

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**2. The Construction.** Ostrom ([13] section III, pp. 7, 8, 9) develops derivation in finite planes. The planes are of order $q^2$ and the procedure involves the relabeling of certain subplanes of order $q$ (Baer subplanes) as lines. Ostrom's arguments depend strongly on finiteness. However, it will be shown that the essential assumption is not of finiteness but is simply that the subplanes used in the process are *Baer* subplanes.

DEFINITION 2.1. Let $\pi$ be a projective plane. A proper subplane $\pi_0$ of $\pi$ is a *Baer* subplane of $\pi$ if and only if every point of $\pi$ is on a line of $\pi_0$ and every line of $\pi$ is on a point of $\pi_0$.

REMARK 2.2. *A Baer subplane is maximal.*

*Proof.* Let $\pi_0$ be a Baer subplane of a projective plane $\pi$ and let $P^*$ be a point of $\pi - \pi_0$. Any subplane $\tau$ of $\pi$ containing $P^*$ and $\pi_0$ contains the joins of $P^*$ with points of $\pi_0$. Let $l$ be an arbitrary line of $\pi$ incident with $P^*$. By assumption $l$ intersects $\pi_0$ and therefore $\tau$ contains all lines of $\pi$ incident with the point $P^*$. Similarly, $\tau$ contains all lines of $\pi$ incident with any point of $\tau - \pi_0$. Let $Q$ be a point of $\pi$. Every line of $\pi$ incident with $Q$ intersects $\tau$. If $Q \notin \tau$ then $QP^*$ is either a line of $\tau - \pi_0$ or is the unique line of $\pi_0$ incident with $P^*$. Since there is a line of $\pi_0$ incident with $Q$, it follows that $Q \in \tau$ in the former case. In the latter case, if $R$ is a point of $QP^*$, choose a quadrangle whose cross joins contain $R$. Thus, all points of $\pi$ are in $\tau$.

DEFINITION 2.3. Let $\pi$ be a projective plane. Let $l_\infty$ be a line of $\pi$. A *derivation set $\delta$ in $l_\infty$* is a set of points of $l_\infty$ such that if $P, Q$ are distinct points of $\pi - l_\infty$ such that $PQ \cap l_\infty \in \delta$ then there is a Baer subplane $\pi_{P,Q,\delta}$ of $\pi$ containing $P, Q, \delta$ such that $\delta$ is a line of $\pi_{P,Q,\delta}$.

We shall assume in the following that $\delta$ is a derivation set in $l_\infty$ for a projective plane $\pi$ and $\pi_{P,Q,\delta}$ is a Baer subplane containing $P, Q$ and $\delta$ as a line where $PQ \cap l_\infty \in \delta$. Also a point of $\pi_{P,Q,\delta} - \delta$ will be called an affine point of $\pi_{P,Q,\delta}$.

LEMMA 2.4. *$\pi_{P,Q,\delta}$ is the unique proper subplane containing $P$ and $Q$ which contains $\delta$ as a line.*

*Proof.* Let $\Sigma_{P,Q,\delta}$ be any subplane of $\pi$ containing points $P, Q$ which contains $\delta$ as a line. Let $PQ \cap l_\infty = \delta_1 \in \delta$. Let $T = \{Q\delta_2 \cap P\delta_3$ where $\delta_1, \delta_2, \delta_3$ are distinct elements of $\delta\}$. Let $\pi_0 = \{Q\delta_i \cap P\delta_j \mid \forall\delta_i, \delta_j \in \delta$ and not both $\delta_i, \delta_j$ equal to $\delta_1\} \cup \{Q\delta_i \cap T\delta_j \mid \forall\delta_i, \delta_j \in \delta$ and not both $\delta_i$, $\delta_j$ equal to $\delta_2\}$.

Assume there is a point $R \in \Sigma_{P,Q,\delta} - \pi_0$. Since $P, Q, R \in \Sigma_{P,Q,\delta}$, either $RP \cap l_\infty$ and $RQ \cap l_\infty$ are distinct points in $\delta$ or $P, Q, R$ are collinear. In the latter case, $R = P(RP \cap l_\infty) \cap T(RT \cap l_\infty)$ and in the former $R = P(RP \cap l_\infty) \cap Q(RQ \cap l_\infty)$. Therefore, the point sets of $\Sigma_{P,Q,\delta}$ and $\pi_0$ are equal since clearly $\pi_0 \subseteqq \Sigma_{P,Q,\delta}$.

Similarly, let $\tilde{\pi}_0 = \{M\delta_i \mid M \in \pi_0 \forall \delta_i \in \delta\}$. If $l$ is a line of $\Sigma_{P,Q,\delta}$ then $l \cap l_\infty \in \delta$ and there are at least two distinct points $L$, $U \in (l - l \cap l_\infty) \cap \pi_{P,Q,\delta}$. Thus, $L$, $U \in \pi_0 - \delta$ so that $l \in \tilde{\pi}_0$.

$\therefore$ Any two subplanes of $\pi$ which contain points $P$ and $Q$ and contain $\delta$ as a line have the same point sets and the same line sets and hence are identical.

LEMMA 2.5. *Any two points of $\pi_{P,Q,\delta} - \delta$ uniquely determine the subplane. Thus, any two distinct Baer subplanes $\pi_{P,Q,\delta}$ and $\pi_{R,S,\delta}$ intersect in $\delta$ or $\delta \cup \{M\}$ for some affine point $M$.*

*Proof.* Let $M, S$ be any two distinct points of $\pi_{P,Q,\delta} - \delta$. If $M$ and $S$ are $\notin PQ$, $\exists \delta_i$, $i = 1, 2, 3, 4$, $\in \delta \ni M = P\delta_1 \cap Q\delta_2$, $S = P\delta_3 \cap Q_4$. Clearly $P$, $Q \in \pi_{M,S,\delta}$ so that $\pi_{P,Q,\delta} = \pi_{M,S,\delta}$ by (2.4).

The remaining situations where $M$ or $S \in PQ$ are equally clear.

DEFINITION 2.6. If $\pi_{P,Q,\delta} \cap \pi_{R,T,\delta} = \delta$ or $\pi_{P,Q,\delta}$, we shall say that $\pi_{P,Q,\delta}$ is parallel to $\pi_{R,T,\delta}(\pi_{P,Q,\delta} \parallel \pi_{R,T,\delta})$.

LEMMA 2.7. *If $\pi_{P,Q,\delta} \parallel \pi_{R,T,\delta}$ there is an element $\delta^*$ of $\delta \ni$ the set of lines of $\pi_{P,Q,\delta}$ incident with $\delta^*$ is equal to the set of lines of $\pi_{R,T,\delta}$ incident with $\delta^*$.*

*Proof.* Assume $\pi_{P,Q,\delta} \neq \pi_{R,T,\delta}$. Every affine point of $\pi_{R,T,\delta}$ is on a *unique* line of $\pi_{P,Q,\delta}$. $\therefore$ Every affine point of $\pi_{R,T,\delta}$ is on a line common to both subplanes. If $l, p$ are common lines, $l \cap p$ is a common point. Thus, $l$ and $p$ are concurrent on $\delta$. Let the point of concurrency be $\delta_1 \in \delta$.

Let $M$ be an arbitrary point of $\pi_{R,T,\delta}$. $\exists$ line $l$ of $\pi_{R,T,\delta}$ which is incident with $\delta_1$ and $M$. Also $\exists$ line $p \in \pi_{P,Q,\delta}$ which is incident with $M$ and hence $\delta_1$. Therefore, the lines $p$ and $l$ are identical.

Thus, the lines of $\pi_{R,T,\delta}$ incident with $\delta_1 \subseteqq$ a set of lines of $\pi_{P,Q,\delta}$ incident with $\delta_1$.

The argument is symmetric, so (2.7) is proved.

LEMMA 2.8. *Let $\pi_{R,T,\delta}$ be any Baer subplane and $P$ a point $\notin \pi_{R,T,\delta}$. Then there is a Baer subplane $\pi_{P,S,\delta}$ containing $\delta$ as a line $\ni$*

$\pi_{P,S,\delta} \| \pi_{R,T,\delta}$.

*Proof.* Assume without loss of generality that $PR$ is the unique line of $\pi_{R,T,\delta}$ incident with $P$ and assume that $T \notin PR$ (see (2.5)).

Let $S = (RT \cap l_\infty)P \cap (PR \cap l_\infty)T$. Now consider $\pi_{P,S,\delta}$.

Suppose $\pi_{P,S,\delta} \nparallel \pi_{R,T,\delta}$. Let $M$ be a common affine point. Then $M\delta_i$; $\delta_i \in \delta$ are common lines. If $M \notin ST$ then, since $ST$ and $M\delta_i$ are common lines, $M\delta_i \cap ST$ is a common affine point distinct from $M$ for some $\delta_i \in \delta$. But, this is a contradiction by (2.5). $\therefore$ $M \in ST$ and similarly $M \in PR$, which is a contradiction if $M$ is an *affine* point.

LEMMA 2.9. *Let $\pi_{P,Q,\delta}$ be a Baer subplane and suppose $R$ is an affine point $\notin \pi_{P,Q,\delta}$. Then $\exists$ a unique Baer subplane parallel to $\pi_{P,Q,\delta}$ and containing $R$.*

*Proof.* By (2.8) there exists a subplane $\pi_{R,T,\delta} \| \pi_{P,Q,\delta}$.

Suppose $\pi_0$ is a Baer subplane with line $\delta$, containing $R$, and $\| \pi_{P,Q,\delta}$. $\pi_{P,Q,\delta}$ and $\pi_{R,T,\delta}$ have a common concurrent set of lines. Let the point of concurrency be $\delta_1 \in \delta$. $R\delta_1$ is a line common to $\pi_0$, $\pi_{R,T,\delta}$ and $\pi_{P,Q,\delta}$ so the point of concurrency for the common set of lines of $\pi_0$ and $\pi_{P,Q,\delta}$ is also $\delta_1$. We can assume without loss of generality that $PQ \cap l_\infty \neq \delta_1$ since $\pi_{P,\bar{Q},\delta} = \pi_{P,Q,\delta}$ for any affine point $\bar{Q} \neq P$ of $\pi_{P,Q,\delta}$. By (2.7), $Q\delta_1$ is a line of $\pi_0$, $\pi_{R,T,\delta}$ and $\pi_{P,Q,\delta}$ so $(PQ \cap l_\infty)R \cap Q\delta_1 = D$ is an affine Point $\neq R$ of $\pi_0$ and of $\pi_{R,T,\delta}$. $\therefore$ $\pi_0 = \pi_{R,D,\delta} = \pi_{R,T,\delta}$ by (2.5).

Thus, (2.9) is proved.

THEOREM 2.10. (*Compare with Ostrom* [13], *Theorem* 5.)

*Let $\pi$ be a projective plane. Let $l = l_\infty$ be a line of $\pi$ and $\delta$ a derivation set on $l_\infty$. Form $\bar{\pi}$ as follows*:

$$\textit{points } \bar{\pi} = \textit{affine points of } \pi\,.$$

$$\textit{Lines}\begin{cases}\textit{type } 1 = \textit{the affine Baer subplanes } \pi_{P,Q,\delta} \\ \textit{type } 2 = \textit{affine restrictions of lines } l \textit{ of } \pi \ni l \cap l_\infty \notin \delta\,.\end{cases}$$

$$\|\textit{-classes}\begin{cases}\textit{type } 1,\ \pi_{P,Q,\delta} \| \pi_{R,S,\delta} \text{ iff } \pi_{P,Q,\delta} \cap \pi_{R,S,\delta} = \delta \\ \textit{type } 2,\ l \| p \text{ iff } l \cap p \in l_\infty - \delta\,.\end{cases}$$

*Then $\bar{\pi}$ is an affine plane called the plane derived from $\pi$ by $\delta$.*

*Proof.* Let $P$ and $Q$ be distinct points of $\bar{\pi}$. If $P$ and $Q$ are joined in $\pi$ by $(PQ)_\pi \ni (PQ)_\pi \cap l_\infty \in \delta$ then $\exists$ 1 Baer subplane $\pi_{P,Q,\delta}$ containing $P, Q$. If $(PQ)_\pi \cap l \notin \delta$ $\exists$ 1 line $l$ of $\pi$ containing $P, Q$.

Therefore, two distinct points of $\bar{\pi}$ are uniquely joined.

Let $l$ be a line of $\pi$ such that $l \cap l_\infty \notin \delta$ and $\pi_{P,Q,\delta}$ a Baer subplane of $\pi$. Clearly, $l$ must intersect $\pi_{P,Q,\delta}$ in an affine point.

Thus, for each point $P$ of $\bar{\pi}$ and line $\mathscr{L}$ of $\bar{\pi}$ there is a *unique* line incident with $P$ and parallel to $\mathscr{L}$.

Thus, $\bar{\pi}$ is an affine plane.

COROLLARY 2.11. *Let $l$ be a line of $\pi$ containing distinct affine points $P$ and $Q$ such that $l \cap l_\infty \in \delta$. Let $l_{P,Q} = l \cap \pi_{P,Q,\delta} - l \cap l_\infty$. Then the points of $l - l \cap l_\infty$ and the sets $l_{P,Q}$ as lines form an affine Baer subplane $\bar{\pi}_{P,Q}$ of $\bar{\pi}$.*

*Proof.* Let $R$ and $S$ be any distinct affine points of $l$. $l_{R,S} = l \cap \pi_{R,S,\delta} - l \cap l_\infty$ contains $R$ and $S$. Suppose $l_{M,T}$ also contains $R$ and $S$. $l_{M,T} = l \cap \pi_{M,T,\delta} - l \cap l_\in$ so that $\pi_{M,T,\delta}$ contains $R$ and $S$. But $\pi_{M,T,\delta} = \pi_{R,S,\delta}$ by (2.5) so that $l_{M,T} = l_{R,S}$. Thus, $R$ and $S$ are uniquely joined.

Let $R$ be any point of $\bar{\pi}_{P,Q}$ not incident with the line $l_{S,T}$. Since $R \in l$ and $l_{S,T} = l \cap \pi_{S,T,\delta} - l \cap l_\infty$, then $R \notin \pi_{S,T,\delta}$. Thus there is a *unique* Baer subplane $\pi_{R,M,\delta}$ containg $R$ and parallel to $\pi_{S,T,\delta}$. Choose a point $\bar{M}$ of $\pi_{R,M,\delta}$ incident with $l$ ($l \cap l_\infty \in \delta$ and $R \in l$ so $l$ is a line of $\pi_{R,M,\delta}$) and distinct from $R$.

Hence, $\pi_{R,\bar{M},\delta} = \pi_{R,M,\delta}$ and $l \cap \pi_{R,\bar{M},\delta} - l \cap l_\infty = l_{R,\bar{M}}$ is a line of $\bar{\pi}_{P,Q}$ which is parallel to $l_{S,T}$ and contains $R$. Suppose $l_{N,L}$ is a line of $\bar{\pi}_{P,Q}$ containing $R$ and parallel to $l_{S,T}$.

Now $NL = R\bar{M} = l$. $\pi_{N,L,\delta}$ and $\pi_{R\bar{M},\delta}$ have a common line $l$ and a common affine point $R$. Moreover, $\pi_{N,L,\delta}$ and $\pi_{R,\bar{M},\delta}$ contain no affine points of $l$ in common with $\pi_{S,T,\delta}$.

Suppose $\pi_{N,L,\delta}$ is *not* parallel to $\pi_{S,T,\delta}$. Then let $X$ be a common affine point. By assumption, $X \notin l$. Thus, $l$ and $W\delta_i \forall \delta_i \in \delta$ are lines common to $\pi_{N,L,\delta}$ and $\pi_{S,T,\delta}$. It follows that $S$ and $T$ are points of $\pi_{N,L,\delta}$ (see (2.4)) so that $\pi_{N,L,\delta} = \pi_{S,T,\delta}$, which is a contradiction.

Thus, both $\pi_{N,L,\delta}$ and $\pi_{R,\bar{M},\delta}$ are parallel to $\pi_{S,T,\delta}$ and contain $R$ so that $\pi_{N,L,\delta} = \pi_{R,\bar{M},\delta}$ and hence $l_{N,L} = l_{R,\bar{M}}$. Thus, $\bar{\pi}_{P,Q}$ is an affine subplane of $\bar{\pi}$.

Thus, $l_{P,Q} \| l_{S,T}$; $P, Q, S, T$ points of $l$ if and only if $\pi_{P,Q,\delta} \| \pi_{S,T,\delta}$. Furthermore, given a Baer subplane $\pi_{M,N,\delta}$ not containing a point of $l$, there is a Baer subplane $\pi_{X,Y,\delta}$ with $l$ as a line such that $\pi_{X,Y,\delta} \| \pi_{M,N,\delta}$.

Now extend $\bar{\pi}$ to a projective plane $\bar{\pi}^*$. The points on $\bar{l}_\infty$ (line at infinity of $\bar{\pi}^*$) corresponding to the set of all Baer subplanes $\pi_{P,Q,\delta^*}$ are precisely the points of $\bar{\pi}^*_{P,Q}$.

As a point set $\bar{\pi}_{P,Q}$ is $l - l \cap l_\infty$ where $l$ is a line of $\pi$. Therefore, every line of $\bar{\pi}^*$ intersects $\bar{\pi}^*_{P,Q}$ and every point of $\bar{\pi}^*$ is incident with a line of $\bar{\pi}^*_{P,Q}$ (that is, a line of $\bar{\pi}^*_{P,Q}$ extended to $\bar{\pi}^*$. Also note that $l_{P,Q}$ is a subline of $\pi_{P,Q,\delta}$ for $P, Q \in l$ where $\pi_{P,Q,\delta}$ is thought of as a line of $\bar{\pi}$. So the latter statement merely states that every affine point of $\bar{\pi}$ is contained in a Baer subplane $\pi_{P,Q,\delta}$ of $\pi$.)

Thus, $\bar{\pi}^*_{P,Q}$ is a *Baer* subplane of $\bar{\pi}^*$.

COROLLARY 2.12. *Let $\pi$ be a projective plane and $\delta$ a derivation set in $l_\infty$. Let $\bar{\pi}$ be the affine plane derived from $\pi$ by $\delta$. Then there is a derivation set $\bar{\delta}$ in $\bar{l}_\infty$ of the projective extension $\bar{\pi}^*$ such that the plane derived from $\bar{\pi}^*$ by $\bar{\delta}$ is the affine restriction of $\pi$ by $l_\infty$.*

*Proof.* The Baer subplanes $\bar{\pi}^*_{P,Q}$ of $\bar{\pi}^*$ all have the same set of points $\bar{\delta}$ on $\bar{l}_\infty$ (see proof of (2.11)). The affine restrictions of $\bar{\pi}^*_{P,Q}$ are affine lines of $\pi$. Clearly, $\bar{\delta}$ is a derivation set in $\bar{l}_\infty$.

It is trivial to verify that Baer subplanes are carried into Baer subplanes by collineations.

The following theorem is Ostrom's Theorem 7 and its Corollary [13]. His proofs to these results do not use finiteness in any way.

THEOREM 2.13. (*Ostrom* [13]). *Let $\pi$ be a projective plane and $\delta$ a derivation set on $l_\infty$. A collineation $\sigma$* of *$\pi \ni \sigma\delta = \delta$ induces a collineation $\bar{\sigma}$ of $\bar{\pi} \ni \bar{\sigma}$ fixes the set $\bar{\delta}$ (the corresponding derivation set of $\bar{l}_\infty$). If $\sigma$ is a translation of $\pi$, $\bar{\sigma}$ is a translation of $\bar{\pi}$.*

DEFINITION 2.14. Let $\pi$ be a projective plane and let $l$ be a line of $\pi$. We shall say that $\pi$ is a *semi-translation plane with respect to* $l$ if and only if $\pi$ admits a group $\mathscr{G}$ of elations with axis $l$, each of whose point orbits along with the set of elation centers for $l$ form a Baer subplane of $\pi$.

$\pi$ is a *strict semi-translation* (sst) plane with respect to $l$ if $\mathscr{G}$ is the full elation group with axis $l$ and *nonstrict* (nsst) otherwise.

THEOREM 2.15. (*See Ostrom* [13].) *Let $\pi$ be a projective plane and $l_\infty$ a line of $\pi$ and $\delta$ a derivation set in $l_\infty$ and let $\bar{\pi}$ denote the affine plane derived from $\pi$ by $\delta$. If $l$ is a line of $\pi$ whose affine restriction is not a line of $\bar{\pi}$ and $\pi$ admits a group of translations $\mathscr{G}$ (elations with axis $l_\infty$) transitive on the points of $l$, then $\bar{\pi}$ is a semi-translation plane, i.e., $\bar{\pi}^*$ (projective extension) is a semi-translation plane with respect to $\bar{l}_\infty$.*

*Proof.* By (2.13), since $\mathscr{G}\delta = \delta$, $\mathscr{G}$ is a group of translations of $\bar{\pi}$. If $l$ is a line of $\pi$ and the restriction of $l$ is not a line of $\bar{\pi}$ then $l - l \cap l_\infty$ is an affine Baer subplane of $\bar{\pi}$ (see (2.11)).

Thus we have extended Section III of [13] to arbitrary planes admitting derivations sets. We now consider planes possessing Baer subplanes.

We note that Ostrom's sufficient condition for derivation given in Theorem 9 [13] does not directly apply in the infinite case since the indicated affine subplanes are not necessarily *Baer* subplanes.

3. **Baer Subplanes.** It is well known and can be easily established by a counting argument that a finite projective plane of order $n$ has Baer subplanes of order $m$ only if $n$ is a square and the order of the subplane is $m = \sqrt{n}$.

For infinite planes no such characterization of Baer subplanes is known. We wish to develop some conditions which are sufficient for a given subplane to be a Baer subplane. For this will use some concepts of André [2] and Bose and Bruck [5].

DEFINITION. Let $V$ be a vector space. A *congruence* of $V$ is a set $\{V_\alpha\}_{\alpha\in\lambda}$ where $V_\alpha$ is a subspace of $V$ $\forall\alpha\in\lambda$ and

(1) $\bigcup_{\alpha\in\lambda} V_\alpha = V$ and (2) $V_\alpha \oplus V_\beta = V$ for all $\alpha \neq \beta \in \lambda$.

THEOREM 3.2. (*André* [2]). *An affine plane $\pi$ is a translation plane if and only if there is a congruence $\{V_\alpha\}_{\alpha\in\lambda}$ of a vector space $V$ such that the points of $\pi$ are the elements of $V$, the lines of $\pi$ are cosets of elements of $\{V_\alpha\}_{\alpha\in\lambda}$ and the parallel classes are the sets $\{V_\alpha + b, \alpha$ fixed $\in\lambda, b\in V\}$.*

THEOREM 3.3. (*Lüneburg* [11]). *Let $\alpha$ be a collineation of a projective plane with axis $l$ and center $P$. Let $Q$ be a point $\neq P$ and $Q\notin l$. Then every projective subplane containing $P, l, Q, Q\alpha$ is left invariant by $\alpha$.*

LEMMA 3.4. *Let $\pi$ be an affine translation plane and $\pi_0$ any affine subplane of $\pi$. Then there is a congruence $\{V_\alpha\}_{\alpha\in\lambda}$ for $\pi = V$, a subgroup $W$ of $V$, and subgroups $W_\alpha$ of $V_\alpha$ for $\alpha\in\lambda^*\subseteqq\lambda$ such that $\{W_\alpha\}_{\lambda^*}$ is a congruence for $W$ which defines $\pi_0$.*

*Proof.* Let $P, Q$ be points of $\pi_0$. There is a translation $\sigma$ of $\pi$ such that $P\sigma = Q$. By (3.3), $\pi_0$ is invariant under $\sigma$.

Clearly, there is a subgroup $\mathscr{T}_{\pi_0}$ of the translation group $\mathscr{T}$ of $\pi$ which is sharply transitive on the points of $\pi_0$ and leaves $\pi_0$ invariant.

Let $\mathscr{T}(P)$ denote the subgroup of $\mathscr{T}$ with fixed center $P\in l_\infty$ so that $\mathscr{T} = \bigcup_{P\in l_\infty}\mathscr{T}(P)$. $\mathscr{T}_{\pi_0} = \bigcup_{P\in l_\infty}\mathscr{T}(P)\cap\mathscr{T}_{\pi_0}$. Let $\mathscr{T}(P)\cap\mathscr{T}_{\pi_0} = \mathscr{T}_{\pi_0}(P)$. Thus, lines of $\pi$ are $\{\mathscr{T}(P)\}_{P\in l_\infty}$ and translates of these groups. $\{\mathscr{T}(P)\}_{P\in l_\infty}$ and $\{\mathscr{T}_{\pi_0}(P)\}_{P\in l_\infty}$ are congruences of $\pi$ and $\pi_0$, respectively, with the required properties. Note that $W_\alpha$ is not necessarily a vector subspace of $V_\alpha$ for $\alpha\in\lambda^*\subseteqq\lambda$.

Before utilizing (3.4) we mention the following result which depends only on the existence of a particular type of ternary ring.

THEOREM 3.5. *Let $Q = (Q, +, \cdot)$ be a ternary ring with ternary function $T$. Let $F = (F, +, \cdot)$ be a sub-ternary ring of $Q$ such that every element of $Q$ can be uniquely written in the form $t\alpha + \beta$ for some $t \in Q - F$; $\alpha, \beta \in F$. For all $a, m, b \in Q$ let $T(a, m, b) = tf(a, m, b) + g(a, m, b)$; $f, g$ functions from $Q \times Q \times Q$ into $F$. Let $f$ and $g$ satisfy properties* (1) *and* (2):

(1) *If $m$ and $b$ are fixed and $m \notin F$ there exists an element $a \in F$ such that $f(a, m, b) = 0$.*

(2) *If $a \notin F$ is fixed then $\{(f(a, m, b), g(a, m, b))\} = F \times F$ as $m, b$ vary over $F$.*

*Then the subplane $\pi^F$ coordinatized by $F$ of the plane $\pi^Q$ coordinatized by $Q$ is an affine Baer subplane.*

*Proof.* Let $l$ be a line of $\pi^Q$. If $l$ is $\{(x, y) \mid x = c \text{ for } c \in Q\}$ the line $l$ either contains points of $\pi^F$ or (in any case) is $\|$ to $\{(x, y) \mid x = \alpha; \alpha \in F\}$ so the projective extension of $\pi^Q$ contains a point of the projective extension of $\pi^F$.

If $l$ is $\{(x, y) \mid y = T(x, m, b); m, b \in Q\}$ and $m \in F$ then $l \cap l_\infty$ is a point of the projective extension of $\pi^F$. If $m \notin F$ then by (1) $\exists\ \alpha \in F \ni f(\alpha, m, b) = 0$. $\therefore\ (\alpha, g(\alpha, m, b)) \in \{(x, y) \mid y = T(x, m, b)\} \cap \pi^F$.

If $P$ is a point of $\pi^F$ let $P = (tx_1 + x_2, ty_1 + y_2)$; $x_i, y_i \in F$. The lines of $\pi^F$ are $\{(x, y) \mid x = \alpha, \alpha \in F\}$ and $\{(x, y) \mid y = T(x, \alpha, \beta); \alpha, \beta \in F\}$. If $x_1 y_1 = 0$ then $P \in \{(x, y) \mid x = \alpha\}$ or $\{(x, y) \mid y = \beta\}$ for some $\alpha, \beta \in F$. Thus assume $x_1 y_1 \neq 0$. Consider $T(tx_1 + x_2, \alpha, \beta)$ for some $\alpha, \beta \in F$. By (2), $\exists\ \alpha_0, \beta_0 \ni f(tx_1 + x_2, \alpha_0, \beta_0) = y_1$ and $g(tx_1 + x_2, \alpha_0, \beta_0) = y_2$.

COROLLARY 3.6. *Let $Q$ be an alternative field and $F$ the associated quaterion skewfield. Then $\pi^F$ is a Baer subplane.*

*Proof.* (See Pickert [16], s. 172–3.) $\exists\ t \in Q \ni at = t\bar{\alpha}$; $\alpha \in F$ and elements of $Q$ are of the form $t\alpha + \beta$; $\alpha, \beta \in F$ where $\bar{x}$ denotes a certain involuting automorphism.

$T$ is linear, so $T(\alpha, m, b) = \alpha(tm_1 + m_2) + tb_1 + tb_2$ (where $m_i, b_i \in F$, $i = 1, 2$) $= t(\bar{\alpha} m_1) + \alpha m_2 + tb_1 + b_2 = t(\bar{\alpha} m_1 + b_1) + am_1 + b_2$. Choose $\bar{\alpha} = -b_1 m_1^{-1}$, then (1) is satisfied. If $\alpha, \beta, a_i \in F$; $i = 1, 2$, then $T(ta_1 + a_2, \alpha, \beta) = (ta_1 + a_2)\alpha + \beta = (ta_1)\alpha + (a_2\alpha + \beta) = t(\alpha a_1) + a_2\alpha + \beta$. (See [16] Pickert, s. 172.) For $\rho, \chi \in F$ and $a_1 \neq 0\ \exists \alpha, \beta \in F \ni \alpha a_1 = \rho$ and $\chi = a_2\alpha + \beta$. $\therefore$ (2) is also satisfied.

We point out that although the Moufang planes contain Baer subplanes it is not clear whether derivation sets exist.

THEOREM 3.7. *Let $\pi$ be a translation plane and $\pi_0$ a proper subplane. Let $\{V_\alpha\}_\lambda$ and $\{V_\alpha\}_{\lambda^*}$ be congruences for $\pi$ and $\pi_0$, respectively* *where $W_\alpha$ is a subgroup of $V_\alpha$ and $W$ is a subgroup of $V$ for $\alpha \in \lambda^* \subsetneqq \lambda$, then if*

(i) (1) *$W \cap V_\delta = 0 \Rightarrow W + V_\delta = V$ for each $\delta \in \lambda$, or* (2) *$V$ and $W$ are finite dimensional over the same skewfield and there is an element $\delta \in \lambda - \lambda^*$ such that $W \cap V_\delta = 0$ and $W + V_\delta = V$, then every line of the projective extension of $\pi$ is incident with a point of the projective extension of $\pi_0$.*

(ii) *Under the assumptions of* (i) (1) *or* (2), *$\pi_0 \nsubseteqq \bigcup_{\alpha \in \lambda - \lambda^*}(V_\alpha + b)$ for any $b \in V - W$ if and only if $\pi_0$ us a Baer subplane.*

*Proof.* First we observe that $V_\alpha \cap W = W_\alpha$ or 0 depending on whether $\alpha \in \lambda^*$ or $\alpha \in \lambda - \lambda^*$.

Suppose $V_\alpha \cap W \neq 0$ and $\alpha \notin \lambda^*$. $W = \bigcup_{\rho \in \lambda^*} W_\rho \subseteqq \bigcup_{\rho \in \lambda^*} V_\rho$ and $W_\rho \subseteqq V_\rho$. By assumption, $\exists$ an element $w \in W - \{0\} \ni w \in V_\alpha$ and $\alpha \notin \lambda^*$. But $w \in V_\beta$ for some $\beta \in \lambda^*$. $\therefore$ $V_\alpha \cap V_\beta \neq 0$, which is a contradiction since $\alpha \neq \beta$.

$\therefore$ If $\alpha \in \lambda - \lambda^*$, $V_\alpha \cap W = 0$.

Assume $V_\alpha \cap W \neq 0$ and $\alpha \in \lambda^*$. $W_\alpha \subseteqq V_\alpha \cap W$ and $W_\alpha + W_\beta = W$; $\alpha, \beta \in \lambda^*, \alpha \neq \beta$.

If $c \in W - W_\alpha$ then $c = w_\alpha + w_\beta$ for some $w_\alpha \in W_\alpha$ and $w_\beta \in W_\beta - \{0\}$. If $c \in V_\alpha$ then $w_\beta \in V_\alpha$ which is a contradiction. Thus, $W_\alpha = V_\alpha \cap W$ if $\alpha \in \lambda^*$.

For (i) (1), $\delta, \alpha \in \lambda - \lambda^* \Rightarrow V_\delta + W = V_\alpha + W = V$. For (i) (2), $V = V_\delta + W$ is isomorphic to $V_\alpha + W \Rightarrow V_\alpha + W = V$ for all $\alpha \in \lambda - \lambda^*$.

Let $V_\alpha + b$ be any line of $\pi$. If $V_\alpha \cap W = 0$, then $\alpha \in \lambda - \lambda^*$ and $V_\alpha + W = V$ so $V_\alpha + b \cap W \neq \varnothing$. If $V_\alpha \cap W \neq 0$, then $V_\alpha + b$ for $\alpha \in \lambda^*$ is parallel to $V_\alpha + w$, $w \in W$ and since $(V_\alpha \cap W) + w$ is a line of $W = \pi_0$, (i) is proved.

If $\pi_0 = W \nsubseteqq \bigcup_{\alpha \in \lambda - \lambda^*} V_\alpha$ let $b$ be a point of $\pi$. If $b \in \bigcup_{\alpha \in \lambda^*} V_\alpha$, then $b$ is on a line of $W$. So assume $b \in V - \bigcup_{\alpha \in \lambda^*} V_\alpha$. Consider the set of lines $V_\alpha + b$, $\alpha \in \lambda$ on $b$. Each $V_\alpha + b$, $\alpha \in \lambda - \lambda^*$ intersects $W$ uniquely by the previous argument.

If $W \nsubseteqq_{\alpha \in \lambda - \lambda^*}(V_\alpha + b) \exists \delta \in \lambda^* \ni V_\delta + b$ intersects $W$.

$\therefore$ $\pi_0$ is an affine Baer subplane. Thus, (ii) is proved.

Let $PG(3, F)$ denote the projective 3-space over a skewfield $F$. Recall a spread (see Bose and Bruck [4]) $\mathscr{S}$ of $PG(3, F)$ is a covering set of skew lines of $PG(3, F)$.

Barlotti-Bose [3] have studied derivation in dual translation planes of dimension 2 (over their kernels) which correspond to spreads $\mathscr{S}$ of $PG(3, F)$ that have the property that any plane of $PG(3, F)$ contains a line of $\mathscr{S}$ (spreads which are dual spreads). In our terminology this requirement translates to: Let $V_4$ be a 4-dimensional vector space over $F$ and $\{V_\alpha\}_\lambda$ a congruence for $V_4$. Then *any* 3-dimensional subspace $W$ of $V_4$ contains a $V_\alpha$ for some $\alpha \in \lambda$.

REMARK 3.8. Let $V_4$ be a 4-dimensional vector space over a skewfield $F$. Let $\{V_\alpha\}_\lambda$ be a congruence for $V_4$. Then the Barlotti-Bose assumption is equivalent to asserting that *every* 2-dimensional subspace of $V_4$ which is not a $V_\alpha$, $\alpha \in \lambda$ corresponds to a Baer subplane.

*Proof.* Let $\Sigma$ be an arbitrary 3-dimensional vector subspace of $V_4$ Let $\Sigma_0$ be any 2-dimensional subspace of $\Sigma$. Assume $\Sigma_0$ is not a $V_\alpha, \alpha \in \lambda$. $\Sigma_0 = (\bigcup_{\alpha \in \lambda} V_\alpha) \cap \Sigma_0 = \bigcup_{\alpha \in \lambda} (V_\alpha \cap \Sigma_0)$. $V_\alpha \cap \Sigma_0$ is 1 or 0 dimensional. Define $\lambda^*$ as the subset of $\lambda$ such that $V_\alpha \cap \Sigma_0$ is 1-dimensional. Clearly $\{V_\alpha \cap \Sigma_0\}_{\lambda^*}$ is a congruence for $\Sigma_0$.

Assume the subplane $\pi_0$ corresponding to $\{V_\alpha \cap \Sigma_0\}_{\lambda^*}$ is a Baer subplane. Let $b \in \Sigma - \Sigma_0$. Then $b \in V_\alpha + r$ for some $\alpha \in \lambda^*$ and $r \in \Sigma_0$. Since $\Sigma_0$ is 2-dimensional, the subspace generated by $b$ and $\Sigma_0$, $\langle b, \Sigma_0 \rangle = \Sigma$. Since $V_\alpha \cap \Sigma_0$ is 1-dimensional and $b \notin \Sigma_0$ implies that $V_\alpha \subseteqq \langle b, \Sigma_0 \rangle$.

Conversely, assume that every 3-space of $V_4$ contains $V_\alpha$ for some $\alpha \in \lambda$. Let $\pi_0$ be the subplane corresponding to $\{V_\alpha \cap \Sigma_0\}_{\lambda^*}$ as above. Since (3.7) (i) (2) holds, we must show that (3.7) (ii) is satisfied. Let $c \in V_4 - \Sigma_0$. By assumption, the subspace $\langle -c, \Sigma_0 \rangle$ generated by $-c$ and $\Sigma_0$ contains a $V_\delta$ for some $\delta \in \lambda$. Clearly, $\delta \in \lambda^*$ for otherwise $V_\delta \cap \Sigma_0 = 0$. Thus, $c$ is on a line $V_\delta + \bar{c}$ of $\pi_0$, for $\bar{c} \in \Sigma_0$.

We note that Bruen and Fisher [7] have shown that not all spreads of $PG(3, F)$ have the Barlotti-Bose property.

The following theorem also proved by Barlotti and Bose [3] is included. Note that the two arguments are completely distinct.

DEFINITION 3.9. We shall say that a translation plane is of *dimension* 2 if the corresponding congruence is a 4-dimensional vector space over a skewfield $F$. A dual translation plane shall be said to be of dimension 2 if and only if its dual is of dimension 2.

THEOREM 3.10. *Let $\pi$ be any dual translation plane of dimension 2 such that the corresponding congruence has the property that any 3-space contains a 2-space of the congruence. Then $\pi$ is derivable.*

*Proof.* Let $Q$ be a coordinatizing (left) quasifield for $\pi$. $Q$ is a right 2-dimensional vector space over $F$ where $F$ is a skewfield contained in the kernel of $Q$. We assert that $\{(\alpha), (\infty), \alpha \in F\} \subseteqq l_\infty$ of $\pi$ is a derivation set.

It is straightforward to verify that the following sets are subplanes: $\{(a\alpha + b, a\beta + c); a \neq 0, b, c$ fixed elements of $Q \forall \alpha, \beta \in F\}$ (see, e.g., Ostrom [13], Theorem 9). By (2.10) it remains to show that they are Baer subplanes.

It is easy to see that the image of a Baer subplane under a collineation of the plane is a Baer subplane. We may coordinatize $\pi$ so that $(x, y) \to (x, y + c)$ for all $c \in Q$ are translations of $\pi$. We need only to consider the subplanes $\{(a\alpha + b, a\beta)\}$.

Let the lines $\{(x, y) \mid y = xm + b\}$, $\{(x, y) \mid x = c\}$, $\{(x, y) \mid y = c\}$ be denoted simply by $y = xm + b$, $x = c$ and $y = c$, respectively. We may coordinatize the dual plane of $\pi$ by the following: affine points $(m, -b)$ are lines $y = xm + b$ and infinite points $(\infty)$ and $(m)$, $m \in Q$ are lines $l_\infty$ and $x = m$, respectively, and conversely. (See, e.g., Fryxell [9].) That is,

$$(m, -b) \longleftrightarrow y = xm + b$$
$$(m) \longleftrightarrow x = m$$
$$(\infty) \longleftrightarrow l_\infty .$$

The lines of $\{(a\alpha + b, a\beta)\}$ are $l_\infty$, $y = x\alpha + a\beta - b\alpha$ and $x = a\delta + b$ for $a \neq 0$, $b$ fixed $\in Q$ and for all $\alpha, \beta \in F$.

The points of the associated dual subplane may be represented by $(\infty)$, $(a\delta + b)$, $(\alpha, b\alpha - a\beta)$ where juxtaposition denotes multiplication in $Q$. Thus if $*$ denotes multiplication in dual $Q$ then the points are $(\infty)$, $(\delta * a + b)$, $(\alpha, \alpha * b - \beta * a)$. Note that $(1, b)$ and $(0, a)$ form a vector basis for the set of affine points so that the affine subplane is a 2-dimensional vector subspace and hence is an affine Baer subplane. Since the dual of a Baer subplane is a Baer subplane, (3.10) is proved.

Bruen and Fisher [7] have shown that the condition of (3.9) is valid in any regular or subregular spread of $PG(3, F)$ and, of course, the condition is valid if $F$ is finite. In the finite case, Bruck and Bose [4] have pointed out that subregular spreads correspond to the translation planes constructed by a series of derivations in Desarguesian planes. Note that (3.9) in particular implies that Pappian planes coordinatized by fields $K$ that are quadratic extensions of fields $F$ are derivable. Also, finite André planes of order $q^2$ and kern $GF(q)$ may be constructed from Desarguesian planes by a series of derivations. This will be considered in the infinite case.

LEMMA 3.11. *Let $\pi$ be a Pappian plane. Let $\sigma$ be a nontrivial automorphism of the coordinatizing field $K \ni K$ is a 2-dimensional extension of a field $F$ where the fixed field of $\sigma$ is $F$. Then $\pi_0 = \{(x, y) \mid y = x^\sigma m\}$ is the set of points of an affine Baer subplane of $\pi$.*

*Proof.* $\pi$ is of dimension 2 and the spread corresponding to $\pi$ is regular (see [4] or [5]). Since $\pi_0$ is not a line of $\pi$ and is clearly a 2-dimensional vector space over $F$ it follows from the previous remarks and (3.8) that $\pi_0$ is a Baer subplane.

Thus, (3.11) is proved.

Let $L$ be a field and $\rho$ an automorphism of $L$ with fixed field $L_\rho$. If $m \in K$ the norm of $m$ is defined as $\prod_{\tau \in \langle\rho\rangle} m\tau$. If the order of $\rho$ is finite, an André system with kern $L_\rho$ may be defined (see [2] and also [8], p. 355). The lines of the corresponding André plane are cosets (translates) of the sets $\{(x, y) \mid y = x^{\rho(m)}m\}$, $\{(x, y) \mid x = 0\}$ where $\rho(m) \in \langle\rho\rangle$ such that if $m, n \in K$ and $\prod_{\tau \in \langle\rho\rangle} m\tau = \prod_{\tau \in \langle\rho\rangle} m\tau$ then $\rho(m) = \rho(n)$.

LEMMA 3.12. *Let $\pi$ be a Pappian plane coordinatized by a field $K$ which is a 2-dimensional extension of a field $F$. Let $\sigma$ be a non-trivial automorphism of order 2 with fixed field $F$. If $m \in K$ and $\prod_{\tau \in \langle\rho\rangle} m\tau = m^{1+\sigma} = x \in F$ then $\delta_x = \{(m) \in \pi \mid m^{1+\sigma} = x\}$ is a derivation set in $l_\infty$ of $\pi$. The Baer subplanes are the sets $\{(x, y) \mid y = x^\delta m$ for $m^{1+\sigma} = x\}$ and their translates.*

*Proof.* The sets $\{(x, y) \mid y = x^\sigma m\}$ and their cosets are Baer subplanes by (3.11).

Let $P$ and $Q$ be affine points of $\pi$ such that $PQ \cap l_\infty \in \delta_x$. $\therefore$ $PQ$ is a line $y = xm + b$ where $m^{1+\sigma} = x$; $m, b \in K$. $P, Q \in y = xm + b$ if and only if $P\tau_b, Q\tau_b \in y = xm$ where $\tau_b$ is the translation represented by $(x, y) \to (x, y - b)$.

Note that $(c^{\sigma-1}m)^{1+\sigma} = m^{1+\sigma} = x$. Therefore, $(c, d) \in y = x(c^{\sigma-1}m)$ if and only if $(c, d) \in y = x^\sigma m$.

We can assume without loss of generality that $Q\tau_b$ is $(0, 0)$. Thus $P\tau_b, (0, 0) \in y = xm$ if and only if $P\tau_b, (0, 0) \in y = x^\sigma(d^{1-\sigma}m)$ for some $d \in K$.

So there is a Baer subplane containing any two points $P$ and $Q$ such that $PQ \cap l_\infty \in \delta_x$.

LEMMA 3.13. (*See Bruen and Fisher* [7], *Theorems* 2 *and* 3.) *Let $\mathscr{S}$ be a regular spread in $PG(3, F)$ where $F$ is a field. Let $\mathscr{S} = \bigcup_{i \in \lambda} \mathscr{S}_i \cup \mathscr{S}_0$ where the $\mathscr{S}_i$; $i \in \lambda$ are disjoint reguli. Let $\overline{\mathscr{S}_i}$ denote the opposite regulus of $\mathscr{S}_i$ for all $i \in \lambda$. Then $\overline{\mathscr{S}} = \bigcup_{i \in \lambda} \overline{\mathscr{S}_i} \cup \mathscr{S}_0$ is a spread which is a dual spread.*

*Proof.* The argument is essentially the proof of Theorems 2 and 3 of [7]. We shall only sketch the proof.

$\mathscr{S}$ is a dual spread since it is regular. Hence if $\Sigma$ is a plane of $PG(3, F)$, $\Sigma$ contains a line $m$ of $\mathscr{S}$ and hence exactly one. If $m \in \mathscr{S}_0$ then $\Sigma$ contains a line of $\overline{\mathscr{S}}$. Therefore, assume $m \in \bigcup_{i \in \lambda} \mathscr{S}_i$. Let $m \in \mathscr{S}_{i'}$. The lines of $\overline{\mathscr{S}}$ meeting $m$ form a regulus (the opposite regulus to $\mathscr{S}_{i'}$) $\overline{\mathscr{S}_{i'}}$.

Then, if $p$ and $q$ are lines $\in \mathscr{S}_i - \{m\}$ it follows that $(p \cap \Sigma)\cdot(q \cap \Sigma)$ is a line of $\overline{\mathscr{S}_{i'}}$.

By Lemma 12.2 [4], it follows that $\{\{(x, y) \mid y = xm; m^{1+\sigma} = x\}\}$ is a regulus and $\{\{(x, y) \mid y = x^{\sigma}m; m^{1+\sigma} = x\}\}$ its opposite regulus. Thus, each derivation in this case is a matter of "switching" where a regulus is replaced by the opposite regulus. (This is well known in the finite case. See, e.g., [4].)

It appears that there are non-André planes of dim 2 that may be constructed in this way (this is certainly true in the finite case—see Ostrom [12]).

THEOREM 3.14. *Any André plane of dimension* 2 *may be constructed from a Pappian plane by a (possibly infinite) series of derivations.*

COROLLARY 3.15. *Any dual André plane of dimension* 2 *is derivable.*

*Proof.* (3.10), (3.12), (3.13), (3.14).

THEOREM 3.16. *Let $Q$ be any (right) quasifield which is a left 2-dim. vector space over a skewfield $F \subseteqq$ Kernel $Q$. Suppose also that $Q$ is a right 2-dim. vector space over $F$. Let $\pi$ be the translation plane coordinatized by $Q$. Let $\pi_a = \{(a\alpha, a\beta)$, fixed $a \neq 0 \in Q$ for all $\alpha, \beta \in F\}$. $\pi_a$ is a subplane of $\pi$ and $\pi_a$ is a right 2-dim. vector subspace of $\pi$ thought of as a (right) 4-dim. vector space over $F$. Suppose there is a skewfield $R \subseteqq F$ such that $\forall a \in Q - \{0\}$ $\pi_a$ is a left and right vector space of the same finite dimension over $R$. Then $\pi$ is derivable.*

*Proof.* We clearly may extend Ostrom's "homology type" replacement theorem to include the infinite case. (See (3.12), [14].) There is a congruence for $\pi$ which consists of the lines of $\pi$ through the origin. The partial congruence of lines with slopes in $F$ or $(\infty)$ "switches" with the partial congruence of subplanes $\pi_a$. It remains to show that we obtain a new congruence and hence a translation plane $\bar{\pi}$ "derived" from $\pi$.

Since $\pi_a$ is a left and right vector space of finite dimension $k$ over $R \subseteqq F$ and a right vector space of dim. 2 over $F$ then the dimension of $\pi_a$ over $R =$ right $\dim(\pi_a/F) \cdot \dim F/R$. Therefore, $\dim F/R = k/2$.

$\therefore$ Dim $\{(x, y) \mid y = xm\}$ is $k$ and $\pi_a$ and $y = xm, m \notin F$ are independent left $k$-dimensional subspaces over $R$. It follows that we obtain a *new* congruence over $R$.

Note that it was not required that $\pi_a$ be a Baer subplane for the proof. But, since a new congruence is obtained it follows that $\pi_a$ is a Baer subplane.

## 4. Applications.

*Derivation of Desarguesian Planes.*

By § 3, if $\pi$ is a Pappian plane of dim 2 over a field $K$, then $\pi$ is derivable.

Pickert [17] has given an algebraic construction of the Hall planes which does not require finiteness. Following Albert's [1] theory, the following theorem is clear.

THEOREM. *(See Pickert* [17], *Albert* [1].) *If $\pi$ is a Pappian plane of dimension* 2 *over a field K then the plane derived from $\pi$ is a Hall plane.*

Also note that a spread (congruence) corresponding to $\pi$ must be regular since $\pi$ is Pappian. Clearly then the Barlotti-Bose assumption is valid here. Furthermore, a derivation chain may be constructed on $\pi$ by Barlotti-Bose (see [3] and also [9]).

However, if $\pi$ is a Desarguesian, non-Pappian plane it is not clear that a spread for $\pi$ even contains a regulus. (There are finite spreads which do not contain reguli but, of course, are dual spreads (see, e.g., Bruen [6]).)

*The Derivation of the Quaterion Planes.*

The quaterions $Q$ can be considered as a right or left 2-dimensional vector space over the complex $\mathscr{C}$ numbers. Since $\mathscr{C}$ is 2-dimensional over the reals, (3.16) applies. Thus, the quaterion plane $\pi_1$ is derivable.

Consider $\pi_1 \xrightarrow{\text{derive}} \pi_2$. Clearly $\pi_2$ is a translation plane coordinatized by a quasifield $Q_2$ (note also that Ostrom's Theorems 9, 10, 11 [13] clearly extend to the infinite case in this situation) which is a right and left 2-dimensional vector space over the complex numbers.

That is, let $\{1, t\}$ be a basis for $Q/\mathscr{C}$ so that elements of $Q$ are written in the form $t\alpha + \beta$, $\alpha, \beta \in \mathscr{C}$. Let $\{1, i, j, k\}$ be the standard basis for $Q$ over the reals.

Let $*$ denote multiplication in $Q_2$, then $(\alpha + \beta)*t = tz_1 + z_2$ iff $z_1 \cdot t = t(\alpha + \beta) + z_2$ so $z_1 t = t\alpha + t\beta + z_2$.

Let $z_1 = a + bi$, $a, b$ real numbers, and $t = k$ so $(a + bi)k = ak + b(-j) = ka - jb = k(a - ib)$. So $z_1 t = t\bar{z}_1$($\bar{z}_1$ denotes the complex conjugate of $z_1$).

$\therefore\ t\bar{z}_1 = t(\alpha + \beta) + z_2$ so $\alpha + \beta = \bar{z}_1, z_2 = 0\ \overline{\alpha + \beta} = z_1$.

$\therefore\ (\alpha + \beta)*t = t\overline{(\alpha + \beta)} = t\alpha + t\beta = \alpha t + \beta t$. So $\alpha * t = \alpha \cdot t$. It follows also that $\alpha * a = \alpha \cdot a$ for all $\alpha \in \mathscr{C}$ and $a \in Q$.

Thus, $Q_2$ is 2-dim/$\mathscr{C}$, $\mathscr{C}$ is the kernel of $Q_2$ and $Q_2$ is also right 2-dim over $\mathscr{C}$.

It is fairly easy to verify that multiplication $*$ in $\pi_2$ may be obtained as:

$$(t\alpha + \beta)*(t\delta + \gamma) = t\overline{(\beta - \alpha\delta^{-1}\gamma)}\delta + \overline{(\beta - \alpha\delta^{-1}\gamma)}\gamma - \overline{\alpha\delta^{-1}}\,.$$

From this equation the mult $*$ can be defined in terms of the basis $\{1, i, j, k\}$.

Some open questions here are:

(1) Is the full collineation group of $\pi_2$ the group inherited from $\pi_1$?

(2) Is $\pi_2$ a previously known plane?

Let $\pi_a = \{(a\alpha, a\beta)\}$, $a = ta_1 + a_2$; $a_i \in \mathscr{C}$ and $a_1a_2 \neq 0$. Then if $\rho \in \mathscr{C}$, $\rho a = a\delta$ for some $\delta \in \mathscr{C}$ if and only if $\bar{\rho} = \rho$. Thus, $\pi_a$ is a right and left vector subspace of dimension 4 over the reals but is not, in general, a left subspace over the complex numbers.

*The Derivation of André Planes*

I. *Nearfield planes.* Sabharwal [20] has constructed a class of infinite nearfield planes (which are André planes). Each nearfield is of dimension 2 over its kernel where the kernel is a finite extension by radicals of the rationals.

By theorem (3.15) the dual planes are derivable. Actually, Sabharwal shows that a derivation chain can be based on these planes. Moreover, he shows how to construct infinite analogues of the Hughes planes and considers a derivation chain on such planes.

Sabharwal's description is essentially given as follows: Let $F = Q(\sqrt{p})$ where $Q$ is the field of rationals and $p$ is a positive nonsquare in $Q$.

Define multiplication

$$x \circ y = \begin{cases} xy & \text{if the norm } x = x^{1+\sigma} \geqq 0 \\ xy^{\sigma} & \text{if } x^{1+\sigma} < 0 \end{cases}$$

where $\sigma$ is the automorphism $\sqrt{p} \xrightarrow{\sigma} -\sqrt{p}$.

II. *Bol planes.* Burn [8] has given an example of an infinite Bol quasifield $Q$ which is an André system. Both the plane $\pi$ coordinatized by $Q$ and its dual are derivable by (3.15) and (3.16). Moreover, it appears that a derivation chain may be constructed on $\pi$ (see [8], pp. 356–357).

*Semifield planes.* Infinite weak nucleus semifields may be constructed analogous to the Hughes-Kleinfeld-Knuth finite semifields (see [10]) which be derivable by (3.16).

Because of space, we shall postpone explication of the derived planes of this section to a later paper. The discussion of "nets" has been avoided in this treatment, although the set of Baer subplanes of a derivable plane form lines of a net. In the *finite* case the union of two disjoint nets on the same points form a net. However, in the infinite situation this has yet to be proved.

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# THE DISAPPEARING CLOSED SET PROPERTY

V. M. KLASSEN

**A topological space $X$ is said to have the disappearing closed set ($DCS$) property or to be a $DCS$ space, if for every proper closed subset $C$ there is a family of open sets $\{U_i\}_{i=1}^{\infty}$ such that $U_{i+1} \subseteq U_i$ and $\bigcap_{i=1}^{\infty} U_i = \emptyset$, and there is also a sequence $\{h_i\}$ of homeomorphisms on $X$ onto $X$ such that $h_i(C) \subseteq U_i$, for all $i$. Properties of $DCS$ spaces are studied as are connections between this and other related definitions.**

I. Simple examples of sets with the $DCS$ property are the $n$-sphere, $n > 0$, and the open $n$-cell, $n > 0$. This definition was formulated in an attempt to generalize the definition of invertible set which has been extensively studied by Doyle, Hocking and others [1, 2, 3, 4, 6]. A space $X$ is said to be invertible if for every proper closed subset $C$ of $X$ there is a homeomorphism $h$ on $X$ onto $X$ such that $h(C) \subseteqq X - C$. Neither of these definitions implies the other. For example, an open $n$-cell is not invertible, and on the other hand, the 0-sphere is invertible but does not satisfy the $DCS$ property. However, both definitions require that closed sets can be made "small" or "thin."

It is proved in [5] that compact $n$-manifolds have the $DCS$ property. It is the purpose of this paper to investigate some other topological properties of $DCS$ spaces.

II. THEOREM 1. *Any disconnected $DCS$ space $X$ must have an infinite number of components.*

*Proof.* Suppose $X$ has a finite number of components, $A_j$, $j = 1, \cdots, n$. Each $A_j$ is both open and closed. Consider the $DCS$ property applied to $\bigcup_{j=2}^{n} A_j = B$, a closed set. There are open sets $\{U_i\}_{i=1}^{\infty}$ and homeomorphisms $\{h_i\}_{i=1}^{\infty}$ such that $h_i(B) \subseteqq U_i$, $U_{i+1} \subseteqq U_i$, and $\bigcap_{i=1}^{\infty} U_i = \emptyset$. Since there are at most a finite number of components $A_i$ and since the $U_i$ form a decreasing sequence of open sets whose intersection is empty, there must be an $m$ such that for each $j = 1, \cdots, n$, there are $x_j \in A_j$ such that $x_j \notin U_m$. But $X - U_m \subseteqq h_m(A_1)$, since $h_m(B) \subseteqq U_m$ and $X = A_1 \cup B$, $A_1 \cap B = \emptyset$. Thus $x_j \in h_m(A_1)$, $j = 1, \cdots, n$. But this is a contradiction unless $n = 1$, since $h_m(A_1)$ is connected, but intersects all components of $X$.

An example of a $DCS$ space which is not connected is the product space obtained by crossing the real numbers with the rationals.

One method of constructing $DCS$ spaces is given by the following:

THEOREM 2. *If $X$ and $Y$ are DCS spaces, so is $X \times Y$.*

*Proof.* Let $C$ be a proper closed subset of $X \times Y$, and let $P \subseteqq X$, $Q \subseteqq Y$ be open sets in $X$ and $Y$, respectively, such that $P \times Q \subseteqq X \times Y - C$. Let $\{U_i\}_{i=1}^{\infty}$, $\{h_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$, $\{k_i\}_{i=1}^{\infty}$ be the open sets and homeomorphisms for $X - P$ and $Y - Q$ in $X$ and $Y$, respectively. If $(x, y) \in X \times Y$, define $\phi_i(x, y) = \{h_i(x), k_i(y)\}$. Now $\{W_i\}_{i=1}^{\infty} = \{(U_i \times Y) \cup (X \times V_i)\}_{i=1}^{\infty}$ is a decreasing sequence of open sets in $X \times Y$, with empty intersection. Also, $\phi_i(C) \subseteqq W_i$. Thus, $X \times Y$ has the $DCS$ property.

The relation between invertible spaces and spaces with the $DCS$ property can be seen more clearly in the following analysis.

If an invertible $T_1$ space $X$ has the property that the intersection of all neighborhoods of any point is that point, and if any closed set $C$ in an open set $U$ may be "moved" so as to miss any given $x \in U$, without moving outside $U$, then $X$ has the $DCS$ property. (If $U$ is open, $U - \{x\}$ is also.)

III. This suggests a relationship with another concept, also studied by Doyle and Hocking. A space $X$ is near-homogeneous if for any $x \in X$ and any open set $U$ such that $x \in U$, for every $y \in X$ there is a homeomorphism on $X$ onto $X$ such that $h(y) \in U$.

Once again, the 0-sphere is a space that does not satisfy the $DCS$ property, but is near-homogeneous. However, the following converse is true:

THEOREM 3. *Every DCS space $X$ is near-homogeneous.*

*Proof.* Let $x \in X$ and $U$ an open set containing $x$. Let $y \in X$. Consider $C = X - U$, a proper closed subset of $X$. Since $X$ has the $DCS$ property, there is a sequence of homeomorphisms $\{h_i\}_{i=1}^{\infty}$ on $X$ onto $X$ such that $\bigcap_{i=1}^{\infty} h_i(C) = \varnothing$, a somewhat weaker statement than the $DCS$ property allows. There is some $j$ such that $y \notin h_j(C)$. But then $y \in h_j(U)$, so $h_j^{-1}(y) \in U$. Thus, $X$ is near-homogeneous.

In the preceding proof, it is seen that near-homogeneity does not require that closed sets get "thin," but that they move around enough. An equivalent form of the definition of near-homogeneity, related to the $DCS$ property, is of interest here.

THEOREM 4. *Let $H(X)$ be the family of all homeomorphisms on $X$ onto $X$. $X$ is near-homogeneous iff, for every proper closed set $C \subseteqq X$, $\bigcap_{h \in H(X)} h(C) = \varnothing$.*

*Proof.* If $X$ is near-homogeneous, let $C$ be a closed subset of $X$, and let $U = X - C$. Let $y \in C$. Then there is an $h \in H(X)$ such that $h(y) \in U$, by near-homogeneity and thus $\bigcap_{h \in H(X)} h(C) = \varnothing$.

Conversely, let $x, y \in X$, and let $U$ be an open set such that $x \in U$. Let $C = X - U$. If $y \notin C$, there is nothing to show, so suppose $y \in C$. Then there is an $h \in H(X)$ such that $h(y) \notin C$. Otherwise $\bigcap_{h \in H(X)} h(C)$ would not be empty. But this is the desired homeomorphism.

IV. Another definition relating to invertibility that has been studied is that of local invertibility. A space $X$ is said to be invertible at a point $x \in X$ if for every open set $U$ containing $x$ there is a homeomorphism $h$ on $X$ onto $X$ such that $h(X - U) \subseteqq U$. In [2] it was proved that for such a space certain local properties become global properties. For example, if $X$ is invertible and locally compact at $x$, then $X$ is compact. The corresponding definition here is the following. A space $X$ has the $DCS/x$ property for all closed sets which miss $x$. It is evident that a space $X$ has the $DCS$ property, iff it has the $DCS/x$ property for each $x \in X$. Examples of spaces with the $DCS/x$ property include the closed $n$-cell, the $n$-leafed rose and, in fact any space that is invertible at $x$ in such a way that the inverting homeomorphism may be taken to fix $x$. A space that is not invertible at any point but which does have the $DCS/x$ property is the "half-open" annuls $[0, 1) \times S_1$. It will have the $DCS/x$ property for every point of $\{0\} \times S_1$.

Since the $DCS/x$ definition cannot guarantee that any part of the closed set will be carried close to $x$ under any of the homeomorphisms, theorems as sweeping as those of local invertibility cannot be obtained. However, the following is true:

THEOREM 5. *Let $X$ be a space that has the $DCS/x$ property at $x$ and suppose $X$ is locally $T_i$, $i = 0, 1, 2$, in a neighborhood $P$ of $x$. Then $X$ is $T_i$.*

*Proof.* Let $y, z \in X$, $y \neq z$ (perhaps one is $x$). Let $\{U_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ be the open sets and homeomorphisms given by the $DCS/x$ property for the closed set $X - P$. There is a $j$ such that $y, z \notin U_j$. Then $y, z \notin h_j(X - P)$, so $y, z \in h_j(P)$. But then $h_j(y)$ and $h_j(z)$ have the separation property required and thus $y$ and $z$ do also.

Note that this kind of argument is an improvement on near-homogeneity, since it makes it possible to bring two points (or any finite number of points) into a neighborhood of $x$ at once.

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CALIFORNIA STATE UNIVERSITY, FULLERTON

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# ON THE ABSOLUTE MATRIX SUMMABILITY OF FOURIER SERIES

B. KUTTNER AND B. N. SAHNEY

**The paper investigates sufficient conditions under which a summability method of a certain general type absolutely sums the Fourier series of any function of bounded variation. The main theorem includes a recent theorem of M. Izumi and S. Izumi, who considered the problem for the special case of Nörlund summability.**

The summability methods considered are those given by a series-to-series transformation $A = (\alpha_{n,k})$. That is to say, given any series

$$\sum_{k=0}^{\infty} a_k \,, \tag{1}$$

we describe (1) as summable $A$ to $s$ if

$$b_n = \sum_{k=0}^{\infty} \alpha_{n,k} a_k$$

is defined for all $n$, and if

$$\sum_{n=0}^{\infty} b_n \tag{2}$$

converges to $s$. We describe (1) as absolutely summable $|A|$ if (2) converges absolutely. Under certain quite weak restrictions on $A$, necessary and sufficient conditions under which the Fourier series of any function of bounded variation should be absolutely summable $|A|$ have been given by Tripathy [10, Lemma 2]; his result will be stated later as Lemma 1. But the conditions obtained by Tripathy are of such a nature that it is not usually easy in any given example to determine whether they are satisfied or not. The object of the present paper is to obtain sufficient conditions which, while less general, are simpler than those of Tripathy. However, it does not seem possible to obtain reasonably general sufficient condition in any very simple form.

We will be concerned with the case in which $A$ is absolutely conservative, that is to say, it is such that, whenever (1) converges absolutely, so does (2). It is known [4, 6] that in order that this should hold it is necessary and sufficient that, for $k \geqq 0$,

$$\sum_{n=0}^{\infty} |\alpha_{n,k}| = O(1) \,. \tag{3}$$

We remark that, in order that $A$ should be absolutely regular, that is to say, that in order that, whenever (1) converges absolutely then (2) converges absolutely to the same sum, it is necessary and sufficient that (3) should hold and that, further, for all $k \geqq 0$,

$$\sum_{n=0}^{\infty} \alpha_{n,k} = 1 . \tag{4}$$

2. We now state our main result.

THEOREM. *Let $A = (\alpha_{n,k})$ be an absolutely conservative series-to-series transformation, with $\alpha_{n,k} \geqq 0$ for all $n$, $k$. Suppose that either*

(a) *For each fixed $n$, there is a positive integer $r(n)$ such that $\alpha_{n,k}$ is nondecreasing for $1 \leqq k \leqq r(n)$, and nonincreasing for $k \geqq r(n)$, or*

(b) *For each fixed $n$, there is a positive integer $s(n)$, such that $\alpha_{n,k}/k$ is nondecreasing for $1 \leqq k \leqq s(n)$, and nonincreasing for $k \geqq s(n)$. Suppose also in case (a) that, for $K \geqq 1$,*

$$\sum_{r(n) \geqq 2K} \frac{1}{r(n)} \sum_{k=r(n)-K}^{r(n)+K} \alpha_{n,k} = O(1) , \tag{5}$$

*and in case (b) that, for $K \geqq 1$,*

$$\sum_{s(n) \geqq 2K} \frac{1}{s(n)} \sum_{k=s(n)-K}^{s(n)+K} \alpha_{n,k} = O(1) . \tag{6}$$

*Then the Fourier series of any function of bounded variation is absolutely summable $|A|$.*

REMARK. It is clear that (5) is equivalent to

$$\sum_{r(n) \geqq 2K} \sum_{k=r(n)-K}^{r(n)+K} \frac{\alpha_{n,k}}{k} = O(1) \tag{5'}$$

and it is sometimes more convenient to express (5) in this form. Since there are $2K + 1$ terms in the inner sum in (5), and since the middle term is the greatest, a sufficient condition for (5) is that

$$\sum_{r(n) \geqq 2K} \frac{\alpha_{n,r(n)}}{r(n)} = O\left(\frac{1}{K}\right) . \tag{7}$$

However (7), while much simpler than (5) is less general, and, as will be shown later, fails to be satisfied in some important cases. In a similar way, (6) is equivalent to

$$\sum_{s(n) \geqq 2K} \sum_{k=s(n)-K}^{s(n)+K} \frac{\alpha_{n,k}}{k} = O(1) ; \tag{6'}$$

also, a sufficient condition for (6) is that

$$\sum_{s(n)\geqq 2K} \frac{\alpha_{n,s(n)}}{s(n)} = O\left(\frac{1}{K}\right). \tag{8}$$

It is clear that either one of (a), (b) could be satisfied without the other holding. If, however, they both hold, then (5) is a weaker assumption than (6). Thus, in this case, the first form of the theorem is preferable. To prove this assertion, we write $\theta_n$, $\phi_n$ for the inner sums in (5′), (6′) respectively, and shall first show that

$$\theta_n \leqq 2\phi_n . \tag{9}$$

To this end, we first note that $s(n) \leqq r(n)$. Consider first the case in which $r(n) - s(n) \geqq K$. Since $\alpha_{n,k}/k$ is nonincreasing for $k \geqq s(n)$, we have[1], for $\mu = 0, 1, \cdots, K-1$,

$$\begin{aligned} &\frac{\alpha(n, r(n)+K-2\mu)}{r(n)+K-2\mu} + \frac{\alpha(n, r(n)+K-2\mu-1)}{r(n)+K-2\mu-1} \\ &\qquad \leqq \frac{2\alpha(n, s(n)+K-\mu)}{s(n)+K-\mu} . \end{aligned} \tag{10}$$

Also,

$$\frac{\alpha(n, r(n)-K)}{r(n)-K} \leqq \frac{\alpha(n, s(n))}{s(n)} ,$$

whence

$$\theta_n \leqq 2 \sum_{k=s(n)}^{s(n)+K}{}' \frac{\alpha_{n,k}}{k} \leqq 2\phi_n ,$$

where the dash indicates the term $k = s(n)$, is multiplied by 1/2. If $r(n) - s(n) = t(n) < K$, then (10) still holds for $\mu \leqq t(n) - 1$. Hence

$$\theta_n \leqq 2 \sum_{\mu=0}^{t(n)-1} \frac{\alpha(n, s(n)+K-\mu)}{s(n)+K-\mu} + \sum_{\nu=2t(n)}^{2K} \frac{\alpha(n, r(n)+K-\nu)}{r(n)+K-\nu} \tag{11}$$

where the first sum on the right is taken as 0 if $t(n) = 0$. Since the second sum on the right of (11) can be written

$$\sum_{\mu=t(n)}^{2K-t(n)} \frac{\alpha(n, s(n)+K-\mu)}{s(n)+K-\mu} ,$$

we again deduce (9).

It now follows from (9) that

$$\sum_{s(n)\geqq 2K} \theta_n \leqq 2 \sum_{s(n)\geqq 2K} \phi_n .$$

However, since $s(n) \leqq r(n)$, there may be values of $n$ for which $r(n) \geqq 2K$, but $s(n) < 2K$; these values will occur in the sum (5′), but not

[1] To avoid complicated suffixes, we write $\alpha(n, k)$ for $\alpha_{n,k}$ whenever $n$, $k$ are replaced by more complicated expressions

in (6′). If we show that, in any case, the contribution of these terms to the sum (5′) is bounded, the conclusion will now follow. If $r(n) \geqq 2K$ but $s(n) \leqq K$, then, since $\alpha_{n,k}/k$ is nonincreasing for $k \geqq K$ we deduce that

$$\theta_n \leqq \frac{(2K+1)\alpha_{n,K}}{K} .$$

If $r(n) \geqq 2K$ and $K < s(n) < 2K$, then $\alpha_{n,k}$ is nondecreasing for $k \leqq 2K$. Hence, for all $k \geqq 1$

$$\frac{\alpha_{n,k}}{k} \leqq \frac{\alpha_{n,s(n)}}{s(n)} \leqq \frac{\alpha_{n,2K}}{K}$$

so that

$$\theta_n \leqq \frac{(2K+1)\alpha_{n,2K}}{K} .$$

Thus the sum of the terms in question does not exceed

$$\frac{(2K+1)}{K} \sum_{n=0}^{\infty} (\alpha_{n,K} + \alpha_{n,2K}) = O(1) ,$$

by (3).

3. We now state the lemma of Tripathy already mentioned.

LEMMA 1. *Let $A = (\alpha_{n,k})$ be a series-to-series transformation such that*

$$\sum_{n=0}^{\infty} |\alpha_{n,0}| < \infty \tag{12}$$

*and such that, for every fixed $n$,*

$$L_n(t) = \sum_{k=1}^{\infty} \alpha_{n,k} \frac{\sin kt}{k} \tag{13}$$

*converges boundedly in $t$. Then in order that the Fourier series of any function of bounded variation should be absolutely summable $|A|$, it is sufficient that*

$$\sum_{n=0}^{\infty} |L_n(t)| = O(1) , \tag{14}$$

*and necessary that the sum* (14) *should be essentially bounded.*

It may be remarked that the result is not quite correctly stated in [10], where it is asserted that the essential boundedness of (14) is necessary and sufficient. But on examining the proof of sufficiency in [10], we find that it requires the boundedness, and not just the essential boundedness, of (14). The point is not of great importance, since if we assume that, for every fixed $n$,

$$\sum_{k=1}^{\infty} |\alpha_{n,k} - \alpha_{n,k+1}| < \infty ;$$

in other words, that $b_n$ is defined whenever (1) converges, it is easy to prove that the essential boundedness of (14) is equivalent to its boundedness. This result is not, however, required for our present purposes.

In what follows, we will suppose throughout that $0 < t \leqq \pi$. We will apply the hypothesis (5) or (6) with $K = [\pi/t]$; thus, in any equations involving both $K$ and $t$, it will be assumed that this relation holds.

We require two further lemmas.

LEMMA 2. *Let $A = (\alpha_{n,k})$ be an absolutely conservative series-to-series transformation. If, for every fixed $n$, $\alpha_{n,k}/k$ is ultimately nonnegative nonincreasing (and thus, in particular, if the hypotheses of the theorem are satisfied) then the hypotheses of Lemma 1 are satisfied.*

Equation (12) follows at once as a special case of (3). Thus, taking $n$ as fixed, we have only to verify that (13) converges boundedly. Suppose that $\alpha_{n,k}/k$ is nonnegative nonincreasing for $k \geqq M$. Then we have, uniformly in $k_1$, $k_2$ for $K$, $M \leqq k_1 \leqq k_2$,

$$\left| \sum_{k=k_1}^{k_2} \alpha_{n,k} \frac{\sin kt}{k} \right| \leqq \frac{\alpha(n, k_1)}{\left(2 \sin \frac{1}{2} t\right) k_1} . \tag{15}$$

But (3) implies that $\alpha(n, k)$ is bounded; hence the expression on the right of (15) is $O(1)$ uniformly in the range considered, and, for fixed $t$, tends to 0 (uniformly in $k_2$) as $k_1 \to \infty$. Since $M$ is a constant,

$$\sum_{k=1}^{M-1} \left| \alpha_{n,k} \frac{\sin kt}{k} \right|$$

is bounded; also, if $K \geqq M$,

$$\sum_{k=M}^{K} \left| \alpha_{n,k} \frac{\sin kt}{k} \right| \leqq t \sum_{k=M}^{K} |\alpha_{n,k}| = O(1)$$

(by the boundedness of $\alpha_{n,k}$ and the definition of $K$). Hence the result.

LEMMA 3. *Suppose that $\theta_k \geqq 0$. Suppose that $\theta_k$ is nondecreasing* *for* $1 \leqq k \leqq s$, *and nonincreasing for* $k \geqq s$. *Then, for any positive integers* $a$, $b$, *and any* $t$ *with* $0 < t \leqq \pi$,

$$\left|\sum_{k=a}^{b} \theta_k e^{ikt}\right| \leqq A \sum_{k=\max(1,s-K)}^{s+K} \theta_k \,, \tag{16}$$

where $A$ is an absolute constant.

That portion (if any) of the sum on the left for which $s - K \leqq k \leqq s + K$ clearly satisfies the required inequality. Also, by partial summation, that portion (if any) for which $k > s + K$ does not, in modulus, exceed

$$\begin{aligned}\frac{2}{|1-e^{it}|}\theta_{s+K} &\leqq \frac{2}{|1-e^{it}|(K+1)}\sum_{k=s}^{s+K}\theta_k \\ &\leqq \frac{2t}{\pi|1-e^{it}|}\sum_{k=s}^{s+K}\theta_k \,.\end{aligned}$$

That portion of the sum (if any) for which $k < s - K$ may be dealt with in a similar way, and the conclusion follows.

This lemma is a slight generalisation of a lemma due to McFadden [5] which has been much used in investigations on the Nörlund summability of Fourier series.

4. We now come to the proof of the theorem. It follows from Lemmas 1 and 2 that it is enough to show that the hypotheses of the theorem imply (14). Consider first those values of $n$ (if any) for which $r(n) < 2K$ in case (a), and for which $s(n) < 2K$ in case (b). In case (b), we are given that $\alpha_{n,k}/k$ is nonincreasing for $k \geqq 2K$; in case (a), we are given that $\alpha_{n,k}$ is nonincreasing for $k \geqq 2K$; hence, a fortiori, so is $\alpha_{n,k}/k$. Thus, in either case, since the partial sums of $\sum \sin kt$ are $O(1/t)$, we have

$$\sum_{k=2K}^{\infty} \alpha_{n,k}\frac{\sin kt}{k} = O\left(\frac{\alpha_{n,2K}}{2Kt}\right) = O(\alpha_{n,2K})$$

by definition of $K$. For those terms in the sum (13) for which $k \leqq 2K$, we use $|\sin kt| \leqq kt$; and it follows that

$$|L_n(t)| = O\left\{t\sum_{k=1}^{2K-1}\alpha_{n,k}\right\} + O(\alpha_{n,2K}) \,.$$

Hence the contribution to the sum (14) of those values of $n$ now under consideration is

$$O\left\{t\sum_{k=1}^{2K-1}\sum_{n=0}^{\infty}\alpha_{n,k}\right\} + O\left\{\sum_{n=0}^{\infty}\alpha_{n,2K}\right\} = O(1) \tag{17}$$

by (3) and the definition of $K$.

We now investigate the remaining values of $n$. Consider first case (b). For any fixed $n$, we apply Lemma 3 with $\theta_k = \alpha_{n,k}/k$, and take the imaginary part of (16). It follows at once that

$$L_n(t) = O\left\{\sum_{k=s(n)-K}^{s(n)+K} \frac{\alpha_{n,k}}{k}\right\};$$

and (14) therefore follows from (6′) and (17).

Now consider case (a). Since $\alpha_{n,k}$ is nonincreasing for $k \geqq r(n)$ so is $\alpha_{n,k}/k$; thus the part of the sum (13) for which $k > r(n) - K$ may be dealt with as in case (b). The part for which $k < K$ may be dealt with by using $|\sin kt| \leqq kt$, as in the proof of (17). Thus, writing

$$R_n(t) = \sum_{k=K}^{r(n)-K} \alpha_{n,k} \frac{\sin kt}{k},$$

it remains only to show that

$$\sum_{r(n)\geqq 2K} |R_n(t)| = O(1). \tag{18}$$

Now,

$$\begin{aligned} R_n(t) &= \frac{1}{2\sin\frac{1}{2}t} \sum_{k=K}^{r(n)-K} \frac{\alpha_{n,k}}{k}\left[\cos\left(k-\frac{1}{2}\right)t - \cos\left(k+\frac{1}{2}\right)t\right] \\ &= \frac{1}{2\sin\frac{1}{2}t}\left\{-\sum_{k=K}^{r(n)-K} \cos\left(k+\frac{1}{2}\right)t\Delta_k\left(\frac{\alpha_{n,k}}{k}\right)\right. \\ &\quad + \frac{\alpha_{n,K}}{K}\cos\left(K-\frac{1}{2}\right)t \\ &\quad \left. - \frac{\alpha(n, r(n)-K+1)}{r(n)-K+1}\cos\left(r(n)-K+\frac{1}{2}\right)t\right\}. \end{aligned}$$

Since

$$\Delta_k\left(\frac{\alpha_{n,k}}{k}\right) = \frac{\alpha_{n,k}}{k(k+1)} + \frac{\Delta_k(\alpha_{n,k})}{k+1},$$

it follows that

$$\begin{aligned} R_n(t) &= O\left\{\frac{1}{t}\left[\sum_{k=K}^{r(n)-K} \frac{\alpha_{n,k}}{k(k+1)} + \sum_{k=K}^{r(n)-K} \frac{|\Delta_k(\alpha_{n,k})|}{k+1}\right.\right. \\ &\quad \left.\left. + \frac{\alpha_{n,K}}{K} + \frac{\alpha(n, r(n)-K+1}{r(n)-K+1}\right]\right\} \\ &= O\{R_n^1(t) + R_n^2(t) + R_n^3(t) + R_n^4(t)\}, \end{aligned}$$

say. Now, since $\alpha_{n,k}$ is nondecreasing in the relevant range

$$
\begin{aligned}
R_n^{n_2}(t) &= -\frac{1}{t}\sum_{k=K}^{r(n)-K}\frac{\Delta_k(\alpha_{n,k})}{k+1} \\
&= \frac{1}{t}\sum_{k=K}^{r(n)-K}\frac{\alpha_{n,k}}{k(k+1)} - \frac{1}{t}\frac{\alpha_{n,k}}{K} + \frac{1}{t}\frac{\alpha(n,\, r(n)-K+1)}{r(n)-K+1} \\
&= R_n^1(t) - R_n^3(t) + R_n^4(t)\ ,
\end{aligned}
$$

so that

$$R_n(t) = O\{R_n^1(t) + R_n^4(t)\}\ .$$

Next,

$$\sum_{r(n)\geqq 2K} R_n^1(t) \leqq \frac{1}{t}\sum_{k=K}^{\infty}\frac{1}{k(k+1)}\sum_{n=0}^{\infty}\alpha_{n,k} = O(1)$$

by (3) and the definition of $K$. Finally, if $r(n) \geqq 2K$,

$$
\begin{aligned}
R_n^4(t) &= O\left\{\frac{\alpha(n,\, r(n)-K+1)}{tr(n)}\right\} \\
&= O\left\{\frac{1}{tKr(n)}\sum_{k=r(n)-K+1}^{r(n)}\alpha_{n,k}\right\} \\
&= O\left\{\frac{1}{r(n)}\sum_{k=r(n)-K+1}^{r(n)}\alpha_{n,k}\right\}\ ,
\end{aligned}
$$

so that

$$\sum_{r(n)\geqq 2K} R_n^4(t) = O(1)\ ,$$

by (5). The proof of the theorem is thus completed.

5. We now consider an application of our general theorem to the special case of Nörlund summability. We recall that, given a sequence $p = \{p_n\}$, Nörlund summability $(N, p)$ is defined as given by the sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k\ , \tag{19}$$

where we write

$$P_n = p_0 + p_1 + \cdots p_n\ ;$$

it is assumed that $p$ is such that, for all $n$, $P_n \neq 0$. If we write

$$t_n = b_0 + b_1 + \cdots b_n;\ s_k = a_0 + a_1 + \cdots a_k$$

we see that (19) can be expressed as the series-to-series transformation

$$b_0 = a_0\ ;$$

$$b_n = \sum_{k=1}^{n} \left( \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) a_n \qquad (n \geqq 1) ,$$

where we adopt the convention that $P_{-1} = 0$. Thus we have, with the notation of our main theorem, $\alpha_{n,k} = 0$ for $k > n$, while, for $1 \leqq k \leqq n$

$$\alpha_{n,k} = \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} = \frac{P_n p_{n-k} - P_{n-k} p_n}{P_n P_{n-1}} . \tag{20}$$

Now consider the case in which $\{p_n\}$ is nonnegative nonincreasing. We remark that, since $P_0 \neq 0$, we then have $p_0 > 0$. Further (since $p_n \geqq 0$) $\{P_n\}$ is nondecreasing; thus it follows from (20) that $\alpha_{n,k} \geqq 0$. Thus we may omit the modulus signs in (3); and it is now easy to see that (4), and hence (3), holds. Thus, in the case now considered, $(N, p)$ is absolutely regular. Further, for fixed $n$, $p_{n-k}$ is nondecreasing and $P_{n-k}$ nonincreasing as $k$ increases from 1 to $n$. Since $\alpha_{n,k} = 0$ for $k > n$, it follows that condition (a) is satisfied, with $r(n) = n$. Also equation (5) becomes

$$\sum_{n=2K}^{\infty} \frac{1}{nP_n P_{n-1}} \sum_{k=n-K}^{n} (P_n p_{n-k} - P_{n-k} p_n) = O(1) . \tag{21}$$

The inner sum in (21) does not exceed

$$\sum_{k=n-K}^{n} P_n p_{n-K} = P_n P_K ,$$

and thus a sufficient condition for (21) to hold is that

$$\sum_{n=2K}^{\infty} \frac{1}{nP_{n-1}} = O\left(\frac{1}{P_K}\right) . \tag{22}$$

However, since the hypotheses on $p$ imply that $P_{n-1} \sim P_n$, and that ${}_K P \leqq P_{2K} \leqq 2P_K$, it is easily seen that (22) is equivalent to the slightly simpler condition

$$\sum_{n=K}^{\infty} \frac{1}{nP_n} = O\left(\frac{1}{P_K}\right) . \tag{23}$$

Thus our theorem includes the following result;

THEOREM A. *Suppose that $\{p_n\}$ is nonnegative nonincreasing, and that* (23) *holds. Then the Fourier series of any function of bounded variation is absolutely summable* $|N, p|$.

The assumption that $\{p_n\}$ is nonnegative nonincreasing is not, without some further condition, sufficient for the conclusion, for it has been shown by Pati [8] that, when $p_n = 1/(n+1)$, it is not true that the Fourier series of any function of bounded variation is absolutely summable $|N, p|$. This example also shows that, in our main theorem, the assumptions that $A$ is absolutely conservative and that (a) holds would not alone suffice for the conclusion.

Theorem A is included in a recent, slightly more general, theorem of M. Izumi and S. Izumi [3]. It includes earlier theorems of H. P. Dikshit [2, Theorem 2] and T. Singh [9]; the result of Singh itself generalises a theorem of Pati [7]. The theorems of Dikshit and of Singh are respectively as follows.

THEOREM B. *Suppose that* $p_n > 0$, *and that* $p_{n+1}/p_n$ *is non decreasing, and less than or equal to* 1 *for all* $n$. *Suppose that* (23) *holds. Then the Fourier series of any function of bounded variation is absolutely summable* $|N, p|$.

THEOREM C. *Suppose that, for all* $n$, $p_n \geqq p_{n+1} > 0$, *and that* $p_n - p_{n+1}$ *is nonincreasing. Suppose also that*

$$\sum_{n=0}^{K} \frac{P_n}{n+1} = O(P_K) . \tag{24}$$

*Then the fourier series of any function of bounded variation is absolutely summable* $|N, p|$.

It is immediately evident that Theorem A includes Theorem B. The result that Theorem A includes Theorem C follows from the following lemma, which shows that, in Theorem C, we may replace (24) by (23).

LEMMA 4. *Suppose that* $p_0 > 0$, $p_n \geqq 0$. *Then* (23), (24) *are equivalent. In fact, either is equivalent to the assertion*

(c) *There is a constant integer* $r > 1$, *and a constant* $\lambda > 1$ *such that, for all sufficiently large* $n$,

$$P_{rn} \geqq \lambda P_n . \tag{25}$$

We first prove that (23) implies (c). Suppose, then, that (23) holds. Thus there is a constant $M$ such that, for all sufficiently large $K$,

$$\sum_{n=K}^{\infty} \frac{1}{nP_n} \leqq \frac{M}{P_K} .$$

Since $P_n$ is nondecreasing, this gives

$$\frac{M}{P_K} > \sum_{n=K}^{rK} \frac{1}{nP_n} \geqq \frac{1}{P_{rK}} \sum_{n=K}^{rK} \frac{1}{n} .$$

But

$$\sum_{n=K}^{rK} \frac{1}{n} \longrightarrow \log r$$

as $K \rightarrow \infty$, and (c) therefore follows if $r$ has been chosen so that

$$\log r > M\,. \tag{26}$$

If (24) holds, we have, for all sufficiently large $K$,

$$\sum_{n=0}^{K} \frac{P_n}{n+1} \leqq MP_K\,.$$

Thus, replacing $K$ by $rK$,

$$MP_{rK} > \sum_{n=K}^{rK} \frac{P_n}{n+1} \geqq P_K \sum_{n=K}^{rK} \frac{1}{n+1}\,,$$

and we again deduce (c) if $r$ has been chosen so that (26) holds.

We now consider the converse implications. Suppose, then, that (25) holds for $n \geqq n_0$. Then, for $\nu \geqq n_0$

$$\sum_{n=r\nu}^{r(\nu+1)-1} \frac{1}{nP_n} \leqq \frac{r}{r\nu P_{r\nu}} \leqq \frac{1}{\nu\lambda P_\nu}\,.$$

Hence, for $K \geqq n_0$ and $s \geqq 1$,

$$\sum_{n=r^sK}^{r^{s+1}K-1} \frac{1}{nP_n} \leqq \frac{1}{\lambda} \sum_{n=r^{s-1}K}^{r^sK-1} \frac{1}{nP_n}\,. \tag{27}$$

By successive applications of (27), we deduce that for $s \geqq 0$,

$$\sum_{n=r^sK}^{r^{s+1}K-1} \frac{1}{nP_n} \leqq \frac{1}{\lambda^s} \sum_{n=K}^{rK-1} \frac{1}{nP_n} \leqq \frac{1}{\lambda^s P_K} \sum_{n=k}^{rK-1} \frac{1}{n} = O\left(\frac{1}{\lambda^s P_K}\right).$$

Hence

$$\sum_{n=K}^{\infty} \frac{1}{nP_n} = O\left(\frac{1}{P_K} \sum_{s=0}^{\infty} \frac{1}{\lambda^s}\right) = O\left(\frac{1}{P_K}\right),$$

which gives (23). To prove (24), we have, for $\nu \geqq n_0$,

$$\sum_{n=r\nu}^{r(\nu+1)-1} \frac{P_n}{n+1} \geqq \frac{rP_{r\nu}}{r(\nu+1)} \geqq \frac{\lambda P_\nu}{\nu+1}\,.$$

Hence, for $s \geqq 1$,

$$\sum_{n=r^sn_0}^{r^{s+1}n_0-1} \frac{P_n}{n+1} \geqq \lambda \sum_{n=r^{s-1}n_0}^{r^sn_0-1} \frac{P_n}{n+1}\,,$$

so that, for $0 \leqq s \leqq t-1$,

$$\sum_{n=r^s n_0}^{r^{s+1}n_0-1} \frac{P_n}{n+1} \leqq \frac{1}{\lambda^{t-1-s}} \sum_{n=r^{t-1}n_0}^{r^t n_0-1} \frac{P_n}{n+1}. \tag{28}$$

Now take any $K \geqq n_0$. Choose $t$ so that $r^t n_0 \leqq K < r^{t+1}n_0$. Then, by (28),

$$\sum_{n=0}^{K} \frac{P_n}{n+1} \leqq \sum_{n=0}^{n_0-1} \frac{P_n}{n+1} + \sum_{s=0}^{t-1} \frac{1}{\lambda^{t-1-s}} \sum_{n=r^{t-1}n_0}^{r^t n_0-1} \frac{P_n}{n+1} + \sum_{n=r^t n_0}^{K} \frac{P_n}{n+1}, \tag{29}$$

where the second term on the right is omitted when $t = 0$. The first term on the right of (29) is a constant, and is thus certainly $O(P_K)$, since $P_K \geqq p_0 > 0$. Also

$$\sum_{n=r^{t-1}n_0}^{r^t n_0-1} \frac{P_n}{n+1} \leqq P_K \sum_{n=r^{t-1}n_0}^{r^t n_0-1} \frac{1}{n+1} = O(P_K) ;$$

$$\sum_{n=r^t n_0}^{K} \frac{P_n}{n+1} \leqq P_K \sum_{n=r^t n_0}^{K} \frac{1}{n+1} = O(P_K)$$

(since $K < r^{t+1}n_0$). Thus (24) follows.

The conditions (7), (8) have been mentioned as giving simple sufficient conditions. But, while simpler than (5) or (6), they appear to be insufficiently general to be of great use. Consider, for example, the case of Cesàro summability $(C, \delta)$. This is a Nörlund method with

$$p_n = \binom{n+\delta-1}{n}.$$

If $0 < \delta \leqq 1$, then the conditions of Theorem A are satisfied. Thus that theorem includes the result that the Fourier series of any function of bounded variation is absolutely summable $|C, \delta|$; this result was long ago proved by Bosanquet [1]. Now, in this case, $\alpha_{n,k} = 0$ for $k > n$, while, for $1 \leqq k \leqq n$,

$$\alpha_{n,k} = \frac{k}{n} \frac{\binom{n-k+\delta-1}{n-k}}{\binom{n+\delta}{n}}.$$

Thus (a), (b) are both satisfied, with $r(n) = s(n) = n$. But either (7) or (8) reduces to

$$\sum_{n=2K}^{\infty} \frac{1}{n\binom{n+\delta}{n}} = O\left(\frac{1}{K}\right),$$

and this is satisfied only if $\delta = 1$.

**6.** As another application of our main theorem, we let $\{k(n)\}$ be an increasing sequence of nonnegative integers, with $k(0) = 0$, and define

$$\alpha_{n,k} = \begin{cases} 1 & (k(n) \leqq k < k(n+1); \\ 0 & \text{otherwise.} \end{cases}$$

Thus absolute summability $|A|$ of a given series reduces to the absolute convergence of the series formed from it by bracketing together, for every $n$, those terms whose suffixes $k$ satisfy $k(n) \leqq k < k(n+1)$. It is clear that (a), (b) are both satisfied, with $r(n) = s(n) = k(n)$ (except when $n = 0$). In this case, the weaker conditions (7), (8) still give a significant result. Either of these conditions is equivalent to

$$\sum_{k(n) \geqq K} \frac{1}{k(n)} = O\left(\frac{1}{K}\right). \tag{30}$$

We note that (30) is satisfied, in particular, if

$$\liminf_{n\to\infty} \frac{k(n+1)}{k(n)} > 1 . \tag{31}$$

Thus our theorem includes the following result. *Suppose that* (31) *holds. Let us bracket together, in the way indicated, the terms of the Fourier series of any function of bounded variation. Then the resulting series is absolutely convergent.*

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THE UNIVERSITY BIRMINGHAM, B15 2TT, ENGLAND
AND
THE UNIVERSITY OF CALGARY, CALGARY, ALBERTA, T2N-1N4 CANADA.

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# ALGEBRAS OF NORMAL MATRICES

GEORGE MAXWELL

**A classical theorem of matrix theory asserts that a commuting set of complex normal matrices can be simultaneously unitarily diagonalised. In this paper, this result is generalised, both for the field of complex numbers and for more general fields. Namely, a commuting set of normal matrices is replaced by a subalgebra composed entirely of normal matrices. The structure of such subalgebras is determined and results on simultaneous diagonalisation are deduced. In the complex case, these subalgebras turn out to be commutative. However, even in the real case there are noncommutative examples.**

1. Normal subalgebras. Let $F$ be a field with an involution $J$, $V$ a finite dimensional vector space over $F$ and $\phi$ a left hermitian form on $V$ such that

$$\phi(x, x) = 0 \quad \text{implies } x = 0 . \tag{1}$$

In particular, $\phi$ is nondegenerate so that every endomorphism $T$ of $V$ has a unique adjoint w. r. t. $\phi$, defined by the equation

$$\phi(Tx, y) = \phi(x, T^*y) . \tag{2}$$

We call a subalgebra $A$ of $\mathrm{End}_F(V)$ normal if it satisfies

$$\begin{aligned} &\text{(a)} \quad T \in A \quad \text{implies} \quad T^* \in A \\ &\text{(b)} \quad T^*T = TT^* \quad \text{for all} \quad T \in A . \end{aligned} \tag{3}$$

Our first aim is to determine the structure of such normal subalgebras.

The purpose of assuming (1) is to obtain the property

$$T^*T = 0 \quad \text{implies} \quad T = 0 . \tag{4}$$

Indeed, if $T^*T = 0$, we have $\phi(Tx, Tx) = \phi(x, T^*Tx) = 0$ so that $Tx = 0$ for all $x \in V$. From properties 3(a) and (4), a well known argument [6] leads to the fact that $A$ has no nil ideals. In our context, this means that $A$ must be semisimple. Furthermore, if $B$ is a minimal ideal of $A$, so is $B^*$, and thus either $B^* = B$ or $B^*B = 0$, but the latter possibility is precluded by (4). It is therefore sufficient to determine the structure of a simple normal subalgebra.

PROPOSITION 1. *Suppose $R$ is a ring with unit element $1 \neq 0$ and $*$ is an involution of a matrix ring $M_n(R)$ with the property $XX^* =$*

*$X^*X$ for all $X \in M_n(R)$. Then either* (i) $n = 1$ *or* (ii) $n = 2$, *$R$ is commutative and $*$ is the involution*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{5}$$

*Proof.* Linearing the identity $XX^* = X^*X$, we obtain

$$XY^* + YX^* = X^*Y + Y^*X;$$

replacing $Y$ by $Y^*$, this can be written as

$$[X, Y]^* = -[X, Y]. \tag{6}$$

Let $E_{ij}(r)$ be the matrix with $r$ in the $(i, j)$th position and zeros elsewhere. Suppose $n \geqq 3$; if $i \neq j$, we can write $E_{ij}(r) = E_{ik}(1)E_{kj}(r) = [E_{ik}(1), E_{kj}(r)]$ for some $k \neq i, j$. Therefore $E_{ij}(r)^* = -E_{ij}(r)$ by (6); but then $E_{ij}(r)^* = E_{kj}(r)^*E_{ik}(1)^* = E_{kj}(r)E_{ik}(1) = 0$, an absurdity.

If $n = 2$, we can write $E_{12}(r) = [E_{11}(1), E_{12}(r)]$ so that $E_{12}(r)^* = -E_{12}(r)$. Since $E_{11}(r) = E_{12}(1)E_{21}(r)$, we have $E_{11}(r)^* = E_{22}(r)$; the involution is thus given by (5). Furthermore, writing $E_{11}(rs) = E_{11}(r)E_{11}(s)$ and applying $*$, we obtain $E_{22}(rs) = E_{22}(sr)$ so that $rs = sr$ and $R$ must be commutative.

PROPOSITION 2. *Suppose $D$ is a division ring, finite dimensional over its center $Z$ and $*$ is an involution of $D$ such that $dd^* = d^*d$ for all $d \in D$. Then either $D = Z$ or $D$ is a quaternion algebra over $Z$ and $*$ is the standard involution.*

*Proof.* Let $K$ be the subfield of $Z$ left fixed by $*$ and $L$ some algebraic closure of $K$. The extended involution $(d \otimes \alpha)^* = d^* \otimes \alpha$ on $D \otimes_K L$ has the same property as $*$.

If $K = Z$, $D \otimes_K L$ is isomorphic to $M_p(L)$ for some integer $p$. By Proposition 1, $p \leqq 2$ so that $D$ is either $Z$ or a quaternion algebra over $Z$ (see, e.g., [1. p. 146]). If $K \neq Z$, we have $Z \otimes_K Z \cong Z \oplus Z$, so that $D \otimes_Z L \cong D \otimes_Z (Z \otimes_K Z) \otimes_Z L \cong D \otimes_Z L \oplus D \otimes_Z L \cong M_p(L) \oplus M_p(L)$ for some integer $p$. If $*$ induces an involution on each of the factors $M_p(L)$, we again have $p \leqq 2$. However, if $p = 2$, we see from (5) that $*$ must leave central elements fixed, which is not true for $D \otimes_K L$. Therefore $p = 1$, i.e. $D = Z$. If $*$ interchanges the two factors $M_p(L)$, then each is forced to be commutative so that once again $p = 1$.

It remains to verify that in case $D$ is a quaternion algebra over $Z$ and $K = Z$, $*$ can only be the standard involution. If $\operatorname{char}(Z) \neq 2$, $D$ has a basis $\{1, i, j, ij\}$ such that $i^2 = \alpha$, $j^2 = \beta$ and $ij = -ji$ for some $\alpha, \beta \in Z$. Since $2\beta i = [ij, j]$ and $2\alpha j = [i, ij]$, (6) implies that $i^* = -i, j^* = -j$ so that * must be the standard involution. If char $(Z) = 2$, the relations are instead $i^2 = \alpha, j^2 = j + \beta$ and $ij = ji + i$ for some $\alpha, \beta \in Z$. Since $i = [i, j]$ and $ij = [j, ij]$ we have $i^* = i$ and $(ij)^* = ij$; but $\alpha j = i(ij)$ so that $\alpha j^* = (ij)^* i^* = iji = \alpha j + \alpha$ i.e. $j^* = j + 1$, showing that * is again the standard involution.

The preceding proofs could have been somewhat shortened by appealing to a recent result of Amitsur [3], which says that a semi-prime ring with an involution * satisfying a polynomial identity $p(X_1, \cdots X_n, X_1^*, \cdots, X_n^*) = 0$ of degree $d$ satisfies a "standard identity" of degree $2d$. In our case, the polynomial identity is $X_1^* X_1 - X_1 X_1^* = 0$, of degree 2, so that the standard identity is of degree 4. Now a well-known result of Kaplansky [7] implies that if the ring is also primitive, it is at most 4-dimensional over its center. However, we would still have to determine, as above, the possibilities for *, the knowledge of which is important in the sequel.

PROPOSITION 3.
(a) *If $J$ is non-trivial, a simple normal subalgebra $A$ is a finite field extension of $F$; its involution * extends $J$.*
(b) *If $J$ is trivial, $A$ can also be a quaternion division algebra over a finite field extension of $F$, in which case * must be the standard involution.*

*Proof.* Suppose $A$ is isomorphic to $M_n(D)$, where $D$ is some finite dimensional division algebra over $F$. By Proposition 1, either (i) $n = 1$ or (ii) $n = 2$, $D$ is a field and * corresponds to the involution (5). However, the latter violates (4) since, for example, $E_{11}(1)^* E_{11}(1) = 0$; therefore $A$ is a division algebra. By Proposition 2, $A$ is either a field or a quaternion algebra over its center. Furthermore, in the latter case * must be the standard involution, which is certainly trivial on $F$, so that $J$ itself had to be trivial.

Turning to the classical cases, let us suppose that $F$ is either $\boldsymbol{R}$ or $\boldsymbol{C}$ and $\phi$ is the standard hermitian form on $V = F^n$.

COROLLARY 1, *In the complex case, a normal subalgebra is isomorphic to a product of copies of $\boldsymbol{C}$, each with the standard involution.*

COROLLARY 2. *In the real case, a normal subalgebra is isomorphic to a product of copies of $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$, the latter two occurring with the standard involution.*

*Proof.* It is only necessary to explain why a factor consisting of $\boldsymbol{C}$ with the trivial involution could not occur in the real case. This is a consequence of a property stronger than (4):

$$\sum T_i^* T_i = 0 \quad \text{implies that all} \quad T_i = 0\,, \tag{7}$$

enjoyed by * but violated by such a factor. Indeed, if $\sum T_i^* T_i = 0$, we have $\phi(\sum T_i^* T_i x, x) = \sum \phi(T_i x, T_i x) = 0$ for all $x \in V$; since all summands are non-negative, we must have $\phi(T_i x, T_i x) = 0$ and hence $T_i = 0$.

**2. Simultaneous diagonalisation.** Let $A$ be a normal subalgebra of $\mathrm{End}_F(V)$ and consider $V$ as a left $A$-module. One sees at once from (2) that if $W$ is a submodule of $V$, so is $W^\perp$; in view of (1), we have $V = W \oplus W^\perp$. Induction now shows that $V$ is the orthogonal sum of simple submodules, which are isomorphic to simple factors of $A$.

Using Corollaries 1 and 2 of Proposition 3, we can immediately obtain diagonalisation results in the classical situations.

PROPOSITION 4. *In the complex case, there exists an orthonormal basis of $V$ w.r.t. which the matrices of all elements of $A$ are diagonal.*

PROPOSITION 5. *In the real case, there exists a partition* $\dim V = n_1 + 2n_2 + 4n_3$ *and an orthonormal basis of $V$ w.r.t. which the matrices of all elements of $A$ consist of $n_1$ diagonal elements, followed by $n_2$ blocks of the form*

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \tag{8}$$

*and $n_3$ blocks of the form*

$$\begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & -\delta & \gamma \\ \gamma & \delta & \alpha & -\beta \\ \delta & -\gamma & \beta & \alpha \end{pmatrix}. \tag{9}$$

*Proof.* If a simple $A$-submodule is isomorphic to $\boldsymbol{C}$, it has a basis of the form $\{x, i \cdot x\}$, which is orthogonal since $\phi(x, i \cdot x) = \phi(i^* \cdot x, x) = -\phi(i \cdot x, x) = -\phi(x, i \cdot x)$. We may suppose that $\phi(x, x) = 1$, but then $\phi(i \cdot x, i \cdot x) = \phi(x, i^* i \cdot x) = \phi(x, x) = 1$, so that the basis is orthonormal. The action of $\boldsymbol{C}$ on such a basis is given by blocks of the form (8). Similarly, if an $A$-submodule is isomorphic to $\boldsymbol{H}$, it has a basis of the form $\{x, i \cdot x, j \cdot x, ij \cdot x\}$, which can once again be assumed orthonormal and yields blocks of the form (9).

Such diagonalisation results are usually stated for a commuting set $\{T_i\}$ of normal endomorphisms rather than for a normal subalgebra. To deduce them from our results, we first enlarge the set $\{T_i\}$ to $\{T_i, T_i^*\}$, which is still commuting in view of the following well-known result [9]:

PROPOSITION 6. *In the real or complex case, if a normal endomorphism $T$ commutes with an endomorphism $S$, it also commutes with $S^*$.*

Secondly, we form the commutative subalgebra generated by $\{T_i, T_i^*\}$, which is clearly normal, and apply propositions 4 and 5. In the non-classical situations, the results of §1 still enable us to produce diagonalisation theorems, although these can of necessity be more complicated. We shall confine ourselves to some remarks about the case when $F = \boldsymbol{Q}$ and $\phi$ is the standard hermitian form on $V = \boldsymbol{Q}^n$.

PROPOSITION 7. *The possible factors of a normal subalgebra must be of the following types*:
(a) *a totally real finite extension $K/\boldsymbol{Q}$, with the trivial involution.*
(b) *an extension $K(\sqrt{-\alpha})/\boldsymbol{Q}$, where $K$ is as in* (a) *and $\alpha$ is totally positive, with the involution $\sqrt{-\alpha} \to -\sqrt{-\alpha}$.*
(c) *a quaternion algebra $(-\alpha, -\beta)$ over $K$, where $K$ is as in* (a) *and $\alpha, \beta$ are totally positive, with the standard involution.*

*Proof.* Let $A$ be a simple factor. We go back to proposition 3. If $^*$ induces the trivial involution on $A$, every $T \in A$ is hermitian and therefore has totally real eigenvalues-hence $A$ is of type (a). When $^*$ is not trivial, the fixed subfield $K$ of $^*$ is of type (a) by the same argument. If $K \to \boldsymbol{R}$ is some imbedding, then, regarding $\boldsymbol{R}$ as a $K$-algebra, one proves as before that the involution $(a \otimes \lambda)^* = a^* \otimes \lambda$ on the extended algebra $A \otimes_K \boldsymbol{R}$ enjoys property (7). Therefore the images of $\alpha$ or $\alpha$ and $\beta$ must be positive in $\boldsymbol{R}$.

The problem of determining which totally real extensions $K/\boldsymbol{Q}$ can actually occur as factors of type (a), say, has been studied by Bender [4] and seems quite difficult. For example, $\boldsymbol{Q}(\sqrt{d})/\boldsymbol{Q}$ occurs if and only if $d$ is a sum of 2 squares in $\boldsymbol{Q}$.

**3. The infinite dimensional case.** In this paragraph, we shall prove that in some infinite dimensional situations normal subalgebras are necessarily commutative.

Firstly, suppose that $H$ is a complex Hilbert space and $B(H)$ is the algebra of bounded operators on $H$. The analogue of Proposition 6 for elements of $B(H)$ has been proved by Fuglede [5] and later generalised by Putnam [10] to

PROPOSITION 8. *If $S$ and $T$ are normal operators and $R$ is an operator such that $TR = RS$, then $T^*R = RS^*$.*

One can use this result to prove

PROPOSITION 9. *A normal subalgebra $A$ of $B(H)$ such that $A^2$ is dense in $A$ (for example if $1 \in A$) must be commutative.*

*Proof.* Suppose $S, T \in A$; since $(ST^*)S = S(T^*S)$, Proposition 8 implies that $(ST^*)^*S = S(T^*S)^*$ or $T(S^*S) = (S^*S)T$ (this idea occurs in Kaplansky [8]). Now replace $S$ by $S + R^*$, with $R \in A$. After subtraction, one concludes that $T$ commutes with $(RS)^* + RS$ i.e. with all the hermitian elements of $A^2$. Since $A^2$ is dense in $A$ and every element of $A$ can be written in the form $S + iT$ where $S$ and $T$ are hermitian elements of $A$, we conclude that $T$ commutes with every element of $A$.

Secondly, we return to an arbitrary field $F$ and consider an arbitrary $F$-algebra $\Omega$ with an involution $*$, satisfying $(\alpha . x)^* = \alpha^J . x^*$. Let $b(\Omega)$ be the quotient of $\Omega \otimes_k \Omega$ by the subspace generated by all elements of the form $ab \otimes c - a \otimes bc$ and $ba \otimes c - a \otimes cb$. The obvious map $\beta_\Omega \colon \Omega \times \Omega \to b(\Omega)$ is called the universal bitrace on $\Omega$. It may happen that $b(\Omega)$ is not isomorphic to $K$, for example if $\Omega^2 = 0$. Since $\Omega$ has an involution, it is actually more convenient to work with a "twisted" version of the bitrace: $\langle a, b \rangle = \beta_\Omega(a^*, b)$. This is a left sesquilinear (w.r.t. $J$) map on $\Omega$, universal w.r.t. the properties $\langle ab, c \rangle = \langle b, a^*c \rangle$ and $\langle ba, c \rangle = \langle b, ca^* \rangle$. By analogy with [2], $\Omega$ may be termed an $H^*$-algebra if

$$\langle a, a \rangle = 0 \quad \text{implies} \quad a = 0 \,. \tag{10}$$

For such algebras, the analogue of Proposition 8 can be proved purely formally from the identity

$$\begin{aligned} &\langle c^*a - bc^*, c^*a - bc^* \rangle - \langle ac - cb, ac - cb \rangle \\ &\quad = \langle aa^* - a^*a, cc^* \rangle - \langle bb^* - b^*b, c^*c \rangle \,, \end{aligned} \tag{11}$$

a special case of which goes back to von Neumann [11]. For its proof, note first that $\langle ab, cd\rangle = \langle bd^*, a^*c\rangle = \langle c^*a, db^*\rangle$. Then

$$\begin{aligned}&\langle c^*a - bc^*, c^*a - bc^*\rangle \\ &\quad = \langle c^*a, c^*a\rangle - \langle bc^*, c^*a\rangle - \langle c^*a, bc^*\rangle + \langle bc^*, bc^*\rangle \\ &\quad = \langle aa^*, cc^*\rangle - \langle cb, ac\rangle - \langle ac, cb\rangle + \langle c^*c, b^*b\rangle .\end{aligned}$$

Similarly, $\langle ac - cb, ac - cb\rangle = \langle a^*a, cc^*\rangle - \langle ac, cb\rangle - \langle cb, ac\rangle + \langle c^*c, bb^*\rangle$. Subtraction yields (11).

PROPOSITION 10. *If $J$ is nontrivial, a normal subalgebra $A$ of an $H^*$-algebra $\Omega$ such that $A^2 = A$ must be commutative.*

*Proof.* One can use the same argument used in the proof of Proposition 9, with the following remark. Since $J$ is nontrivial, there exists $\theta \in F$ such that $\theta^J \neq \theta$; then every $x \in A$ can be written in the form $x_1 + \theta \cdot x_2$, where $x_1 = (\theta \cdot x^* - \theta^J \cdot x)/(\theta - \theta^J)$ and $x_2 = (x - x^*)/(\theta - \theta^J)$ are hermitian elements of $A$.

In conclusion, we add a remark regarding the property

$$aa^* = a^*a,\ bb^* = b^*b,\ ac = cb \quad \text{implies} \quad c^*a = ab^* \tag{12}$$

in arbitrary rings with involution. Two of its special cases are

$$aa^* = a^*a,\ ac = ca \quad \text{implies } c^*a = ac^* \tag{13}$$

and

$$aa^* = a^*a,\ ac = 0 \quad \text{implies} \quad c^*a = 0 . \tag{14}$$

However, one can get an example in which both (13) and (14) hold but (12) does not, by taking $K = \boldsymbol{Q}$, $\alpha = 2$ in

PROPOSITION 11. *Let $K$ be a field of characteristic $\neq 2$, $\alpha$ a nonzero element of $K$ and $*$ the involution*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a & -\alpha c \\ -b/\alpha & d \end{pmatrix}$$

*of $M_2(K)$. Then* ( i ) (13) *is true in $M_2(K)$,* (ii) (14) *is true iff $\alpha$ is not a square and* (iii) (12) *is true iff $\alpha$ is not a sum of* 2 *squares.*

We omit the full proof, but give the counterexample for (12): suppose $\alpha = \beta^2 + \gamma^2$ and let

$$X = \begin{pmatrix} 1 & \beta \\ \beta/\alpha & 1 \end{pmatrix},\ Y = \begin{pmatrix} 2 & -\gamma \\ \gamma/\alpha & 0 \end{pmatrix},\ Z = \begin{pmatrix} \alpha & -\alpha\gamma \\ \beta & 0 \end{pmatrix}.$$

Then $XX^* = X^*X$, $YY^* = Y^*Y$, $XZ = ZY$ but $X^*Z \neq ZY^*$.

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UNIVERSITY OF BRITISH COLUMBIA

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# MULTIPLIERS OF TYPE $(p, p)$

KELLY MCKENNON

**It will be shown in this paper that the Banach algebra of all continuous multipliers on $L_p(G)$ ($G$ a locally compact group, $p \in [0, \infty[$) may be viewed as the set of all multipliers on a natural Banach algebra with minimal approximate left identity.**

Let $G$ be an arbitrary locally compact group, $\lambda$ its left Haar measure, and $p$ a number in $[1, \infty[$. Write $\mathfrak{B}_p$ for the Banach algebra of all bounded linear operators on $L_p$ and write $\mathfrak{M}_p$ for the subset of $\mathfrak{B}_p$ consisting of those operators which commute with all left translation operators; elements of $\mathfrak{M}_p$ are called *multipliers of type* $(p, p)$. If $A$ is a Banach algebra, then a bounded linear operator $T$ on $A$ such that $T(a \cdot b) = T(a) \cdot b$ for all $a, b \in A$ is called a *multiplier on* $A$; write $\mathfrak{m}(A)$ for the set of all such. By $C_{00}$ will be meant the set of all continuous complex-valued functions on $G$ which have compact support. A function $f$ in $L_p$ such that for each $g$ in $L_p$, the function $g * f(x) = \int g(t) f(t^{-1}x) d\lambda(t)$ exists $\lambda$-almost everywhere, $g * f$ is in $L_p$, and $\| g * f \|_p \leqq \| g \|_p \cdot k$ where $k$ is a positive number independent of $g$, is said to be *p-tempered*; write $L_p^t$ for the set of all such. Evidently $L_p^t$ is closed under convolution and $C_{00}$ is a subset of $L_p^t$. Thus, for each $f$ in $L_p^t$ and $h$ in $C_{00}$, there is precisely one operator $W$ in $\mathfrak{B}_p$ such that $W(g) = g * f * h$ for all $g$ in $L_p$; write $\mathfrak{A}_p$ for the norm closure in $\mathfrak{B}_p$ of the linear span of all such $W$. The principal result of this paper is that $\mathfrak{A}_p$ is a Banach algebra with minimal approximate left identity and that $\mathfrak{m}(\mathfrak{A}_p)$ and $\mathfrak{M}_p$ are isomorphic isometric Banach algebras.

THEOREM 1. *Let $f$ be a function in $L_p$ and $k$ a positive number such that $\| g * f \|_p \leqq \| g \|_p \cdot k$ for all $g$ in $C_{00}$. Then $f$ is in $L_p^t$.*

*Proof*. First of all, suppose that $h$ is a function in $L_1 \cap L_p$. As is well known, $h * f$ is in $L_p$ and $\| h * f \|_p \leqq \| h \|_1 \cdot \| f \|_p$. Let $\{h_n\}$ be a sequence in $C_{00}$ which converges to $h$ in the $L_p$ and $L_1$ norms both. It follows from the above that $\{h_n * f\}$ converges to $h * f$ in $L_p$. This fact and the hypothesis for $f$ imply

$$\| h * f \|_p = \lim_n \| h_n * f \|_p \leqq \overline{\lim_n} \| h_n \|_p \cdot k = \| h \|_p \cdot k .$$

Let $h$ be now an arbitrary function from $L_p$. We may assume that $h$ vanishes off some $\sigma$-finite set $A$. Let $\{A_n\}$ be an increasing nest of $\lambda$-finite and $\lambda$-measurable subsets of $G$ such that their union is $A$. Let for each $n \in N$, $h_n$ be the product of $h$ with the characteristic function of $A_n$. Let $\pi_j$ $(j = 0, 1, 2, 3)$ be the minimal non-negative functions on the complex field $K$ such that $z = \sum_{j=0}^{3} i^j \pi_j(z)$ for each $z \in K$.

Fix $j$ in $\{0, 1, 2, 3\}$. For each $x \in G$, define the measurable function $w^x$ in $[0, \infty]^G$ by letting $w^x(t) = \pi_j[h(t) \cdot f(t^{-1}x)]$ for all $t \in G$. For each $x \in G$ and $n \in N$, define the measurable function $w_n^x$ in $[0, \infty]^G$ by letting $w_n^x(t) = \pi_j[h_n(t) \cdot f(t^{-1}x)]$ for all $t \in G$. Since the sequence $\{w_n^x\}$ converges upwards to $w^x$ for each $x \in G$, it follows from the monotone convergence theorem that $\lim_n \int w_n^x d\lambda = \int w^x d\lambda$. Define the function $F$ in $[0, \infty]^G$ by letting $F(x) = \int w^x d\lambda$ for all $x \in G$. For each $n \in N$, define the function $F_n$ in $[0, \infty]^G$ by letting $F_n(x) = \int w_n^x d\lambda$ for all $x \in G$. Thus, $\{F_n\}$ converges upwards to $F$ at each point $x \in G$.

For each $n \in N$, $h_n$ is in $L_1 \cap L_p$; it follows that $\pi_j[h_n * f]$ is in $L_p$, and so equals $F_n$ almost everywhere. Hence, each $F_n$ is measurable whence $F$ is measurable. Further, by the monotone convergence theorem and the inequality which concludes the initial paragraph of this proof,

$$\begin{aligned} \| F \|_p &= \lim_n \| F_n \|_p \\ &= \lim_n \| \pi_j[h_n * f] \|_p \leqq \overline{\lim_n} \| h_n * f \|_p \leqq \overline{\lim_n} \| h_n \|_p \cdot k = \| h \|_p \cdot k \ . \end{aligned}$$

Recalling that $F(x) = \int \pi_j[h(t) \cdot f(t^{-1}x)]dt$ almost everywhere and $j$ was arbitrary, we see that $h*f$ exists almost everywhere, is in $L_p$ and $\| h*f \|_p \leqq \| h \|_p \cdot 4k$. This proves that $f$ is $p$-tempered.

The condition given in Theorem 1 for a function in $L_p$ to be in $L_p^t$ is clearly necessary as well as sufficient. Another such condition was proved in [4], Theorem 1.3:

THEOREM 2. *Let $f$ be a function in $L_p$ such that $g*f$ is defined and in $L_p$ for all $g$ in $L_p$. Then $f$ is in $L_p^t$.*

For each $f \in L_p^t$, there is precisely one operator $W_f \in \mathfrak{B}_p$ such that

$$W_f(g) = g*f \tag{1}$$

for all $g \in L_p$. For $f \in C_{00}$, we have as well (see [1] 20.13)

$$\| W_f \| \leqq \int \Delta^{-(p-1)/p} | f | d\lambda \ . \tag{2}$$

It is easy to check that

$$W_{f*h} = W_h \circ W_f \tag{3}$$

for all $f$ and $h$ in $L_p^t$.

THEOREM 3. *The set $\mathfrak{A}_p$ is a complete subalgebra of $\mathfrak{M}_p$ and it possesses a minimal left approximate identity (i.e., a net $\{T_\alpha\}$ such that $\overline{\lim}_\alpha \| T_\alpha \| \leqq 1$ and $\lim \| T_\alpha \circ T - T \| = 0$ for all $T \in \mathfrak{A}_p$).*

*Proof.* A simple calculation shows that, when $f$ is in $L_p^t$, then $W_f$ is in $\mathfrak{M}_p$. Evidently, $\mathfrak{M}_p$ is a Banach algebra; hence, $\mathfrak{A}_p$ is a subset of $\mathfrak{M}_p$. That $\mathfrak{A}_p$ is a Banach space is an elementary consequence of its definition. That $\mathfrak{A}_p$ is a Banach algebra is a consequence of the fact that $L_p^t * C_{00}$ is closed under convolution.

For each compact neighborhood $E$ of the identity of $G$, let $f_E$ be a nonnegative function in $C_\infty$ which vanishes outside $E$ and such that $\int f_E d\lambda = 1$. Directing the family of compact neighborhoods of the identity by letting $E > F$ when $E \subset F$, we obtain a net $\{f_E\}$ which is a minimal approximate identity for $L_1$. If $\{h_\gamma\}$ denotes the product net of $\{f_E\}$ with itself, then $\{h_\gamma\}$ is again a minimal approximate identity for $L_1$ and the net $\{W_{h_\gamma}\}$ is in $\mathfrak{A}_p$. Since $\Delta$ is unity and continuous at the identity of $G$, we have by (2),

$$\overline{\lim_\gamma} \| W_{h_\gamma} \| \leqq \overline{\lim_\gamma} \int \Delta^{-(p-1)/p} h_\gamma d\lambda \leqq 1 .$$

For $f \in L_p^t$ and $g \in C_{00}$, (3) and (2) imply

$$\begin{aligned}
\overline{\lim_\gamma} \| W_{h_\gamma} \circ W_{f*g} - W_{f*g} \| &= \overline{\lim_\gamma} \| (W_{g*h_\gamma} - W_g) \circ W_f \| \\
&\leqq \overline{\lim_\gamma} \| W_{g*h_\gamma} - W_g \| \cdot \| W_f \| \leqq \Big( \overline{\lim_\gamma} \int | g*h_\gamma - g | \cdot \Delta^{-(p-1)/p} d\lambda \Big) \cdot \| W_f \| \\
&\leqq \overline{\lim_\gamma} \| g*h_\gamma - g \|_1 \cdot \sup \{ \Delta^{-(p-1)/p}(x) \colon g*h_\gamma(x) \neq g(x) \} \cdot \| W_f \| = 0
\end{aligned}$$

since $\overline{\lim}_\gamma \|g*h_\gamma - g\|_1 = 0$ and since the net of sets $\{x \in G \colon g*h_\gamma(x) \neq g(x)\}$ is eventually contained in some fixed compact set. Since $L_p^t * C_{00}$ generates a dense subset of $\mathfrak{A}_p$, we have $\lim \| W_{h_\gamma} \circ T - T \| = 0$ for all $T \in \mathfrak{A}_p$. Thus, $\{ W_{h_\gamma} \}$ is a minimal left approximate identity for $\mathfrak{A}_p$.

We now turn to $\mathfrak{M}_p$. We shall need a theorem proved in [3] 4.2.

THEOREM 4. *Let $\mu$ and the elements of a net $\{\mu_\alpha\}$ be bounded, complex, regular Borel measures on $G$ such that*

$$\lim_\alpha \| \mu_\alpha \| = \| \mu \| \tag{a}$$

*and*

(b) $$\lim_\alpha \int f d\mu_\alpha = \int f d\mu \quad \textit{for each} \quad f \in C_{00}\,.$$

*Then, for each* $g \in L_p$ ($p \in [1, \infty[$), $\lim_\alpha \| \mu_\alpha * g - \mu * g \|_p = 0$.

COROLLARY. *For each multiplier* $T$ *in* $\mathfrak{M}_p$ *and each bounded, complex, regular Borel measure* $\mu$, *we have*

(i) $T(\mu * g) = \mu * T(g)$

*for all* $g \in L_p$. *In particular, for* $f \in L_1$, *we have*

(ii) $T(f * g) = f * T(g)$.

*Proof.* Since $T$ commutes with left translation operators, it is evident that (i) holds when $\mu$ is a linear combination of Dirac measures. Now let $\mu$ be arbitrary. Since the extreme points of the unit ball of the conjugate space $C_{00}^*$ (where $C_{00}$ bears the uniform or supremum norm) are Dirac measures, and since Alaoglu's Theorem implies that the unit ball of $C_{00}^*$ is $\sigma(C_{00}^*, C_{00})$-compact, it follows by the Krein-Milman Theorem that there exists a net $\{\mu_\alpha\}$ consisting of linear combinations of Dirac measures such that the hypotheses (a) and (b) of Theorem 4 are satisfied. By Theorem 4, we have $\lim_\alpha \| \mu_\alpha * g - \mu * g \|_p = 0$ for all $g \in L_p$. This implies that $\lim_\alpha \| T(\mu_\alpha * g) - T(\mu * g) \|_p = 0$ for all $g \in L_p$. Consequently,

$$\| T(\mu * g) - \mu * T(g) \|_p \leqq \overline{\lim_\alpha} \| T(\mu * g) - T(\mu_\alpha * g) \|_p$$
$$+ \overline{\lim_\alpha} \| T(\mu_\alpha * g) - \mu * T(g) \|_p = 0 + \overline{\lim_\alpha} \| \mu_\alpha * T(g) - \mu * T(g) \|_p = 0\,.$$

This proves part (i). Part (ii) is a special case of (i).

THEOREM 5. *For each multiplier* $T$ *in* $\mathfrak{M}_p$ *and each function* $f$ *in* $C_{00}$, *the function* $T(f)$ *is in* $L_p^t$ *and* $W_{T(f)} = T \circ W_f$.

*Proof.* Because $f$ is in $L_p$, it follows from the corollary to Theorem 4 and (1) that $g * T(f) = T(g * f) = T \circ W_f(g)$ for all $g \in C_{00}$. This implies that $\| g * T(f) \|_p \leqq \| T \| \cdot \| W_f \| \cdot \| g \|_p$ for all $g \in C_{00}$. Thus, by Theorem 1, $T(f)$ is in $L_p^t$. Since $C_{00}$ is dense in $L_p$, we have that $W_{T(f)} = T \circ W_f$.

We purpose to identify the multipliers on $\mathfrak{A}_p$. To accomplish this, we shall set down a general multiplier identification theorem.

Let $B$ be a normed algebra with identity and let $A$ be any subalgebra of $B$ which is $\| \;\|_B$-complete and which has a minimal left approximate identity. Define $\mathfrak{R}(B, A)$ to be the coarsest topology with respect to which each of the seminorms ${}^a\| \;\|$ ($a \in A$) is continuous where ${}^a\| b \| = \| b \cdot a \|_B$ for all $b \in B$. It is known (see [3] 1.4. (ii)) that

(4) the map $(a, b) \longrightarrow a \cdot b$ is $\mathfrak{K}(B, A)$-continuous

when $a$ and $b$ run through any $\| \|_B$-bounded subset of $B$.

THEOREM 6. *Let $A$ and $B$ be as above and suppose that the following hold:*

(i) *the unit ball $A_1$ of $A$ is $\mathfrak{K}(B, A)$-dense in the unit ball $B_1$ of $B$;*

(ii) $\| b \|_B = \sup \{\| b \cdot a \|_B : a \in A_1\}$ *for each* $b \in B_1$;

(iii) *$B_1$ is $\mathfrak{K}(B, A)$-complete.*

*Then $\mathfrak{m}(A)$ is isomorphic to $B$.*

*Proof.* By [3] 1.8. (iv), $A$ is a left ideal in $B$. Define the map $T | \rightarrow m(A)$ by letting $T_b(a) = b \cdot a$ for all $b \in B$ and $a \in A$. That $T$ is an algebra homomorphism of $B$ into $m(A)$ is easy to check. That $T$ is an isometry follows from (ii). That $T$ is onto is a consequence of [3] 1.12.

LEMMA 1. *The unit ball of $\mathfrak{A}_p$ is $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$-dense in the unit ball of $\mathfrak{M}_p$.*

*Proof.* Let $T$ be any operator in the unit ball of $\mathfrak{M}_p$. Let $\{W_{h_\gamma}\}$ be the minimal left approximate identity for $\mathfrak{A}_p$ chosen in Theorem 3. For each index $\gamma$, we know from Theorem 5 and (3) that $T(h_\gamma)$ is in $L_p^t$ and $W_{h_\gamma} \circ T \circ W_{h_\gamma} = W_{h_\gamma} \circ W_{T(h_\gamma)} = W_{T(h_\gamma) * h_\gamma}$. From (4), we see that $\{W_{h_\gamma} \circ T \circ W_{h_\gamma}\}$ converges to $I \circ T \circ I = T$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$: in other words, $\lim W_{T(h_\gamma) * h_\gamma} = T$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$.

Thus, we must have $\underline{\lim}_\gamma \| W_{T(h_\gamma) * h_\gamma} \| \geqq \| T \|$, as is easily seen. But $\overline{\lim}_\gamma \| W_{T(h_\gamma) * h_\gamma} \| = \overline{\lim}_\gamma \| W_{h_\gamma} \circ T \circ W_{h_\gamma} \| \leqq \overline{\lim}_\gamma \| W_{h_\gamma} \|^2 \cdot \| T \| \leqq \| T \|$. Thus, we have $\lim_\gamma \| W_{T(h_\gamma) * h_\gamma} \| = \| T \|$. It follows that $\lim_\gamma \| W_{T(h_\gamma) * h_\gamma} \|^{-1} \cdot W_{T(h_\gamma) * h_\gamma} = T$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$. We have shown that $T$ is the $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$-limit of operators in the unit ball of $\mathfrak{A}_p$.

LEMMA 2. *Let $\{T_\alpha\}$ be any $\mathfrak{K}(\mathfrak{B}_p, \mathfrak{A}_p)$-Cauchy net in $\mathfrak{B}_p$ such that* $\sup_\alpha \| T_\alpha \| < \infty$. *Then there is an operator $T$ in $\mathfrak{B}_p$ such that* $\lim_\alpha T_\alpha = T$ *in both the strong operator topology and the topology $\mathfrak{K}(\mathfrak{B}_p, \mathfrak{A}_p)$.*

*Proof.* Let $S$ be the subspace of $L_p$ spanned by the set $L_p * L_p^t * C_{00}$. If $g$ is in $L_p$ and $\{h_\gamma\}$ is the net in $L_p^t * C_\infty$ constructed in the proof of Theorem 3, then $\lim_\gamma \| g * h_\gamma - g \|_p = 0$ (see [1] 20.15. ii). It follows that $S$ is dense in $L_p$.

Let $\sum_{j=1}^m f_j * h_j * g_j$ be a typical element of $S$ where $f_j \in L_p$, $h_j \in L_p^t$, and $g_j \in C_{00}$ $(j = 1, 2, \cdots, m)$. Then $W_{h_j * g_j}$ is in $\mathfrak{A}_p$ $(j = 1, 2, \cdots, m)$ so that, by hypothesis, the net $\{T_\alpha \circ W_{h_j * g_j}\}$ is $\| \|$-Cauchy in $\mathfrak{B}_p$. Since

$T_\alpha(f_j*h_j*g_j) = T_\alpha \circ W_{h_j*g_j}(f_j)$ for each $j = 1, 2, \cdots, m$ and each index $\alpha$, it follows that the net $\{T_\alpha(f_j*h_j*g_j)\}$ is $\|\ \|_p$-Cauchy for each $j = 1, 2, \cdots, m$. Thus, $\{T_\alpha(\sum_{j=1}^m f_j*h_j*g_j)\}$ is $\|\ \|_p$-Cauchy and so has some limit in $L_p$ which we shall write as $T_0(\sum_{j=1}^m f_j*h_j*g_j)$. The operator $T_0 \mid S \to L_p$ thus defined is clearly linear and, by the hypothesis $\sup_\alpha \| T_\alpha \| < \infty$, is also bounded. Since $S$ is dense in $L_p$, $T_0$ is the restriction to $S$ of a unique operator $T$ in $\mathfrak{B}_p$. Since the net $\{T_\alpha\}$ converges to $T$ on the dense subspace $S$ of $L_p$, and since $\sup_\alpha \| T_\alpha \| < \infty$, it follows that $\lim_\alpha T_\alpha = T$ in the strong operator topology.

Let $f$ be any function in $L_p^t * C_{00}$. By hypothesis, the net $\{T_\alpha \circ W_f\}$ is $\|\ \|$-Cauchy and so has some $\|\ \|$-limit $V$ in $\mathfrak{B}_p$. For each $g \in L_1 \cap L_p$, we have

$$V(g) = \lim_\alpha T_\alpha \circ W_f(g) = \lim_\alpha T_\alpha(g*f) = T(g*f) = T \circ W_f(g) .$$

Since $L_1 \cap L_p$ is dense in $L_p$, it follows that $V = T \circ W_f$. Thus, $\lim_\alpha \| (T_\alpha - T) \circ W_f \| = 0$. Since $\{ W_f : f \in L_p^t * C_{00}\}$ spans a dense subset of $\mathfrak{A}_p$ and since $\sup_\alpha \| T_\alpha \| < \infty$, it follows that $\lim_\alpha T_\alpha = T$ in $\mathfrak{K}(\mathfrak{B}_p, \mathfrak{A}_p)$.

THEOREM 7. *Let $\pi \mid \mathfrak{M}_p \to \mathfrak{B}_p^{\mathfrak{A}_p}$ be defined by, for each $T \in \mathfrak{M}_p$, letting the function $\pi_T \mid \mathfrak{A}_p \to \mathfrak{B}_p$ be given by $\pi_T(W) = T \circ W$ for all $W \in \mathfrak{A}_p$. Then $\pi$ is an isometric algebra isomorphism $\mathfrak{M}_p$ onto $\mathfrak{m}(\mathfrak{A}_p)$.*

*Proof.* We shall apply Theorem 6 for $B = \mathfrak{M}_p$ and $A = \mathfrak{A}_p$. That $\mathfrak{A}_p$ has a minimal left approximate identity follows from Theorem 3. That condition (i) of Theorem 6 is satisfied follows from Lemma 1. That condition (iii) of Theorem 6 is satisfied follows from Lemma 2. To invoke Theorem 6 and so prove Theorem 7, it will suffice to show that $\| T \| = \sup \{ \| T \circ W \| : W \in \mathfrak{A}_p, \| W \| = 1\}$ for each $T \in \mathfrak{M}_p$.

Let then $T$ be any multiplier in $\mathfrak{M}_p$. That $\| T \| \geqq \sup \{ \| T \circ W \| : W \in \mathfrak{A}_p, \| W \| = 1\}$ is obvious. Let $\varepsilon$ be any positive number. Choose $f \in L_p$ such that $\| f \|_p \leqq 1$ and $\| T(f) \|_p > \| T \| - \varepsilon/2$. Let $\{ W_\gamma \}$ be a minimal left approximate identity for $\mathfrak{A}_p$. Then $\lim_\gamma W_\gamma = I$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$ where $I$ is the identity operator on $L_p$. By (4) we have $\lim_\gamma T \circ W_\gamma = T \circ I = T$ in $\mathfrak{K}(\mathfrak{M}_p, \mathfrak{A}_p)$. By Lemma 2 we know that $\lim_\gamma T \circ W_\gamma = T$ in the strong operator topology. In particular, there exists some index $\gamma$ such that $\| T \circ W_\gamma(f) - T(f) \| < \varepsilon/2$. It follows that

$$\begin{aligned} \| T \circ W_\gamma(f) \|_p &\geqq \| T(f) \|_p - \| T(f) - T \circ W_\gamma(f) \|_p \\ &\geqq \| T \| - \varepsilon/2 - \varepsilon/2 = \| T \| - \varepsilon ; \end{aligned}$$

but $\| T \circ W_\gamma(f) \|_p \leqq \| T \circ W_\gamma \| \cdot \| f \|_p \leqq \| T \circ W_\gamma \|$, so that $\| T \circ W_\gamma \| \geqq \| T \| - \varepsilon$. Since $\varepsilon$ was arbitrary and $\| W_\gamma \| \leqq 1$, we have shown that

$$\| T \| = \sup \{ \| T \circ W \| \colon W \in A, \| W \| \leqq 1 \}.$$

We shall identify $L_p^t$ and $\mathfrak{A}_p$ for several particular cases.

*Case* I. $p = 1$. Since $L_1$ is a Banach algebra with 2-sided minimal approximate identity, it follows that $L_1^t = L_1$ and $\| W_f \| = \| f \|_1$ for all $f \in L_1$. Because $L_1 * C_{00}$ is dense in $L_1$, it follows that $\mathfrak{A}_p$ is isomorphic to $L_1$ as a Banach algebra. Thus, in this case, Theorem 7 is the well-known fact that a bounded linear operator on $L_1$ commutes with all left translation operators if and only if it commutes with all left multiplication by elements of $L_1$.

*Case* II. *$G$ is Abelian and $p = 2$.* Let $X$ be the character group of $G$ and $\theta$ the Haar measure on $X$ such that $\| \hat{f} \|_2 = \| f \|_2$ for all $f \in L_2$. In this case there is an isometric isomorphism $\widehat{\ } \mid M_2 \rightarrow L_\infty(X)$ which is onto $L_\infty(X)$ and such that $\widehat{T(f)} = \hat{T} \cdot \hat{f}$ for all $g \in L_2$. Evidently, $L_2^t$ is just $\{ f \in L_2 \colon \hat{f} \in L_\infty(X) \}$. It is known that there is a net $\{ g_\alpha \}$ in the set $\{ \hat{f} \colon f \in C_{00}(G) \}$ such that $\| g_\alpha \|_\infty = 1$ for each index $\alpha$ and $\lim g_\alpha(\chi) = 1$ uniformly on compact subsets of $X$. Consequently, the set $\{ \widehat{h * f} \colon h \in L_2^t, f \in C_{00} \}$ is dense in the set $\{ g \in L_2(X) \cap L_\infty(X) \colon g$ vanishes at $\infty \}$. It follows that $\mathfrak{A}_2$ is isomorphic in this case to $\{ f \in L_\infty(x) \colon f$ vanishes at $\infty \}$.

*Case* III. *$G$ is compact and $p \neq 1$.* In this case $L_p$ is a convolution algebra ([2] 28.64). Thus, $L_p^t = L_p$ and $W$ may be viewed as a non norm-increasing linear operator from $L_p$ into $\mathfrak{A}_p$. Since $C_{00} \subset L_p \cap L_1$, it is not difficult to show that $W$ is an isomorphism into $\mathfrak{A}_p$.

Let $f \in L_p$ and choose a minimal approximate identity $\{ f_\alpha \}$ for $L_1$ out of $C_{00}$. Then $\{ f * f_\alpha \}$ converges to $f$ in $L_p$. Consequently, $\{ W_{f * f_\alpha} \}$ converges to $W_f$ in $\mathfrak{A}_p$. All this shows that, in this case, $\mathfrak{A}_p$ is the closure in $\mathfrak{B}_p$ of the set $\{ W_f \colon f \in L_p \}$.

Suppose now that $G$ is also infinite. Then $L_p$ has no minimal 1-sided identity (see [2] 34.40. b); since $\mathfrak{A}_p$ does have one, it follows that $W$ is not a homeomorphism. Since $W$ is a continuous isomorphism, the open mapping theorem implies that $W \mid L_p \rightarrow \mathfrak{A}_p$ is not onto $\mathfrak{A}_p$.

*Case* IV. *$G$ is compact and $p = 2$.* Let $\Sigma$ be the dual object of $G$ as in [2]. For the spaces $\mathfrak{E}_0(\Sigma)$, $\mathfrak{E}_\infty(\Sigma)$, and $\mathfrak{E}_2(\Sigma)$ and the norms $\| \ \|_\infty$ and $\| \ \|_2$ on these spaces, see [2] 28.34. It is an easy consequence of [2] D. 54 that

$$(5) \qquad \| E \|_\infty = \sup \{ \| A \circ E \|_2 \colon A \in \mathfrak{E}_2(\Sigma), \| A \|_2 \leqq 1 \}$$

for all $E \in \mathfrak{C}_\infty(\Sigma)$. For the definition of the Fourier-Stieltjes transform $\hat{f}$ of a function $f \in L_2$, see [2] 28.34. By [2] 28.43, the mapping $\hat{}\,| L_2 \rightarrow \mathfrak{C}_2(\Sigma)$ is a surjective linear isometry and, by [2] 28.40, $\widehat{f*g} = \hat{f}\circ\hat{g}$ for all $f, g \in L_2$. Consequently, by (5),

$$\| W_f \| = \| \hat{f} \|_\infty \quad \text{for all} \quad f \in L_2 \ . \tag{6}$$

Since $C_{00} \subset L_2$, it follows from [2] 28.39, 28.27, and 28,40 that the set $\{\hat{f}: f \in L_2\}$ is a dense subspace of $\mathfrak{C}_0(\Sigma)$. Since $\mathfrak{A}_p$ is just the closure in $\mathfrak{B}_p$ of the set $\{W_f: f \in L_2\}$, it follows from (6) that $\mathfrak{A}_p$ is isomorphic to $\mathfrak{C}_0(\Sigma)$ as a Banach algebra.

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WASHINGTON STATE UNIVERSITY

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# SEQUENCES OF QUASI-SUBORDINATE FUNCTIONS

JAMES MILLER

**In this paper a theorem is proved which connects bounded analytic functions in the unit disk and sequences of quasi-subordinate functions. As an application a necessary and sufficient condition for certain sequences of quasi-subordinate functions to converge is found.**

Let $f$ and $F$ be analytic functions in $|z| < R$. If there exist two functions $\phi$ and $\omega$ which are analytic in $|z| < R$ and satisfy $\omega(0) = 0$, $|\phi(z)| \leqq 1$, $|\omega(z)| < R$, and $f(z) = \phi(z)F(w(z))$ for $|z| < R$, then we say that $f$ is quasi-subordinate to $F$ in $|z| < R$ and write $f \prec_q F$. Without loss of generality we may assume that $R = 1$. This class was introduced by Robertson [2, 3].

We note that there are two special cases of quasi-subordination which are of interest: If $\phi$ is the constant function one, then $f$ is subordinate to $F$, and on the other hand, if $\omega$ is the identity function, then $f$ is majorized by $F$.

Let $B$ denote the class of functions $\theta$ which are analytic in $|z| < 1$ and satisfy $|\theta(z)| \leqq 1$ for $|z| < 1$. Then the functions $\phi$ and $\omega$ which are defined above are elements of $B$. In this paper we prove a theorem which connects functions in $B$ and sequences of quasi-subordinate functions. As an application we find necessary and sufficient conditions for certain sequences of quasi-subordinate functions to converge. This is a generalization of Pommerenke's results [1] on sequences of subordinate functions.

Let $\{f_n\}$, $n = 1, 2, \cdots$, be a sequence of functions which are analytic in $|z| < 1$ such that $f_n \prec_q f_{n+1}$ for each $n$ or $f_{n+1} \prec_q f_n$ for each $n$. When considering the convergence of such sequences we need to require that either the sequence $\{f_n(0)\}$ converges or the functions agree at a single point. In this paper we shall assume that the functions agree at a single point. Further we may assume that the point is $z = 0$ for if the functions $f_n$ agree at the point $a \neq 0$ then we could consider the functions $g_n(z) = f_n((z-a)/(1-az))$. We will use $f_n(0) = 0$ for all $n$, otherwise the function $\phi$ would be identically one. The proof for the case where $\{f_n(0)\}$ is convergent is similar.

THEOREM 1. *Let $\{f_n\}$ be a sequence of functions which are analytic in $|z| < 1$ and satisfy $f_n(0) = 0$, $\alpha_n = f_n'(0) \neq 0$, and $f_n(z) \prec_q f_{n+1}$, and let $\phi_{n+1}, \omega_{n+1} \in B$ and $\omega_{n+1}(0) = 0$ be such that*

$$f_n(z) = \phi_{n+1}(z) f_{n+1}(\omega_{n+1}(z))$$

*for* $|z| < 1$. *If* $\sum_{n=2}^{\infty} \arg \phi_n(0)$ *converges and* $\lim_{n\to\infty} \alpha_n = \alpha$, $|\alpha| < \infty$, *then* $\prod_{n=2}^{\infty} \phi_n(0)$ *converges.*

*Proof.* We observe that if $m < n$, then we have $f_m \prec_q f_n$. Thus for $m < n$ there are functions $\phi_{mn}, \omega_{mn} \in B$ where $\omega_{mn}(0) = 0$ such that

$$f_m(z) = \phi_{mn}(z) f_n(\omega_{mn}(z))$$

for $|z| < 1$. Let $\phi_{n\, n+1}(z) = \phi_{n+1}(z)$. We now observe that

$$f'_m(0) = \phi_{mn}(0)\omega'_{mn}(0) f'_n(0)$$

or

$$\alpha_m = \phi_{mn}(0)\omega'_{mn}(0)\alpha_n \,. \tag{1}$$

Since $0 < |\alpha_m| \leqq |\alpha_n|$ for $m < n$ and $\alpha_n \to \alpha$, there exists an integer $K$ such that if $n > m > K$, then

$$\left| \frac{\alpha_m}{\alpha_n} - 1 \right| < \varepsilon \,. \tag{2}$$

From (1) and (2) we see that

$$1 - \varepsilon < \left| \frac{\alpha_m}{\alpha_n} \right| = |\phi_{mn}(0)\omega'_{mn}(0)| \leqq |\phi_{mn}(0)| \leqq 1 \,.$$

We now observe that

$$\phi_{mn}(0) = \prod_{k=m+1}^{n} \phi_k(0) \,.$$

Thus we have

$$1 - \varepsilon < \left| \prod_{k=m+1}^{n} \phi_k(0) \right| \leqq 1$$

for $n > m > K$. Since $\sum_{n=2}^{\infty} \arg \phi_n(0)$ converges this says that $\prod_{k=2}^{\infty} \phi_k(0)$ converges. Further we have that $\omega'_n(0) \to 1$ and $\omega'_{mn}(0) = 1$.

In applying Theorem 1 to sequences of quasi-subordinate functions we will also need two lemmas for functions in $B$. The proofs of the lemmas are essentially in **[1]**.

LEMMA 1. *Let* $\phi \in B$, $\phi(0) = 0$, *and satisfy* $|\phi(0)| \geqq \sigma > 0$. *Then the mapping* $w = \phi(z)$ *maps the disk*

$$|z| < \rho = \frac{\sigma}{1+\sqrt{1-\sigma^2}}$$

*univalently onto a region that contains* $|w| < \rho^2$.

LEMMA 2. *For* $\varepsilon > 0$ *and* $0 < r < 1$, *there exists an* $\eta > 0$ $(\eta(\varepsilon, r))$ *such that if* $\phi \in B$ *satisfies* $\phi(z) = \sum_{n=0}^{\infty} \beta_n z^n$ *and* $|\beta_k - 1| \leqq \eta$, *then*

$$|\phi(z) - z^k| < \varepsilon , \qquad \text{for } |z| < r .$$

THEOREM 2. *Let* $\{f_n\}$ *be a sequence of analytic functions in* $|z| < 1$ *such that* $f_n(0) = 0$, $f_n \prec_q f_{n+1}$, *and* $\alpha_n = f_n'(0) \neq 0$, and let $\phi_{n+1}$, $\omega_{n+2} \in B$ and $\omega_{n+1}(0) = 0$ be such that $f_n(z) = \phi_{n+1}(z) f_{n+1}(\omega_{n+1}(z))$ for $|z| < 1$ and $\sum_{n=2}^{\infty} \arg \phi_n(0)$ converges. *Then the sequence* $\{f_n\}$ *converges uniformly in* $|z| < r$ *for every* $0 \leqq r < 1$ *if and only if*

$$\lim_{n\to\infty} \alpha_n = \alpha , \qquad |\alpha| < \infty .$$

PROOF. If $\{f_n\}$ converges uniformly in $|z| \leqq r$ for every $0 < r < 1$ then $\alpha_n = f_n'(0)$ converges. Further since $|\alpha_n| \leqq |\alpha_{n+1}|$, $f_n(0) = 0$, and $\alpha_n \neq 0$ we see that $\lim_{n\to\infty} \alpha_n = \alpha \neq 0$ and $|\alpha| < \infty$.

Let $\omega_{n+1}, \phi_{n+1} \in B$, and $\omega_{n+1}(0) = 0$ be as defined in Theorem 2. Further for $m < n$, let $\phi_{mn}, \omega_{mn} \in B$ with $\omega_{mn}(0) = 0$ be such that

$$f_m(z) = \phi_{mn}(z) f_n(\omega_{mn}(z)) . \tag{3}$$

Suppose that $\alpha_n \to \alpha$, $|\alpha| < \infty$. Then by Theorem 1 the product $\prod_{k=2}^{\infty} \phi_k(0)$ converges. We will first show that $\{f_n\}$ is a normal family in $|z| < 1$.

Let $r$, $0 < r < 1$, be fixed and $\sigma$ determined by

$$\sqrt{r} = \frac{\sigma}{1+\sqrt{1-\sigma^2}} .$$

Since $\sigma < 1$ and $\alpha_n \to \alpha \neq 0$, there exists an integer $N_1$ such that

$$\left|\frac{\alpha_m}{\alpha_n}\right| > \sigma , \qquad \text{for } n > m > N_1 .$$

Further, since $|\phi_{mn}(z)| \leqq 1$, we have $|\phi_{mn}(0)|^{-1} \geqq 1$. For $n > m > N_1$ we have $\omega'_{mn}(0) = \alpha_m/(\alpha_n \phi_{mn}(0))$ or

$$|\omega'_{mn}(0)| = \left|\frac{1}{\phi_{mn}(0)} \frac{\alpha_m}{\alpha_n}\right| > \sigma . \tag{4}$$

Thus by Lemma 1 the mapping $\zeta = \omega_{mn}(z)$ for $n < m < N_1$ maps $|z| < \sqrt{r}$ univalently onto a domain that contains $|\zeta| < r$. Let $\psi_{mn}$ be the inverse of $\zeta = \omega_{mn}(z)$ in $|\zeta| < r$, then

$$| \psi_{mn}(\zeta) | \leqq \sqrt{r}\ .$$

From (3) we may write

$$f_n(\zeta) = \frac{1}{\phi_{mn}(\psi_{mn}(\zeta))} f_m(\psi_{mn}(\zeta))\ , \qquad \text{for } | \zeta | < r\ .$$

For $| \zeta | \leqq r$ we have

$$|f_n(\zeta)| \leqq \max_{|z| \leqq \sqrt{r}} \left| \frac{f_m(z)}{\phi_{mn}(z)} \right| \leqq \frac{1}{\min\limits_{|z| \leqq \sqrt{r}} | \phi_{mn}(z) |} \max_{|z| \leqq \sqrt{r}} |f_m(z)|\ .$$

From Lemma 2 with $k = 0$, given $\varepsilon > 0$, there exists an $\eta$ such that if $| \beta_0 - 1 | < \eta$ then $| \phi(z) - 1 | < \varepsilon$ for $| z | < r$. Since $\prod_{k=2}^{\infty} \phi_k(0)$ converges by Theorem 1 and $\phi_{mn}(0) = \prod_{k=m+1}^{n} \phi_k(0)$, there exists an integer $N_2$ such that if $n > m > N_2$ then $| \phi_{mn}(0) - 1 | < \eta$. Let $N = \max(N_1, N_2)$. Thus, by Lemma 2 we have that $| \phi_{mn}(z) - 1 | < \varepsilon$ for $| z | \leqq r$ and $n > m > N$ or

$$\min_{|z| \leqq r} | \phi_{mn}(z) | \geqq 1 - \varepsilon\ .$$

Hence, for $n > N$ and $| \zeta | \leqq r$ we have

$$| f_n(\zeta) | \leqq \frac{1}{1 - \varepsilon} \max_{|z| \leqq \sqrt{r}} | f_{N+1}(z) |\ .$$

Thus there exists $M(r)$ such that

$$(5) \qquad | f_n(z) | \leqq M(r)$$

for all $n$, that is, $\{f_n\}$ is locally uniformly bounded. Therefore $\{f_n\}$ is normal.

Let $\{f_{n_\nu}\}$ be a subsequence of $\{f_n\}$ which is uniformly convergent in $| z | \leqq r_0$, for every $r_0 < 1$. Let $f$ be the limit function of $\{f_{n_\nu}\}$. Let $\varepsilon > 0$ and $r < 1$. Then choose $\nu_0$ such that

$$| f_{n_\nu}(z) - f(z) | < \varepsilon/3$$

for $\nu \geqq \nu_0$ and $| z | \leqq r$. From inequality (5) we have that the sequence $\{f_n\}$ is bounded in $| z | \leqq r$ and thus equicontinuous in $| z | \leqq r$. Therefore there exists a $\delta > 0$ such that

$$| f_n(z_1) - f_n(z_2) | < \varepsilon/3$$

for $| z_1 - z_2 | < \delta$, $| z_1 | \leqq r + \delta$, $| z_2 | \leqq r + \delta$, and for all $n$.

Using (4), the convergence of $\sum_{n=2}^{\infty} \arg \phi_n(0)$, and applying Lemma 2 we have that there exists an integer $M_1$ such that if $n \geqq m \geqq M_1$, then

$$| \omega_{mn}(z) - z | < \delta\ , \qquad \text{for } | z | \leqq r$$

where $M_1$ is chosen so that $|\omega'_{mn}(0)-1|<\eta$ for a suitable $\eta$. Again making use of Lemma 2 we have that there exists an integer $M_2$ such that if $n>m>M_2$ then

$$|\phi_{mn}(z)-1|<\varepsilon/3M(r), \quad \text{for} \quad |z|<r\,.$$

Let $M=\max\{M_1, M_2, n_{\nu_0}\}$. If $M \leqq k < n_\nu$ and $|z|<r$ then

$$\begin{aligned}|f_k(z)-f(z)| &\leqq |f_k(z)-f_{n_\nu}(z)| + |f_{n_\nu}(z)-f(z)| \\ &< \varepsilon/3 + |f_{n_\nu}(z) - \phi_{kn_\nu}(z) f_{n_\nu}(w_{kn_\nu}(z))| \\ &\leqq \varepsilon/3 + |f_{n_\nu}(z) - f_{n_\nu}(\omega_{kn_\nu}(z))| \\ &\quad + |f_{n_\nu}(\omega_{kn_\nu}(z))\,[1-\phi_{kn_\nu}(z)]| \\ &< \varepsilon/3 + \varepsilon/3 + M(r)\,\varepsilon/3M(r) = \varepsilon\end{aligned}$$

for $|z| \leqq r$ and $k>M$. This completes the proof of Theorem 2.

THEOREM 3. *Let $\{f_n\}$ be a sequence of functions analytic in $|z|<1$ such that $f_n(0)=0$, $\alpha_n=f'_n(0)\neq 0$, and $f_{n+1} \prec_q f_n$, and let $\phi_{n+1}$, $\omega_{n+1} \in B$ and $\omega_{n+1}(0)=0$ be such that*

$$f_{n+1}(z)=\phi_{n+1}(z)f_n(\omega_{n+1}(z))$$

*for $|z|<1$ and $\sum_{n=2}^{\infty} \arg \phi_n(0)$ converges. Then the sequence $\{f_n\}$ converges uniformly in $|z| \leqq r$ for every $r<1$ if the sequence $\{\alpha_n\}$ converges. The limit function is constant if and only if*

$$\lim_{n\to\infty} \alpha_n = 0\,.$$

The proof of this theorem is similar to that of Theorem 2 and Pommerenke's Theorem 2 [1].

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TEXAS A AND M UNIVERSITY

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# THE HASSE-WITT-MATRIX OF SPECIAL PROJECTIVE VARIETIES

LEONHARD MILLER

**The Hasse-Witt-matrix of a projective hypersurface defined over a perfect field $k$ of characteristic $p$ is studied using an explicit description of the Cartier-operator. We get the following applications. If $L$ is a linear variety of dimension $n+1$ and $X$ a generic hypersurface of degree $d$, which divides $p-1$, then the Frobenius-operator $\mathscr{F}$ on $H^n(X\cdot L; \mathscr{O}_{L\cdot Y})$ is invertible.**

As another application we prove the invertibility of the Hasse-Witt-matrix for the generic curve of genus two. We don't study the Frobenius $\mathscr{F}$ directly, but the Cartier-operator [1]. It is well-known, that for curves Frobenius and Cartier-operator are dual to each other under the duality of the Riemann-Roch theorem. A similar fact is true for higher dimension via Serre duality. We have therefore to extend to the whole "De Rham" ring the description of the Cartier-operator given in [4] for 1-forms. We give this extention in §1. Diagonal hypersurfaces are studied in §2 and the invertibility of the Hasse-Witt-matrix is proved, if the degree divides $p-1$. The same theorem for the generic hypersurface follows then from the semicontinuity of the matrix rank. The §3 is devoted to hyperelliptic curves and is intended as a preparation for a detailed study of curves of genus two.

**1. The Cartier-operator of a projective hypersurface.** We extend the explicit construction of the Cartier-operator given in [4] to the whole "De Rham" ring, but restrict ourself to projective hypersurfaces.

As an application we show: Let $V$ be a projective hypersurface of dimension $n-1$, defined by a diagonal equation $F(X) = \sum_{i=0}^{n} a_i X_i^r$, $a_i \in k$ a perfect field of $\operatorname{char} k = p > 0$, $a_i \neq 0$. Let $X$ be a linear variety of dimension $t+1$. If $r$ divides $p-1$, then

$$\mathscr{F}: H^t(X\cdot V, \mathscr{O}_{X\cdot V}) \rightarrow H^t(X\cdot V, \mathscr{O}_{X\cdot V})$$

is invertible, $\mathscr{F}$ being the induced Frobenius endomorphism. We have to rely on a technical proposition, which is a collection of some lemmas in [4]. We give first the proposition.

PROPOSITION 1. *Let*

$$\psi: k[T] \rightarrow k[T] \qquad (T = (T_1, \cdots, T_n))$$

*be $k$ $p^{-1}$—linear and*

$$\psi(T^{\mu}) = \begin{cases} T^{\nu} & \text{if} \quad \mu = p\cdot\nu \\ 0 & \text{else} . \end{cases}$$

*Then the following holds:*

(1) $\psi(T_{\mu_1} \cdots T_{\mu_r} h) = T_{\mu_1} \cdots T_{\mu_1}\bar{h}$, *for some* $\bar{h} \in k[T]$

(2) *Let* $D_\mu = T_\mu\,(\partial/\partial T_\mu)$ *and* $D_\mu g = 0$ *for a given* $1 \leqq \mu \leqq n$, *then* $\psi(D_\mu h\cdot g) = 0$

(3) *Let* $D_\mu g = 0$, *then* $\psi(h^{p-1}D_\mu h\cdot g) = D_\mu h \psi(g)$.

*Proof.*

(1) By the $p^{-1}$-linearity of $\psi$ we may assume $h$ to be a monomial. The statement follows then directly from the definition of $\psi$.

(2) $\psi$ is $p^{-1}$-linear, so we may assume $h$ to be a monomial

$$h = T_1^{r_1} \cdots T_n^{r_n}, \qquad 0 \leqq r_i \leqq p-1$$

(say $\mu = n$), then $D_n h = r_n\cdot h$. If $r_n = 0$ then (2) is trivially true. So $r_n \neq 0$. Again because of $p^{-1}$-linearity we may also assume $g$ to be monomial.

But $D_n g = 0$, so

$$g = T_1^{v_1} \cdots T_{n-1}^{v_{n-1}} \qquad 0 \leqq v_i \leqq p-1 .$$

So the exponent of $T_n$ in $D_n h\cdot g$ is $r_n$ and $0 < r_n \leqq p-1$, therefore not divisible by $p$. The definition of $\psi$ gives

$$\psi(D_n h\cdot g) = 0 .$$

(3) We may write

$$h = f_0 + f_1\cdot T_\mu + \cdots + f_r\cdot T_\mu^r, \qquad 0 \leqq r \leqq p-1$$

and

$$D_\mu f_i = 0 .$$

We proceed by induction on $T$. $r = 0$ clear. Let $r \geqq 1$, then $h = f + T_\mu \bar{h}$ with $D_\mu f = 0$ $\deg_{T_\mu}\bar{h} < r$. Now

$$T_\mu^{p-1}\bar{h}^{p-1}D_\mu(T_\mu\bar{h}) = (T_\mu\bar{h})^p\Big(\frac{D_\mu T_\mu}{T_\mu} + \frac{D_\mu\bar{h}}{\bar{h}}\Big) .$$

By $p^{-1}$-linearity of $\psi$ and induction assumption for $\bar{h}$ we get

$$\begin{aligned}\psi(g\cdot T_\mu^{p-1}\bar{h}^{p-1}D_\mu(T_\mu\bar{h})) &= T_\mu\bar{h}\psi(g) + T_\mu\psi(g\cdot\bar{h}^{p-1}D\bar{h}) \\ &= \psi(g)(T_\mu\bar{h} + T_\mu D_\mu\bar{h}) \\ &= D_\mu(T_\mu\bar{h})\cdot\psi(g) .\end{aligned}$$

On the other hand

$$T_\mu^{p-1}\bar{h}^{p-1} = (h-f)^{p-1} = h^{p-1} + \frac{\partial P}{\partial h},$$

where $P$ is a polynomial in $f$ and $h$. We have

$$D_\mu(T_\mu \bar{h}) = D_\mu(h-f) = D_\mu h .$$

So

$$T_\mu^{p-1}\bar{h}^{p-1}D_\mu(T_\mu\bar{h}) = h^{p-1}D_\mu h + D_\mu P .$$

Multiply by $g$ and apply $\psi$, then one gets

$$D_\mu h\cdot\psi(g) = D_\mu(T_\mu\bar{h})\psi(g) = \psi(h^{p-1}D_\mu h\cdot g) + \psi(D_\mu P\cdot g) .$$

But by (2)

$$\psi(D_\mu P\cdot g) = 0 .$$

Let $F(X_0 \cdots X_n)$ define a absolutely irreducible hypersurface $V/k$ in $\mathscr{P}_{n,k}$ $\operatorname{char} k = p > 0$. We denote by $f(X_1 \cdots X_n)$ an affinization of $F$. Let $F_\mu = (\partial/\partial X_\mu)F$, similar $f_\mu$ $1 \leqq \mu \leqq n$. We assume $f_n$ not to be the zero function on $V$. Let $K = K(V)$ be the function field of $V$. We assume that $K = K^p(x_1 \cdots \check{x}_j \cdots x_n)$ for any index $j$. The $x_i$ are the coordinate functions and $\check{x}_j$ means omit $x_j$. As a consequence of these assumptions, we have that for a given index $j$ any function $z \in K$ can be represented modulo $F$ by a rational function $G(X_1 \cdots X_n)$, which is $X_j$-constant, i.e. such that $\partial G/\partial X_j = 0$. Write

$$F_{i_1,\cdots,i_r,n} = (X_{i_1} \cdots X_{i_r}\cdot X_n)^{-1}F .$$

DEFINITION 1. Let

$$\psi_{F_{i_1,\cdots,i_r,n}} = F_{i_1,\cdots,i_r,n}\circ\psi\circ F^{-1}_{i_1,\cdots,i_r,n} .$$

Let $\omega = \sum_{i_1\cdots i_r} h_{i_1,\cdots,i_r}\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_r}$ be $r$-form on $V$. Put

$$\omega_{i_1,\cdots,i_r} = \frac{dx_{i_1}\wedge\cdots\wedge dx_{i_r}}{f_n} .$$

Define

$$C(\omega) = \sum_{i_1,\cdots,i_r} \psi_{Fi_1,\cdots,i_r,n}(h_{i_1,\cdots,i_r} - f_n)\omega_{i_r,\cdots,i_r} .$$

The definition is justified by the following theorem.

THEOREM 1. (1) *$C$ is $p^{-1}$-linear*
(2) *If $\omega = d\varphi$, then $C(\omega) = 0$*

(3) *If* $\omega = z_{i_1}^{p-1} \cdots z_{i_r}^{p-1} dz_{i_1} \wedge \cdots \wedge dz_{i_r}$ *then* $C(\omega) = dz_{i_1} \wedge \cdots \wedge dz_{i_r}$. *In other words, if one restricts* $C$ *to* $Z_{V/k}^r$, *the closed forms, then*

$$C\colon Z_{V/k}^r \to \Omega_{V/k}^r$$

*is the Cartier-operator of* $V$ [1].

*Proof of the theorem.*

(1) The $p^{-1}$-linearity follows from the $p^{-1}$-linearity of $\psi$.
(2) Let $\varphi = \sum_{i_1,\cdots,i_{r-1}} \varphi_{i_1,\cdots,i_{r-1}} dx_{i_1} \wedge \cdots \wedge dx_{i_{r-1}}$ be a $(r-1)$-form, then

$$d\varphi = \sum_j \sum_{i_1,\cdots,i_{r-1}} \frac{\partial}{\partial x_j}(\varphi_{i_1,\cdots,i_{r-1}}) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{r-1}} .$$

To simplify the notation we put for the moment

$$\varphi_{i_1,\cdots,i_{r-1}} = \tilde{\varphi}$$

and

$$F_{j_1 i_1,\cdots,i_{r-1},n} = \tilde{F} .$$

To compute $C(d\varphi)$ we have to compute

$$\varphi_{\tilde{F}}\Big(\frac{\partial}{\partial x_j}\tilde{\varphi}\cdot f_n\Big)$$

for every system $(j, i, \cdots, i_{r-1})$.

Now remembering the definition of $\psi^{\tilde{F}}$ we have to show

$$\psi(F^{p-1} D_n F X_{i_1} \cdots X_{i_{r-1}} D_j \varphi) = 0$$

in order to get $C(d\varphi) = 0$.

We have to use the above proposition. We apply first (3) and then (2) and get:

$$\psi(F^{p-1} D_n F X_{i_1} \cdots X_{ir-1} D_j \varphi) = D_n F \psi(X_{i_1} \cdots X_{i_{r-1}} D_j \varphi) = 0 .$$

Remark, that we assume $j \neq (i_1, \cdots, i_{r-1})$ otherwise

$$dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{r-1}} = 0 .$$

That shows $C(d\varphi) = 0$

(3) Let $\omega = z_{i_1}^{p-1} \cdots z_{ir}^{p-1} dz_{i_1} \wedge \cdots \wedge dz_{i_r}$.

We have

$$dz_{i_1} \wedge \cdots \wedge dz_{i_r} = \sum_{j_1 \cdots j_r} D_{j_1} z_{i_1} \cdots D_{j_r} z_{i_r} \frac{dx_{j_1} \wedge \cdots \wedge dx_{j_r}}{x_{j_1} \cdots x_{j_r}},$$

$$D_j = x_j \frac{\partial}{\partial x_j}.$$

To Compute $C(\omega)$, we have to work out

$$U = \psi(F^{p-1} D_n F \cdot Z_{i_1}^{p-1} \cdot D_{j_1} Z_{i_1} \cdots Z_{i_r}^{p-1} D_{j_r} Z_{i_r}) \text{ modulo } F.$$

$$Z_j \bmod F = z_j.$$

We apply several times (3) of the propositition and get

$$U \equiv D_n F D_{j_1} Z_{i_r} \cdots D_{j_r} Z_{i_r} \bmod (F).$$

Therefore

$$C(\omega) = \sum_{j_r j_r} D_n f D_{j_1} z_{i_1} \cdots D_{j_r} z_{i_r} \frac{dx_{j_1} \wedge \cdots \wedge dx_{j_r}}{x_n f_n x_{j_1} \cdots x_{j_r}}$$
$$= dz_i \wedge \cdots \wedge dz_{i_r}.$$

All forms of highest degree $n-1$ are closed. We use the fact, that $H^\circ(V, \Omega^{n-1})$ has a basis of the following form

$$\omega_u = x_1^{u_1} \cdots x_n^{u_n} \omega_0.$$

where

$$\omega_0 = \frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{x_1 \cdots x_n f_n}$$

$$\sum_{i=1}^{n} u_i \leqq r; r = \deg V \quad \text{and} \quad 1 \leqq u_i.$$

Recall $x_i = X_i/X_0$ are coordinate functions on $V$ and of the affinization of $F, f_n = \partial f/\partial x_n$.

We get the important corollary to the theorem.

COROLLARY 1. *Let $A_{u,v}$ be the matrix of the Cartier-operator on $H(V, \Omega^{n-1})$ with respect to the above basis $\omega_u$. Then*

$$A_{u,v} = \textit{coefficient of } X^v \textit{ in } \psi(F^{p-1} \cdot X^u)$$

$$X^u = X_0^{u_0} \cdots X_n^{u_n}, \quad \sum_{i=0}^{n} u_i = \sum_{i=0}^{n} v_i = r$$

$$\left.\begin{matrix} 1 \leqq u_i \\ 1 \leqq v_i \end{matrix}\right. \quad \textit{for} \quad i = 1 \cdots n.$$

*Proof.* By definition

$$C(\omega_u) = \psi_{F_{1\cdots n}}(x_1^{u_1-1} \cdots x_n^{u_n-1})\frac{dx_1 \wedge \cdots \wedge dx_{n-1}}{f_n}$$
$$= \psi(f^{p-1}\cdot x^u)\omega_0 .$$

Now recall

$$\psi(f^{p-1}\cdot x^u) = \psi\left(\frac{F^{p-1}X_0^{u_0} \cdots X_r^{u_r}}{X_0^{pr}}\right) \bmod F$$
$$\sum_{i=0}^{n} u_i = r , \quad 1 \leqq u_i , \quad i = 1 \cdots n .$$

If $A_{u,v}$ is the coefficient of $X^v$ in $\psi(F^{p-1}\cdot X^u)$.
Then

$$C(\omega_u) = \sum_{\substack{1 \leqq v_i \leqq r \\ i=1\cdots n}} A_{u,v}x_1^{v_1} \cdots x_n^{v_n}\omega_0 = \sum_v A_{u,v}\omega_v .$$

Notice

$$\sum_{i=0}^{n} u_i = \sum_{i=0}^{n} v_i = r , \quad 1 \leqq u_i, 1 \leqq v_i, i = 1 \cdots n .$$

REMARK. We have now on explicit description for the Cartier-operator on $H^0(V, \Omega^{n-1}_{V/k})$. We can use Serre duality $H^0(V, \Omega^{n-1}_{V/k})^\vee \cong H^{n-1}(V, \mathcal{O}_U)$. Under this duality $\check{C}$ is the Frobenius $\mathcal{F}$ on $H^{n-1}(V, \mathcal{O}_V)$. We have therefore also an explicit description for $\mathcal{F}$.

**2. The Cartier-operator of a diagonal hypersurface.** Let $F(X) = \sum_{i=0}^{n} a_i X_i^r$ define a "generic" hypersurface. To compute the Cartier-operator, by the preceding discussion we have to analyse

$$\psi(F^{p-1}X^u) \qquad \left(\sum_{i=0}^{n} u_i = r , \quad u_i > 0\right) .$$

Let us adapt the following notation:

$$\rho^! = \rho_0^! \cdots \rho_n^! , \quad a^\rho = \prod_{i=0}^{n} a_i^{\rho_i} , \quad X^{u+1} = \prod_{i=0}^{n} X_i^{u_i+1} ,$$
$$|u| = \sum_{i=0}^{n} u_i , \quad u > 0 \Leftrightarrow u_i > 0 \quad (i = 0 \cdots n) .$$

THEOREM 2. *Let*

$$\operatorname{char} k = p > 0, F(X) = \sum_{i=0}^{n} a_i X_i^r , \quad \prod_{i=0}^{n} a_i \neq 0 \in k$$

*$V/k$ is defined by $F$. Suppose $r$ divides $p - 1$. Then the Cartier-operator*

$$C\colon H^\circ(V, \Omega^{n-1}_{V/k}) \to H^\circ(V, \Omega^{n-11}_{V/k})$$

*is invertible.*

*Proof.*

$$F^{p-1} = \sum_{|m|=p-1} \frac{(p-1)!}{m!} a^m X^{rm} .$$

Using $p^{-1}$-linearity of $\psi$ we get

$$\psi(F^{p-1}X^u) = \sum_{|m|=p-1} \frac{-1}{m!} \bar{a}^m \psi(X^{rm+u}) = \sum_{|m|=p-1} \frac{-1}{m!} \bar{a}^m X^v .$$

We put $\bar{a} = a^{1/p}$, and $rm + u = pv$. Notice if $u > 0$ and $|u| = r$, then also $v > 0$ and $|v| = r$. If we write

$$\psi(F^{p-1}X^u) = \sum_{\substack{|v|=r \\ v>0}} A_{u,v} X^v ,$$

then we have

$$A^p_{u,v} = \begin{cases} -\dfrac{1}{m!} a^m & \text{if} \quad rm = (p-1)v + v - u \\ & \qquad |u| = |v| = r \quad u > 0 \quad v > 0 \\ 0 & \text{else} . \end{cases}$$

Let us now assume:

$$p - 1 = r \cdot s .$$

If $r$ divides $v - u$ put $v - u = r \cdot E(u, v)$ then

$$A^p_{u,v} = \begin{cases} -\dfrac{1}{m!} a^m & \text{if} \quad r \mid v - u \quad \text{and} \quad m = sv + E(u, v) \\ 0 & \text{else} . \end{cases}$$

We fix now a total ordering of $u, v$. Let us order the $n$-tuples $(u_1 \cdots u_n)$ resp $(v_1 \cdots v_n)$ lexicographically and put

$$u_0 = r - \sum_{i=1}^n u_i \quad \text{resp.} \quad v_0 = r - \sum_{i=1}^n v_i$$

$v < u$ means now, that either $v_1 < u_1$ or $v_i = u_i$ for $i = 1 \cdots j - 1$ but $v_j < u_j$. If any case, if $v < u$, then $v_j < u_j$ for some $j$. We claim if $v < u$, the $A_{u,v} = 0$.

*Case* 1. $r$ does not divide $u - v$, then $A_{u,v} = 0$.

*Case* 2. $r$ divides $u - v$. Now if $v < u$ then for some $j$ $u_j - v_j > 0$

and $r$ divides $u_i - v_j$. But $r \geqq u_j$ and $v_j \geqq 1$, so $r - 1 \geqq u_j - v_j$, therefore $r$ cannot divide $u_j - v_j$. This contradiction shows, if $v < u$, then $A_{u,v} = 0$. $A_{u,v}$ is therefore a triangle matrix.

What is the diagonal?

$$A^p_{u,u} = -\frac{1}{m!} a^m$$

with $m = s \cdot u$. Therefore

$$(\det A_{u,v})^p = \prod_u \left(-\frac{1}{(su)!}\right) a^{s\sum_u u} \neq 0$$

Corollary 2. *The assumptions are the same as in the theorem. Then*

$$\mathscr{F}: H^{n-1}(V, \mathscr{O}_V) \to H^{n-1}(V, \mathscr{O}_V) \quad (\mathscr{F} \text{ is the Frobenius morphism})$$

*is invertible.*

*Proof.* Clear by Serre duality and the fact that $\check{C} = \mathscr{F}$.

*The Cartier-operator of $W \cdot H$.* The differential operator $C$ as given in Definition 1 on $\Omega^1$ is by $p^{-1}$-linearity completely determined on $\Omega^1$ by its value on $\omega = h \cdot dx$, where $x$ runs through a set of coordinate functions.

We have $C(\omega) = x^{-1}\psi(xh)dx$, that notation is only intrinsic, if $d\omega = 0$, because $\psi$ depends on the coordinate system. If we choose a different coordinate system, then we get in general a different operator; but for $\omega$ with $d\omega = 0$, we get the same, namely the Cartier-operator.

That fact can be exploited in the following way. Suppose

$$W = \{x_1 = x_2 \cdots = x_t = 0\} \cap H .$$

We write now $C_H$ resp. $C_W$ for the the operators. The above definition shows $\bigoplus_{i=1}^t Kdx_i$ is stable under $C_H$. But by the property of $\psi$, $\psi(X_i H) = X_i \bar{H}$ for some $\bar{H}$, we have for

$$\omega = x_i h dx_j \quad i \neq j \quad i, j \text{ arbitrary}$$
$$C_H(\omega) = x_i \bar{h} dx_j .$$

Let $\mathfrak{A} = \{x_1 \cdots x_t\}$, then $\mathfrak{A}\Omega^1_{H/k} \oplus \bigoplus_{i=1}^t \mathscr{O}_H dx_i$ is stable under $C_H$. By the exact sequence

$$0 \to \mathfrak{A}\Omega^1_{H/k} + \bigoplus_{i=1}^t \mathscr{O}_H dx_i \to \Omega^1_{H/k} \to \Omega^1_{W/k} \to 0$$

$C_H$ induces an operator $C_W$ on $\Omega^1_{W/k}$. $C_W$ has again the properties

(1) $C_W$ is $p^{-1}$-linear
(2) $C_W(dh) = 0$
(3) $C_W(h^{p-1}dh) = dh$ .

If we restrict $C_W$ to the closed forms on $W$, then $C_W$ is the Cartier-operator.

Let now $L$ be an arbitrary linear variety. After a suitable coordinate change we may assume $L$ is the intersection of some coordinate hyperplanes. $W = L \cdot H$ has then the above shape.

Let us assume that the hypersurface $H$ has a diagonal defining equation of degree $d$ diving $p-1$, $p = \operatorname{char} k$. Then the above Theorem 1 shows that $C_W$ is semisimple on $Z^1_{W/k}$. In the same way as before we can extend $C_W$ to any $\Omega^r_{W/k}$, in particular to $\Omega^m_{W/k}$, where $m = \dim W$. As result of this discussion we get:

THEOREM 3. *If $L$ is a linear variety of dimension $m+1$, then there exists a hypersurface $H$ of degree $d$, which divides $p-1$, such that*

$$\mathscr{F}: H^m(L\cdot H, \mathscr{O}_{L\cdot H}) \to H^m(L\cdot H, \mathscr{O}_{L\cdot H})$$

*is invertible.*

**3. The Cartier-operator of plane curves.** For curves the explicit description of the Cartier-operator is of special interest if one wants to study, how the Cartier-operator varies with the moduli of the curve. Unfortunately one is restricted to plane curves, because the above explicit form of the Cartier-operator is available only for hypersurfaces.

If one specializes the above results to plane curves, one has to assume, that the curve is singularity free.

The space $W = \{\text{homogenous forms of degree } d-3\}$ is for nonsingular curves $V$ of degree $d$ isomorphic to $H^\circ(V, \Omega^1_{V/k})$ under

$$\begin{aligned} W &\xrightarrow{\sim} H^0(V, \Omega^1_{V/k}) \\ P(X) &\to P(x)\omega_0 \end{aligned}$$

where the coordinate functions are given by

$$x = X_1/X_0\,, \quad y = X_2/X_0 \mod F\,,$$

$F$ being the defining equation for $V$ and $f(x, y)$ the affinization, $f_y$ denotes $\partial f/\partial y$. With that notation $\omega_0 = dx/f_y$.

But it is important to know, that one can give a similar description also for singular curves. Then $W$ is the space of $P(X)$, which define the "adjoint" curves to $V$. These are those curves, which cut out at least the "double point divisor".

To give an explicit basis depends on nature of the singularities.

*Hyperelliptic curves*: Let $p = \operatorname{char} k > 2$.

For a detailed study of the Hasse-Witt-matrix of hyperelliptic curves one needs the explicit Cartier-operator with respect to various "normal forms".

Let the hyperelliptic $V$ be given by $y^2 = f(x)$, $\deg f(x) = 2g + 1$ and such that $f(x)$ has no multiple roots. $V$ has a singularity at "infinity". One could apply the above method and work out the adjoint curves in order to get a basis for $H^0(V, \Omega^1_{V/k})$. But we have already a basis, namely if $\omega = dx/y$ then $\{x^i\omega \mid i = 0 \cdots g - 1\}$ form a basis.

We specialize the results of §2 and get from Corollary 1 as matrix for the Cartier-operator with respect to the above basis (let us put $p - 1/2 = m$):

$$A_{u,v} = \text{coefficient of } x^{v+1} \text{ in } \psi(f(x)^m x^{u+1}) \quad 0 \leqq {u \atop v} \leqq g - 1 .$$

*Legendre form*: We assume now the defining equation in Legendre form.

$$f(x) = x(x-1)\prod_{i=1}^{r}(x - \lambda_i) \qquad \begin{array}{l} r = 2g - 1 \\ \lambda_i \neq \lambda_j \neq 0, 1 . \end{array}$$

*Notation*: Let

$$\begin{aligned} |\rho| &= \rho_1 + \cdots + \rho_r \\ \lambda^\rho &= \lambda_1^{\rho_1} \cdots \lambda_r^{\rho_n} . \end{aligned}$$

The permutation group of $r$ elements $S_r$ operates on the monomials

$$\lambda^\rho \longrightarrow \lambda^{\pi(\rho)}, \pi \in S_r .$$

Let $G_\rho$ be the fix group of $\lambda^{m-\rho}$ and $G^{(\rho)} = S_r/G_\rho$. Let

$$H^{(\rho)}(\lambda) = \sum_{\pi \in G^{(\rho)}} \lambda^{m-\pi(\rho)} .$$

Apparently

$$H^{(\rho)} = H^{(\bar\rho)} , \quad \text{iff} \quad \bar\rho = \bar\pi(\rho) .$$

We may therefore assume

$$0 \leqq \rho_1 \leqq \rho_2 \leqq \rho_r \leqq m .$$

For given

$$0 \leqq {u \atop v} \leqq g - 1 \quad \text{let} \quad \rho_0 = |\rho| - vp + u .$$

Put

$$a_{u,v}^{(\rho)} = (-1)^{u+v+m}\binom{m}{\rho_0}\cdots\binom{m}{\rho_r}$$

and

$$A_{u,v}^p = \sum_{\rho} a_{u,v}^{(\rho)} H^{(\rho)}(\lambda) \qquad 0 \leqq {u \atop v} \leqq g-1,\ r = 2g-1$$

the summation condition being:

$$0 \leqq \rho_1 \leqq \cdots \leqq \rho_r \leqq m\ , \quad \rho_0 = |\rho| - vp + u\ , \quad 0 \leqq \rho_0 \leqq m$$

$$vp - u + m \geqq |\rho| \geqq vp - u\ .$$

We state as a proposition

PROPOSITION 2. *Let be* $A_{u,v}$, $0 \leqq {u \atop v} \leqq g-1$, *as defined above, and* $\omega = dx/y$, *then*

$$C(x^u\omega) = \sum_{0 \leqq v \leqq g-1} A_{u,v} x^v \omega$$

*is the Cartier-operator.*

*Applications*: We want to investigate, when the Cartier-operator is invertible. It seems that an answer to that question, without any restrictions is not available. It is therefore worthwhile to have various methods even in special cases.[1]

We restrict ourself to genus 2, although the method could be applied to higher genus, but the calculations would be very easy. Let $p > 2$ and $g = 2$

$$\text{i.e. } y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)\ , \quad \lambda_i \neq \lambda_j \neq 0, 1 \quad i \neq j\ .$$

The notation is the same as above.

$H^{(\rho)}(\lambda)$ is homogeneous in the $\lambda$'s of degree $3m - |\rho|$, $m = (p-1)/2$. We have

$$A_{u,v}^p = \sum_{0 \leqq \rho_0 \leqq \rho_1 \leqq \rho_2 \leqq \rho_3 \leqq m} a_{u,v}^{(\rho)} H^{(\rho)}(\lambda) \qquad 0 \leqq {u \atop v} \leqq 1$$

$$\rho_0 = |\rho| - vp + u \quad vp - u \leqq |\rho| \leqq vp - u + m\ .$$

We want to know of $A_{u,v}^p$, what the forms of lowest homogeneous degree in the $\lambda$'s are. We have to give $|\rho|$ the maximal possible value.

We use the shorthands

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[1] *Added in proof*: We settled this question in the meantime, see [**6**].

$$\binom{m}{\rho} = \prod_{i=1}^{3} \binom{m}{\rho_i}$$

and $D(u, v)$ = degree of the lowest homogeneous term in $A^p_{u,v}$. In the list below is $\rho_0 = \max|\rho| - vp + u$.

| $(u, v)$ | $\max\|\rho\|$ | $\rho_0$ | $D(u, v)$ |
|---|---|---|---|
| (0, 0) | $m$ | $m$ | $p-1$ |
| (0, 1) | $3m$ | $m-1$ | 0 |
| (1, 0) | $m-1$ | $m$ | $p$ |
| (1, 1) | $3m$ | $m$ | 0 |

We get therefore:

$$A^p_{0,0}A^p_{1,1} = \text{terms of degree } p-1 + \text{higher terms}$$
$$A^p_{0,1}A^p_{1,0} = \text{terms of degree } p + \text{higher terms}\,.$$

The lowest degree term $L$ in $\det(A_{u,v})^p$ is given by

$$L = m\sum \binom{m}{\rho} H^{(\rho)}(\lambda)$$
$$\rho_1 + \rho_2 + \rho_3 = m$$
$$0 \leqq \rho_1 \leqq \rho_2 \leqq \rho_3\,.$$

Notice, if $\rho \neq \bar{\rho}$, then $H^{(\rho)}$ and $H^{(\bar{\rho})}$ have no monomial in common. Therefore $L$ is not the zero polynomial. We are able to specialize the variables $(\lambda_1, \lambda_2, \lambda_3)$ in the algebraic closure of $k$, such that $\det(A_{u,v}) \neq 0$. In other words, there exist curves of genus two with invertible Cartier-operator.

We do not know, what the smallest finite field is, over which such a curve exists.

REMARK. For large $p$ we could push through a similar discussion for higher genus. We omit that, because there is a more elegant method for large $p$ by Lubin (unpublished). Let $y^2 = x^{2g+1} + ax^{g+1} + x$. The claim is, that for large $p$ (depending on $g$) and variable $a$ the Hasse-Witt-matrix of that curve is a permutation matrix.

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# A THEOREM ON BOUNDED ANALYTIC FUNCTIONS

MICHAEL C. MOONEY

**The purpose of this paper is to prove the following**

THEOREM: **Let $\phi_1, \phi_2, \cdots$ be an infinite sequence of functions in $L^1([0, 2\pi])$ such that $L(f) = \lim_{n\to\infty} \int_0^{2\pi} f(e^{i\theta})\phi_n(\theta)d\theta$ exists for every $f \in H^\infty$. Then there is a $\phi \in L^1([0, 2\pi])$ such that $L(f) = \int_0^{2\pi} f(e^{i\theta})\phi(\theta)d\theta$ for all $f \in H^\infty$.**

Throughout this paper we will use the following notation and conventions: $D$ will denote the unit disc and $T$ its boundary. In order to save time we will avoid making distinctions between $T$ and $[0, 2\pi]$ if no confusion results. Similarly, it will be convenient to treat elements of $H^\infty[= H^\infty(D)$, the bounded analytic functions on $D]$ as though they were the same as those functions on $T$ with which they are naturally identified.

If $w \in D$, the symbol $g_w$ will stand for the function $z \rightarrow g(wz)$. $C(T)$ will stand for the usual space of continuous functions on $T$. $A$ will denote the subspace of $C(T)$ of functions analytically extendable to $D$. $\lambda$ will denote ordinary Lebesgue measure divided by $2\pi$ and "WLOG" means "without loss of generality".

In their paper [4] Piranian, Shields, and Wells observed that the theorem stated above would imply their result, namely that if $a_0, a_1, \cdots$ was a sequence of complex constants such that $\lim_{r\to 1} \sum_{n=0}^{\infty} a_n b_n r^n$ exists for all $f \in H^\infty$ [with Taylor coefficients $b_0, b_1, \cdots$], then the $a_n$'s are the the nonnegative Fourier coefficients of an $L^1([0, 2\pi])$ function. They also mentioned that our result here was a question raised in [1].

Kahane [3], using a somewhat different method than that in [4] showed that under the hypothesis of our main theorem, there was a $\phi \in L^1([0, 2\pi])$ such that the conclusion held for all $f \in A$. He went further to show that the subset of $H^\infty$ for which the conclusion held was large in some sense. Our proof here makes use of Kahane's result.

## 2. Remarks and lemmas.

First, given the hypothesis of the main theorem we may assume WLOG that the $\phi_n$'s are uniformly bounded in $L^1$ norm. To see why this is so we observe that for each $n$, $g \longrightarrow L_n(g) = \int_T g\phi_n$ is a bounded linear functional on $A$. By the uniform boundedness principle, the norms of the $L_n$'s as elements of $A^*$ are uniformly bounded, say by $M$. By the Hahn-Banach Theorem, each $L_n$ may be extended to an element of $C(T)^*$ with norm less than

$M$. This extended functional corresponds in the usual way to a Borel measure $\mu_n$ on $T$ having variation norm less than $M$. For each $n$, $\mu_n - \int\phi_n$ is also a finite Borel measure on $T$. Since this measure is orthogonal to $A$, it must be absolutely continuous [by the classical F. and M. Riesz Theorem] and, in turn, so must $\mu_n$. Hence we may replace $\phi_n$'s with $d\mu_n$'s if necessary. From here on we assume $||\phi_n||_1 \leqq 1$, for all $n$.

Suppose now for purposes of contradiction that there is an $f \in H^\infty$ such that $L(f) \neq \int_T f\phi$ where $\phi$ is the function referred to in Kahane's result. We may assume WLOG that $\phi \equiv 0$ [simply subtract $\phi$ from $\phi_n$'s beforehand and that $|f|_\infty = 1$. We also assert WLOG:

LEMMA 1. *There exists a bounded, increasing function $\beta$ on $T$ such that*

$$\lim_{n\to\infty} \int_E |\phi_n| = \int_E d\beta$$

*whenever $E$ is a finite union of closed subintervals of $T$.*

*Proof.* Since all our previous assertions remain valid if the $\phi_n$'s are replaced by an infinite subsequence, we will do this if necessary so that the functions $\int|\phi_n|$'s converge pointwise on $T$ to a function which we call $\beta$. This construction and the conclusion of the lemma follow from the Helly's Theorem. [See Zygmund [5] IV-4.6-(p. 137).]

We consider the fact that:

$$\lim_{r\to 1-}\lim_{n\to\infty} \int_T f_r\phi_n = 0 \neq \lim_{n\to\infty}\lim_{r\to 1-} \int_T f_r\phi_n = L(f)$$

despite the fact that $f_r$'s are uniformly bounded and converge to $f$ in measure. It is reasonable to subspect that in some useful sense of the word that the support of $\int f\phi_n$ tends to become concentrated on smaller and smaller sets as $n \to \infty$.

To be more specific, our plan at this point is to produce a sequence of pairwise disjoint "nice" closed sets $E_1, E_2, \cdots$ such that $\int_{E_n} f\phi_n$ tends approximately to $L(f)$ while $\int_{T-E_n} |f\phi_n|$ remains uniformly $< \varepsilon \ll |L(f)|$. [We will find that it is expedient to replace $f$ with $f - f_r$ for some $r$ in order to do this.]

Ultimately we will construct $g \in H^\infty$ so that $g$ is approximately $(-1)^n$ on $E_n$. The function $gf$ [actually we will look at $g \times (f - f_r)$] will give us a counterexample to the condition that $L(h)$ exists for all $h \in H^\infty$, and hence we will have a contradiction to the assumption

$L(f) \neq 0$.

Let $\varepsilon_0 = (1/10)|L(f)|$. In order to prove Lemma 2, it will be desirable to keep the singular part of $\beta$ small, say less than $\varepsilon_0/2$. To be sure of this we can choose a closed subset $E$ of the support of the singular part of $\beta$ such that outside of $E$, the singular part of $\beta$ has variation norm less than $\varepsilon_0/2$.

Let $g$ denote a Rudin-Carleson type function such that $g \in A$, $g$ is zero on $E$, and $g$ is close to 1 outside some neighborhood of $E$. Such functions were used in both [3] and [4], and a proof of their existence is available in Hoffman [2] p. 80, 81. [See also [2], Notes on p. 95.] If the original $\phi_n$'s are replaced by $g\phi_n$'s, we may proceed as before with our new set of $\phi_n$'s, $\phi$, $\beta$, etc. The new $d\beta = |g|$ times the old $d\beta$, and hence the singular part of the new $\beta$ will have variation norm less than $\varepsilon_0/2$. This process gives us a new value for $L(f)$, however, and we must be sure that the new value is close enough to the old that our assertion is still valid when the new value of $L(f)$ is used in the expression for $\varepsilon_0$. To do this we observe that the functions $f\phi_n$ also satisfy the hypothesis of our Theorem [in place of the $\phi_n$'s] and that by Kahane's Theorem, there is a $\psi \in L^1([0, 2\pi])$ such that

$$\lim_{n\to\infty} \int_T hf\phi_n = \int_T h\psi \quad \text{for all} \quad h \in A \ .$$

In particular this is true when $h = g$. Since $\psi$ is absolutely continuous and since we can make $g$ uniformly as close to 1 as we like outside neighborhoods of $E$ taken as small as we like, the new $L(f) = \int_T g\psi$ can be taken as close to the old $L(f) = \int_T \psi$ as we like. Hence WLOG we may assume that the singular part of $\beta$ has variation norm less than $\varepsilon_0/2$. Let us now choose $\delta > 0$ such that

$$\lambda(E) < \delta \Rightarrow \int_E d\beta_a < \varepsilon_0/2 - \int_T d\beta_S$$

where $\beta_a$ and $\beta_s$ are the absolutely continuous and singular parts of $\beta$ respectively. We note that if $J$ is a finite union of closed intervals, and $\lambda(J) < \delta$, then for $n$ sufficiently large $\int_J |\phi_n| < \varepsilon_0/2$.

Choose $r \in (0, 1)$ such that $\lambda(F) < \delta$ where

$$F = \{\theta \mid \theta \in [0, 2\pi], |f(e^{i\theta})| - f_r(e^{i\theta})| \geqq \varepsilon_0\} \ .$$

Let $G$ be an open subset of $T$ such that $F \subset G$ and $\lambda(G) < \delta$.

Since

$$L(f_r) = 0, L(f) = L(f - f_r) = \lim_{n\to\infty} \int_G (f - f_r)\phi_n + \lim_{n\to\infty} \int_{T-G} (f - f_r)\phi_n \ .$$

[We may choose subsequences of the original $\phi_n$'s if necessary in order to guarantee the limits exist.] Now for each $n$, $\int_{T-G} |(f - f_r)\phi_n| \leqq \varepsilon_0$. Hence $\left|\int_G (f - f_r)\phi_n - L(f)\right| < \varepsilon_0$ for all sufficiently large $n$.

LEMMA 2. *There exists a sequence of sets $E_1, E_2, \cdots$; a sequence of positive numbers $\delta_1, \delta_2, \cdots$; and an increasing sequence of positive integers $j_1, j_2, \cdots$ such that:*

(a) *Each $E_n$ is a finite union of closed intervals.*

(b) *Let $E'_j$ denote the closure of the $\delta_j$ neighborhood of $E_j$. Then $E'_j \subset G$.*

(c) $j \neq k \Rightarrow E'_j \cap E'_k = \varnothing$. *[Note that this $\Rightarrow \lambda(E_j) \to 0$, and $\lambda(E'_j) \to 0$.]*

(d) $\int_{G-E_k} |\phi_{j_k}| < \varepsilon_0/2$ *for* $k = 1, 2, \cdots$.

(e) $\int_{E_n} (f - f_r)\phi_{j_n} \to x_0$ *where* $|x_0 - L(f)| < 2\varepsilon_0$.

*Proof.* Construction using mathematical induction and the following scheme: After the first $k$, $E_j$'s, $\delta_j$'s and $j_n$'s are constructed, we pick $j_{k+1}$, $E_{k+1}$, and $\delta_{k+1}$ in the order.

Using the fact that $\lim_{n\to\infty} \int_{\cup_{p=1}^k E'_p} |\phi_n| < (1/2)\varepsilon_0$ [since $\lambda(\bigcup_{p=1}^k E'_p) < \lambda(G) < \delta$] and the fact that $\int_G (f - f_r)\phi_n$ eventually comes within $\varepsilon_0$ of $L(f)$, we have that for $j_{k+1}$ sufficiently large: $\int_{\cup_{p=1}^k E'_p} |\phi_{j_{k+1}}| < (1/2)\varepsilon_0$ and $\int_{G-\cup_{p=1}^k E'_p} (f - f_r)\phi_{j_{k+1}}$, is within $2\varepsilon_0$ of $L(f)$.

We now choose $E_{k+1}$ inside the open set $G - \bigcup_{p=1}^k E'_p$. Using the absolute continuity of $\phi_{j_{k+1}}$ we can choose $E_{k+1}$ large enough that (d) holds, and that $\int_{E_{k+1}} (f - f_r)\phi_{j_{k+1}}$ is within $2\varepsilon_0$ of $L(f)$.

$\delta_{k+1}$ will now be chosen so that (b) and (c) satisfied. Obviously our construction will satisfy (a), (b), (c), (d). We may choosen an appropriate subsequence if necessary in order that (e) be satisfied as well.

## 3. Construction of the counterexample function.

LEMMA 3. *Let $E$ be a closed subset of $T$, $\varepsilon > 0$. Then there is a function, $s$, analytic on $D$ such that:*

(a) *$s$ has positive real part and* $|s|_\infty < 1$

(b) $\theta \in E \Rightarrow |s(e^{i\theta}) - 1| < \varepsilon$

(c) $\theta \notin E \Rightarrow |s(e^{i\theta})| < 2\lambda(E)/\varepsilon \cdot \operatorname{dist}(\theta, E)$

(d) $|s(0)| < \lambda(E)/\varepsilon$.

*Proof.* Let $U = (1/\varepsilon)\pi_E$ on $T$ [$\pi_E$ denotes characteristic function for $E$]. Let $u$ be the harmonic function on $D$ corresponding to $U$ on the boundary [$u$ is the integral of $U$ with respect to Poisson's kernel]. Let $v$ be the conjugate harmonic function for $u$ such that $v(0) = 0$. Let $g = u + iv$. [$g$ is analytic on $D$ with positive real part.]

Note that for $\theta \notin E$, $|g(e^{i\theta})| = |v(e^{i\theta})|$ where

$$v(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\phi})\frac{\sin(\theta-\phi)}{1-\cos(\theta-\phi)}d\phi = \frac{1}{2\pi\varepsilon}\int_E \frac{\sin(\theta-\phi)}{1-\cos(\theta-\phi)}d\phi\ .$$

The maximum modulus of the function inside the integral occurs when $|\theta - \phi| = \operatorname{dist}(\theta, E)$. In order not to be troubled by awkward trigonometric expressions in the material to follow, we observe by some elementary calculations that $|\sin x|/(1 - \cos x) < 2/|x|$ for $|x| < \pi$. Hence we may assert that $|v(e^{i\theta})| < 2\lambda(E)/\varepsilon\cdot\operatorname{dist}(\theta, E)$. Now let

$$s = g/(1+g) = 1 - 1/(1+g)\ .$$

(a) Since $g$ is of positive real part, the range of $1/(1+g)$ is contained in the disc $\{z \mid |z - 1/2| < 1/2\}$. So is the range of $s$.

(b) For $\theta \in E$, $\operatorname{Re}(g(e^{i\theta})) = 1/\varepsilon$ and hence $\operatorname{Re}(1 + g(e^{i\theta})) = 1 + 1/\varepsilon$. This makes $|1 + g(e^{i\theta})| \geqq 1 + 1/\varepsilon$ and in turn $|1/(1 + g(e^{i\theta}))| \leqq \varepsilon/(1+\varepsilon) < \varepsilon$ whence $|s(e^{i\theta}) - 1| = |1/(1 + g(e^{i\theta}))| < \varepsilon$.

(c) For $\theta \notin E$, $|s(e^{i\theta})| = |g(e^{i\theta})|/|1 + g(e^{i\theta})|$ where

$$|g(e^{i\theta})| < 2\lambda(E)/\varepsilon\cdot\operatorname{dist}(\theta, E) \quad\text{and}\quad |1 + g(e^{i\theta})| \geqq 1$$

(d) $s(0) = g(0)/(1 + g(0))$, where $g(0) = \lambda(E)/\varepsilon$ and the proof is complete.

*Construction.* Given $\varepsilon_1 > 0$, $\varepsilon_2 > 0$; a sequence of functions $s_1, s_2, \cdots$ is to be constructed as follows:

Suppose $s_1, s_2, \cdots, s_k$ have been chosen and that $S_k = \sum_{j=1}^{k} s_j$ is such that $|S_k|_\infty = M_k < \infty$, $s_{k+1}$ will be of the form $c_{k+1}s$ where $c_{k+1}$ is a positive real number and $s$ is related to $E_{n_{k+1}}$ in the same manner that $s$ is related to $E$ in Lemma 3.

We want $c_{k+1}$ sufficiently large and $\varepsilon$ [in Lemma 3] sufficiently small that:

(a) $\theta \in E_{n_{k+1}} \Rightarrow \varepsilon_2 \log|S_{k+1}(e^{i\theta})| = (-1)^{k+1}(\pi/2)(\operatorname{mod} 2\pi) - \pi/2$ within an error of magnitude not more than $\varepsilon_1$. Note that we can pick $\varepsilon$ dependent only on $\varepsilon_1$ and $\varepsilon_2$ [independent of $k+1$], and $c_{k+1} \gg M_k$ so as to make the ratio between $|s_{k+1} + S_k|$ and $|\operatorname{Re}(s_{k+1})|$ small enough to make $\log|S_{k+1}|$ close enough to $\log(c_{k+1})$ on $E_{n_{k+1}}$ for this purpose. Furthermore, the choice of $c_{k+1}$ depends only on $E_{n_1}, E_{n_2}, \cdots, E_{n_k}$. We wish further to have:

(b) $\theta \in E_{n_k} = \varepsilon_2 \log|S_p(e^{i\theta})| = (-1)^k\pi/2(\operatorname{mod} 2\pi) - \pi/2$ within an error of magnitude not more than $\varepsilon_1$ for all $p > k$. To do this, we use the fact that for $\theta \in E_{n_k}$, $p > k$; then dist $(\theta, E_{n_p}) > \delta_{n_k}$ [independent of $p$-note]. Hence $|s_p(e^{i\theta})| < c_p\lambda(E_{n_p})/\varepsilon \cdot \text{dist}\,(\theta, E_{n_p}) < c_p\lambda(E_{n_p})/\varepsilon\delta_{n_k}$. Recall that the choice of $c_p$ depends only on $S_{p-1}$ and is independent of $E_{n_p}$. Hence we may require that $\lambda(E_{n_p}) \to 0$ sufficiently rapidly to guarantee that $\sum_{p=k+1} c_p\lambda(E_{n_p})/\varepsilon\delta_{n_k}$ is always small enough that (b) is satisfied. The above requirement also guarantees that $\sum_{k=1}^{\infty} c_k\lambda(E_{n_k})/\varepsilon$ converges.

Each $s_p$ has positive real part and hence by Harnack's principal the $S_p$'s must either converge to an analytic function, $S$, of positive real part on $D$, or diverge to $\infty$ on $D$. The latter is impossible since each $|S_p(0)| < \sum_{k=1}^{p} |s_k(0)| \leqq \sum_{k=1}^{p} c_k\lambda(E_{n_k})/\varepsilon < \sum_{k=1}^{\infty} c_k\lambda(E_{n_k})/\varepsilon < \infty$. We also note that our requirement in (b) above also guarantees that the $S_p$'s converge absolutely on each $E_{n_k}$ and hence we also have: $\theta \in E_{n_k} \Rightarrow \varepsilon_2 \log |S| = (-1)^k\pi/2 (\text{mod}\, 2\pi) - \pi/2$ within an error of magnitude not more than $\varepsilon_1$.

Let $g = e^{i\varepsilon_2 \log S}$. Then $g$ is bounded on $D$ [in fact: $e^{-\varepsilon_2\pi/2} < |g(z)| < e^{\varepsilon_2\pi/2}$ for all $z \in D$]. $\theta \in E_{n_p} \Rightarrow$ argument $(g(e^{i\theta})) = ((-1)^p\pi/2)(\text{mod}\, 2\pi) - \pi/2 +$ error not larger than $\varepsilon_1$. This is, given $\varepsilon_3 > 0$ we may choose $\varepsilon_1, \varepsilon_2$ so that $1 - \varepsilon_3 < |g(z)| < 1 + \varepsilon_3$ for all $z \in D$ and such that $|g(e^{i\theta}) - (-1)^p| < \varepsilon_3$ for all $\theta \in E_{n_p}$. Now:

$$\int_T g(f - f_r)\phi_{j_{n_k}} = \int_{E_{n_k}} g(f - f_r)\phi_{j_{n_k}} + \int_{G-E_{n_k}} g(f - f_r)\phi_{j_{n_k}} + \int_{T-G} g(f - f_r)\phi_{j_{n_k}}\,.$$

Recalling Lemma 2, we see that the first of these three integrals is within $2\varepsilon_0(1 + \varepsilon_3)$ of $(-1)^p L(f)$; the second has magnitude less than $\varepsilon_0(1 + \varepsilon_3)$ [by (d), Lemma 2] and the third also has magnitude less than $\varepsilon_0(1 + \varepsilon_3)$ [from the way in which $f_r$ and $G$ were chosen]. If $\varepsilon_3$ is chosen small enough, $\int_T g(f - f_r)\phi_{j_{n_k}}$ fails to have a limit as $k \to \infty$ and we have our contradiction.

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LOUISIANA STATE UNIVERSITY

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# DIFFERENTIAL EQUATIONS ON ABSTRACT WIENER SPACE

M. ANN PIECH

**The main purpose of this paper is to indicate a simple method by means of which the work of L. Gross concerning the Laplacian on an abstract Wiener space may be extended to a certain class of pure second order elliptic operators with constant coefficients. A short proof of uniqueness of the solution semigroup of the heat equation will also be given.**

Our extension method is motivated by the often-used technique of performing a change of variables in order to reduce a pure second order elliptic operator on $R^n$ with constant coefficients to the Laplacian. However, some fundamental dissimilarities between finite dimensional and infinite dimensional potential theory must be taken into account. First let us define an infinite dimensional Laplacian. Let $H$ denote a real separable Hilbert space and $D^2f(x)$ denote the second Fréchet derivative of a real-valued function $f$ on $H$. We may regard $D^2f(x)$ as a bounded linear operator on $H$. We define $\Delta f(x) \equiv \text{trace } D^2f(x)$ whenever $D^2f(x)$ exists and is of trace class. This obviously extends the finite dimensional Laplacian. However, unlike the finite dimensional case, the existence of $D^2f(x)$ is not sufficient to ensure the existence of $\Delta f(x)$. Another dissimilarity is a consequence of the unavailability of a substitute for $n$-dimensional Lebesgue measure which would be countably additive on the Borel field of $H$. Use of Gauss cylinder set measure can provide an integration theory on $H$, but this is not adequate for potential theory, and more particularly for regularity studies. The reason for this inadequacy is that a Brownian motion defined in $H$ in terms of Gauss cylinder set measure would have the property that the probability of a particle starting at the origin and instantly leaving the ball of radius $r > 0$ would be one.

To avoid this inadequacy, the concept of an abstract Wiener space $(H, B, i)$ was introduced by Gross [1]. $B$ denotes the completion of $H$ with respect to a fixed measurable norm $\|\cdot\|$, and $i$ is the natural injection of $H$ into $B$. Gauss cylinder set measure on $H$ determines a cylinder set measure on $B$, which in turn extends to a countably additive Borel measure on $B$ (Wiener measure). The measure on $B$ determined by Gauss measure on $H$ with variance parameter $t > 0$ is denoted by $p_t$. For a Borel set $\Gamma \subset B$ and $x \in B$, let $p_t(x, \Gamma) \equiv p_t(\Gamma - x)$. The measures $p_t(x, \cdot\,)$ give the transition probabilities for a Wiener process with continuous sample paths initiating at the origin of $B$.

Problems in potential theory are stated in terms of a fixed $(H, B, i)$. If $u(t, x)$ is a real valued function on $[0, \infty) \times B$ and if

$$(t_0, x_0) \in [0, \infty) \times B$$

is fixed, then we may consider $h(x) \equiv u(t_0, x_0 + x)$ as a function from $H$ into $R$. The second $H$-derivative of $u$ at $(t_0, x_0)$ is defined as

$$D^2 u(t_0, x_0) \equiv D^2 h(0) .$$

The initial value problem for the heat equation can now be stated as

$$\frac{\partial}{\partial t} u(t, x) = \text{trace } D^2 u(t, x) \qquad t > 0$$

$$u(0, x) = f(x)$$

where $x$ varies over $B$. We note that we are only concerned with differentiation in directions of $H$, even though the space variable ranges over $B$. In an analogous fashion, open sets in $B$ are appropriate for a statement of the Dirichlet problem.

Let $A$ be a fixed member of $L(H)$ (the space of bounded linear operators on $H$) satisfying

(a- i ) $A$ is symmetric,

(a- ii) $A \geqq \varepsilon I$ for some $\varepsilon > 0$,

(a-iii) $A = I + C$ where $C$ is of Hilbert-Schmidt class.

We claim that within the context of a given abstract Wiener space $(H, B, i)$ most of the results of Ref. [2] hold when the Laplacian is replaced by the differential operator trace $AD^2 f(x)$.

Properties (a-i) and (a-ii) guarantee that $\sqrt{A}$ exists as a positive symmetric invertible member of $L(H)$. When $H$ is finite dimensional it is customary to transform trace $AD^2 f(x)$ into the Laplacian of $f$ by making the change of variables $x \to \sqrt{A}^{-1} x$. Now $H = B$ when $H$ is finite dimensional; otherwise $H \subsetneqq B$. Since $x$ is to vary over $B$, this application of a change of variables is meaningless for infinite dimensional $H$. It turns out that, rather than transforming the differential operator, we can meaningfully transform the fundamental solution of the heat equation.

Let $H_A$ be the Hilbert space obtained by replacing the inner product $( , )$ on $H$ by $[h, k] \equiv (\sqrt{A}^{-1} h, \sqrt{A}^{-1} k)$. The invertibility of $\sqrt{A}^{-1}$ ensures that $[ , ]$ and $( , )$ give rise to equivalent norms. Thus $\| \cdot \|$ is a measurable norm on $H_A$, and we may also view $B$ as the completion of $H_A$ with respect to $\| \cdot \|$. If $i_A$ denotes the natural injection of $H_A$ into $B$, then $(H_A, B, i_A)$ is an abstract Wiener space. We will let $p_t^A$ denote that measure on $B$ determined by Gauss cylinder set measure on $H_A$ with variance parameter $t > 0$. $p_t^A$ will be called Wiener measure on $(H_A, B, i_A)$.

Wiener measure $p_t$ on $(H, B, i)$ gives rise to a fundamental solution of the heat equation

$$(1) \qquad \frac{\partial}{\partial t} u(t, x) = \text{trace } D^2 u(t, x)$$

($x$ ranges over $B$, and $t$ over $(0, \infty)$). Specifically, the family

$$\{p_{2t}(x, dy) \colon x \in B, t > 0\}$$

has the following properties [2, Theorem 3 and Porposition 6]:

For each bounded real-valued uniformly Lip 1 function $f$ on $B$, letting

$$p_t f(x) \equiv \int_B f(y) p_t(x, dy) ,$$

(b–i) $p_{2t}f(x)$ satisfies the heat equation (1)—that is, $\sigma/(\sigma t)p_{2t}f(x)$ and $D^2 p_{2t}f(x)$ exist, $D^2 p_{2t}f(x)$ is of trace class and the equality (1) holds;

(b–ii) $p_{2t}f(x) \to f(x)$ as $t \downarrow 0$, uniformly for all $x$ in $B$.

As a consequence of (b–i) and (b–ii), we say that

$$\{p_{2t}(x, dy);\ x \in B, t > 0\}$$

forms a fundamental solution of the heat equation.

By analogy with the finite dimensional situation, we expect the measures $\{q_t(x, dy) \colon x \in B,\ t > 0\}$ defined by

$$(2) \qquad q_t(x, dy) \equiv [\det A]^{-1/2} e^{-([A^{-1}-I](x-y),x-y)/4t}\, p_{2t}(x, dy)$$

to form a fundamental solution of

$$(3) \qquad \frac{\partial}{\partial t} u(t, x) = \text{trace } AD^2 u(t, x) .$$

That is, we expect that for each $f$ in the family $\mathscr{A}$ of bounded real-valued uniformly Lip 1 functions on $B$, the function

$$q_t f(x) \equiv \int_B f(y) q_t(x, dy)$$

satisfies (3) and $q_t f \to f$ in sup norm as $t \downarrow 0$.

REMARK. We must explain the meaning of the exponential term which occurs in the expression for $q_t(x, dy)$. It is to be interpreted as the limit in mean $(p_{2t}(x, dy))$ as $F \to I$ of the net of tame functions $\{\exp[-([A^{-1} - I](x - P_F y), x - P_F y)/4t]: F$ is a finite dimensional subspace of $H$ which is left invariant by $C$ and $P_F$ is projection onto $F\}$. The integral of a tame function with respect to $p_t$ is described in Ref. [1]. We will see later that this net does converge.

A direct verification that $\{q_t(x, dy)\}$ has the properties of a fundamental solution would be both difficult and lengthy. However, Theorems 2 and 3 of Ref. [5] assert that $p_t^A$ is mutually absolutely continuous with respect to $p_t$, with Radon-Nikodym derivative given by

$$p_t^A(dy) = [\det A]^{-1/2} e^{-([A^{-1}-I]y, y)/2t} p_t(dy) ,$$

provided $\sqrt{A} - I$ is of Hilbert-Schmidt class. The latter property is verified by writing $\sqrt{A} = I + C[I + \sqrt{A}]^{-1}$. Setting $p_t^A(x, \Gamma) \equiv p_t^A(\Gamma - x)$ for Borel sets $\Gamma$ in $B$, we see that $q_t(x, dy) = p_{2t}^A(x, dy)$. We may now appeal to the work of Gross [2] to establish many properties of $\{q_t(x, dy)\}$. Before doing so, however, we recall some properties of trace class operators.

We will identify individual elements of $H$ and of $H_A$ via the identity map on the topological vector space $H$. Similarly we will identity individual elements of $L(H)$ and $L(H_A)$. We recall that the family of trace class operators in $L(H)$ is

$$\left\{T \in L(H): \sum_{i=1}^{\infty}([T^*T]^{1/2}e_i, e_i) < \infty \text{ for some orthonormal basis } \{e_i\} \text{ of } H\right\} ,$$

with the trace of $T$ defined as $\operatorname{Tr} T \equiv \sum_{i=1}^{\infty} (Te_i, e_i)$ where $\{e_i\}$ is any orthonormal basis of $H$. The trace class norm of $T \in L(H)$ is

$$| T |_{\operatorname{Tr} L(H)} \equiv \operatorname{Tr} [T^*T]^{1/2} .$$

The completely continuous operators in $L(H)$ with $| \cdot |_{L(H)}$ form the dual of the space of trace class operators in $L(H)$ under the pairing $\langle U, V\rangle \equiv \operatorname{Tr} U^*V$, where $U$ is completely continuous and $V$ is of trace class. Since operators of finite rank are dense in the space of completely continuous operators, we may write $| T |_{\operatorname{Tr} L(H)} = \sup\{| \operatorname{Tr}[TF] |/| F |_{L(H)}: F$ is of finite rank in $L(H)$ and $F \not\equiv 0\}$. For any $S$ in $L(H)$ and $T$ of trace class, $ST$ and $TS$ are of trace class and $\operatorname{Tr} ST = \operatorname{Tr} TS$. Thus the set of trace class operators on $H$ is invariant under a change of inner product. Consequently the set of trace class operators and their traces are the same whether we consider $L(H)$ or $L(H_A)$. The trace class norm does vary with the change of inner product, although $| \cdot |_{\operatorname{Tr} H}$ and $| \cdot |_{\operatorname{Tr} H_A}$ are equivalent norms.

We point out that, by definition, $D^2 q_t f(x)$ is a member of $L(H, H^*)$. The identification of $D^2 q_t f(x)$ with an element of $L(H)$ is dependent on the inner product assigned to $H$. Unless otherwise specified, we will always intend this identification to be via $(\ ,\ )$. If we let $T(t, x)$ denote the operator in $L(H)$ determined by considering $D^2q_tf(x)$ as a member of $L(H_A, H_A^*)$, identifying $H_A^*$ with $H_A$ via $[\ ,\ ]$ and $L(H_A)$ with $L(H)$, then $T(t, x) = AD^2q_tf(x)$. Since

$$\{p_{2t}^A(x, dy) \colon x \in B,\ t > 0\}$$

is a fundamental solution of the heat equation in $(H_A, B, i_A)$, we immediately have the

PROPOSITION 1. *Assume $A$ satisfies* (a–i), (a–ii) *and* (a–iii). *Then* $\{q_t(x, dy) \colon x \in B,\ t > 0\}$ *forms a fundamental solution of the equation*

$$\frac{\partial}{\partial t} u(t, x) = \operatorname{trace} AD^2u(t, x) .$$

REMARK. The existence of fundamental solutions of Eq. (3) in situations where $A$ is nonconstant has been considered by the author in [3]. There $A - I$ was assumed to be of trace class, and this property was relied upon considerably. Proposition 1 allows generalization of the results of Ref. [3] to situations where $A$ is of the form $I + C_1 + C_2$ where $A \geqq \varepsilon I$ for some $\varepsilon > 0$, $I + C_1$ satisfies (a–i) – (a–iii) and $I + C_2$ satisfies the hypotheses made in Ref. [3]. Generally speaking, then, such an $A$ is of the form identity plus a constant Hilbert-Schmidt class operator plus a variable trace class operator. We conjecture that the results of Ref. [3] may be extended to operators of the form identity plus a variable Hilbert-Schmidt class operator.

Now let us assume that $f \in \mathscr{S}$ and that $f$ has bounded support. We may apply the preceding technique to obtain a solution of

$$\operatorname{trace} AD^2u(x) = f(x) . \tag{4}$$

We define the Green's measures $G$ and $G_A$ on Borel sets $\Gamma$ of $B$ by

$$G(\Gamma) \equiv \int_0^\infty p_t(\Gamma)dt$$

and

$$G_A(\Gamma) \equiv \int_0^\infty q_t(\Gamma)dt ,$$

and the potentials $Gh$ and $G_Ah$ of a Borel function $h$ on $B$ by

$$Gh(x) \equiv \int_B h(x + y)\, G(dy)$$

and

$$G_A h(x) \equiv \int_B h(x+y)\, G_A(dy) \; .$$

Then by Ref. [2, Theorem 3], $Gf(x)$ satisfies

$$\frac{1}{2} \operatorname{trace} [(D^2 Gf)(x)] = -f(x)$$

for all $x$ in $B$. We thus immediately have the

PROPOSITION 2. *Assume $A$ satisfies* (a–i), (a–ii) *and* (a–iii). *For $f$ in $\mathscr{A}$ and of bounded support,*

$$u(x) \equiv -G_A f(x)$$

*satisfies Eq.* (4).

REMARK. Many smoothness properties and corresponding estimates concerning $p_t f(x)$ and $Gf(x)$ are given in Ref. [2]. Analogues of these may now trivially be deduced for $q_t f(x)$ and $G_A f(x)$.

From Ref. [2] we see that for $t > 0$ the operators $q_t$: $f \to q_t f$ form a strongly continuous contraction semigroup on the space $\mathscr{C}$ of bounded uniformly continuous functions $f$ on $B$ with $\|f\|_\infty$. Let $\mathscr{L}$ denote the infinitesimal generator of this semigroup. Then [2, Cor. 3.1] for $f$ in $\mathscr{A}$, $q_t f$ is in the domain $\mathscr{D}_{\mathscr{L}}$ of $\mathscr{L}$ and

$$(\mathscr{L} q_t f)(x) = \operatorname{trace} [(AD^2 q_t f)(x)] \equiv Lf(x) \; .$$

A question naturally arises concerning possible uniqueness of the semigroup $\{q_t\colon t > 0\}$ among semigroups on $\mathscr{C}$ whose infinitesimal generators are "related" to $L$. This question for variable coefficients $A(x)$ will be discussed by the author in a forthcoming paper [4]. The method used there could be applied to the case presently under consideration. However Ref. [4] makes use of a theory of stochastic integrals on $(H, B, i)$, which requires a special hypothesis on the abstract Wiener space $(H, B, i)$. Moreover, the approach of [4] is unduly cumbersome in the constant coefficient case. Therefore we will now present a brief uniqueness result for the constant coefficient case. We begin by showing that $\mathscr{L}$ is the closure of $L$. Specifically, we have the

PROPOSITION 3. *Let the set $\mathscr{S}$ consist of real-valued functions $f$ satisfying*

(c–i) *$f$ is in $\mathscr{A}$;*

(c–ii) *$Df$: $B \to H$ exists, is bounded and continuous;*

(c–iii) *$D^2 f$: $B \to$ trace class operators on $H$ with $|\cdot|_{\mathrm{Tr}\, H}$ exists, is bounded and uniformly continuous.*

*Then* $\mathscr{S} \subset \mathscr{D}_{\mathscr{L}}$, *for* $f$ *in* $\mathscr{S}$ $\mathscr{L}f = Lf$, *and* $\{(f, \mathscr{L}f): f \in \mathscr{S}\}$ *is dense in the closed subset* $\{(f, \mathscr{L}f): f \in \mathscr{D}_{\mathscr{L}}\}$ *of* $\mathscr{C} \times \mathscr{C}$.

*Proof.* Assume $f$ is in $\mathscr{S}$. Since

$$(q_t f)(x + sh) = \int_B f(x + sh + y) q_t(dy) ,$$

(c–ii) enables differentiation under the integral sign, yielding

$$(Dq_t f(x), h) = \int_B (Df(x + y), h) q_t(dy)$$

for all $h$ in $H$. Similarly (c–iii) enables us to write

$$((D^2 q_t f)(x)k, h) = \int_B (D^2 f(x + y)k, h) q_t(dy)$$

for all $k$ and $h$ in $H$, and

$$Lq_t f(x) = \int_B Lf(x + y) q_t(dy) .$$

Since $q_t f \in \mathscr{D}_{\mathscr{L}}$, we have

$$\mathscr{L} q_t f = q_t L f .$$

$Lf$ is in $\mathscr{C}$ by (c–iii). Thus $q_t Lf \to Lf$ uniformly as $t \downarrow 0$, and so $\mathscr{L} q_t f \to Lf$ uniformly as $t \downarrow 0$. But $q_t f \to f$ uniformly as $t \downarrow 0$ and, since $\mathscr{L}$ is a closed operator by basic semigroup theory, we conclude that $f$ is in $\mathscr{D}_{\mathscr{L}}$ and $\mathscr{L} f = Lf$.

It is shown in Ref. [2, Cor. 3.2] that functions of the form

$$g(x) = \int_0^\infty e^{-t} (p_t f)(x) dt$$

where $f \in \mathscr{A}$ are dense in the domain of $\mathscr{L}$ in the graph ($\mathscr{C} \times \mathscr{C}$) norm. It is furthermore shown that such functions $g$ satisfy (c–iii). It is trivial to see that $q_t$: $\mathscr{A} \to \mathscr{A}$, and hence that $g \in \mathscr{A}$. To verify (c–ii) we make use of Eq. (8) of Ref. [2]—viz. for $h$ in $H$

$$(Dq_t f(x), h) = (2t)^{-1} \int_B f(x + y) \, [h, y] \, q_t(dy) .$$

Thus we obtain

$$\begin{aligned}(Dg(x), h) &= \int_0^\infty e^{-t} (2t)^{-1} \int_B f(x + y) \, [h, y] \, q_t(dy) dt \\ &= \int_0^\infty e^{-t} (2t)^{-1/2} \int_B f(x + (2t)^{1/2} y) \, [h, y] \, p_1^A(dy) dt .\end{aligned}$$

Therefore

$$| (Dg(x), h) | < \int_0^\infty e^{-t} t^{-1/2} \| f \|_\infty \left\{ \int_B | [h, y] |^2 p_1^A(dy) \right\}^{1/2} dt$$
$$\leqq \| f \|_\infty | h |_{H_A} \int_0^\infty e^{-t} t^{-1/2} dt ,$$

and so

$$| Dg(x) |_H \leqq \text{constant} \cdot \| f \|_\infty .$$

In addition, we see that

$$| (Dg(x) - Dg(z), h) | \leqq \text{constant} \cdot \int_0^\infty e^{-t} t^{-1/2} \| x - z \| \, | h |_H dt ,$$

and we conclude that $g$ satisfies (c-ii).

Thus we have proved that $(\mathscr{L}, \mathscr{D}_{\mathscr{L}})$ is the closure of $(L, \mathscr{S})$.

REMARK. The preceding calculations of $Dq_t f$ and $D^2 q_t f$ were possible because $q_t f$ is a convolution of $f$ with $p_t(dy)$. This is not the case with variable coefficients.

We now give a uniqueness result for the semigroup $\{q_t\}$.

PROPOSITION 4. *If* $\{q_t' \colon t > 0\}$ *is a contraction semigroup on* $\mathscr{C}$ *whose infinitesimal generator* $\mathscr{L}'$ *extends* $(L, \mathscr{S})$, *then* $q_t' = q_t$ *for all* $t > 0$.

*Proof.* If we show that $\mathscr{D}_{\mathscr{L}'} = \mathscr{D}_{\mathscr{L}}$ and that $\mathscr{L}' = \mathscr{L}$ on their common domain, then since $\{q_t\}$ is strongly continuous on $\mathscr{C}$ it follows from basic semigroup theory that $q_t' = q_t$. Since $(\mathscr{L}', \mathscr{D}_{\mathscr{L}'})$ is a closed operator, we have $(\mathscr{L}', \mathscr{D}_{\mathscr{L}'}) \supset (\mathscr{L}, \mathscr{D}_{\mathscr{L}})$. Let

$$f \in \mathscr{D}_{\mathscr{L}'}, \ g \equiv (I - \mathscr{L}')f$$

and

$$\mathscr{C}_0 \equiv \{h \text{ in } \mathscr{C} \colon \| q_t' h - h \|_\infty \to 0 \text{ as } t \downarrow 0\} .$$

$\mathscr{D}_{\mathscr{L}'} \subset \mathscr{C}_0$ and $\mathscr{L}' \colon \mathscr{C}_0 \to \mathscr{C}_0$. Thus $g \in \mathscr{C}_0$. It is well known that for $g$ in $\mathscr{C}_0$ the equation $(I - \mathscr{L}')h = g$ has a unique solution in $\mathscr{D}_{\mathscr{L}'}$. By the strong continuity of $\{q_t\}$, there exists a unique solution $\tilde{f}$ in $\mathscr{D}_{\mathscr{L}}$ of the equation $(I - \mathscr{L})h = g$. Since $\tilde{f}$ is also in $\mathscr{D}_{\mathscr{L}'}$, $\mathscr{L}'\tilde{f} = \mathscr{L}\tilde{f}$ and so $\tilde{f} = f$. Thus $(\mathscr{L}', \mathscr{D}_{\mathscr{L}'}) \subset (\mathscr{L}, \mathscr{D}_{\mathscr{L}})$.

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STATE UNIVERSITY OF NEW YORK AT BUFFALO

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# SESQUILINEAR FORMS IN INFINITE DIMENSIONS

ROBERT PIZIAK

**This paper is concerned with sesquilinear forms defined on vector spaces of arbitrary dimension. Motivation is taken from classical Hilbert space theory, as the orthogonality relation induced by the form is used to replace the topology. First, an algebraic version of the Frechet-Riesz Representation Theorem is proved for linear functionals having an orthogonally closed kernel. Next, the notion of adjoint is formulated, following von Neumann, in the language of linear relations. It is proved that the adjoint of an arbitrary relation is a single valued linear relation precisely when the domain of that relation is orthogonally dense. Finally, an algebraic version of a continuous linear operator is introduced and the relationship with the notion of adjoint and linear functional is studied. The main result here is that an operator is orthogonally continuous precisely when it has an everywhere defined adjoint. These general results of pure algebra imply standard topological results in the context of a Hilbert space.**

There are two directions in which to generalize away from the concept of a Hilbert space. One is the familiar topological generalization via Banach spaces, linear topological spaces. The other direction is algebraic via inner product spaces, sesquilinear forms. The finite dimensional theory of sesquilinear forms is well worked out. However, the infinite dimensional case seems fraught with pathology. Kaplansky and others have initiated a study of the infinite dimensional case [6], [7], [8]. Gross and Fischer [4] have used topological methods. In this paper, we propose an algebraic approach to infinite dimensions motivated by the "happy accidents" in Hilbert space theory that correlate algebraic and topological conditions. In particular, we prove an algebraic version of the Frechet-Riesz Representation Theorem, von Neumann's theorem on the single valuedness of the adjoint relation, and discuss continuity, all in the algebraic context of a vector space over a division ring with no "natural" topology present.

2. Quadratic spaces. We shall follow the terminology of Bourbaki [2] on sesquilinear forms.

By a *quadratic space* we mean a triple $(k, E, \Phi)$ where $E$ is a left vector space over the division ring $k$ and $\Phi$ is a nondegenerate orthosymmetric $\theta$-sesquilinear form on $E$ with respect to the involutive anti-automorphism $\theta$ of $k$. Given vectors $x$ and $y$ in $E$, we say $x$ is *orthogonal* to $y$ and write $x \perp y$ when $\Phi(x, y) = 0$. For any subset $M$ of $E$, we define the *orthogonal of* $M$ by

$$M^{\perp} = \{x \text{ in } E \mid x \perp m \text{ for all } m \text{ in } M\} .$$

It is clear that $M^{\perp}$ is always a subspace of $E$. A vector $x$ of $E$ is called *isotropic* if $x \perp x$ and is *anisotropic* otherwise.

The two main differences between general quadratic spaces and Hilbert space is first in the general nature of the scalars and second, in the possible existence of nonzero isotropic vectors. The role of isotropic vectors is important in physical theories and indeed a good example to hold in mind is the geometry of space-time with the Minkowski metric. Here, of course, $k = \boldsymbol{R}$, $E = \boldsymbol{R}^4$, and

$$\Phi((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 - \alpha_4\beta_4 .$$

The first "happy accident" to note is that in Hilbert space, a subspace $M$ is metrically closed precisely when $M = M^{\perp\perp}$. Thus we are led to consider the closure operator $M \mapsto M^{\perp\perp}$ on the lattice of all subspaces of $E$, Lat $(k, E)$, as an algebraic substitute for the topology. Let $P_c(E, \Phi) = \{M \text{ in Lat } (k, E) \mid M = M^{\perp\perp}\}$. The geometry of $P_c(E, \Phi)$, which is of interest in the study of the logical foundations of quantum mechanics, has been considered in [9].

In a Hilbert space $H$ we have that each closed space $M$ yields an orthogonal direct sum decomposition $H = M \oplus M^{\perp}$. This is not true for a general quadratic space. A subspace $F$ of $E$ is said to be a *splitting subspace* provided $E = F + F^{\perp}$. Let $P_s(E, \Phi)$ be the collection of all splitting subspaces of $E$. It is easy to see that each splitting subspace is $\perp$-closed. We shall show later that the converse need not hold.

It is well known that the lattice of closed subspaces of a Hilbert space is an orthomodular lattice. We have shown elsewhere [10] that orthomodularity actually residues in $P_s(E, \Phi)$ in general and $P_s(E, \Phi)$ is an orthomodular poset which need not be a lattice. Thus the orthomodularity of the lattice of closed subspaces of Hilbert space arises from the "happy accident" that $P_s(H) = P_c(H)$.

**3. Linear functionals.** The next "happy accident" we note is that a linear functional on a Hilbert space is continuous exactly when it has a closed kernel. This motivates our next definition.

Let $(E, \Phi)$ be a quadratic space. Let $f$ be a linear functional on $E$. Call $f$ *orthocontinuous* if $\ker(f) = \ker(f)^{\perp\perp}$. Let $E'$ denote the set of all orthocontinuous linear functionals on $E$ and call it the *orthodual of* $E$. Let $E^*$ denote the algebraic dual space of $E$.

3.1. *Frechet-Riesz Representation Theorem.* Let $(E, \Phi)$ be a quadratic space. Then the induced map $d\colon E \to E^*$ defined by

$$d(y)(x) = \Phi(x, y)$$

is a $\theta$-linear monomorphism and $im(d) = E'$. Moreover, the image under $d$ of all anisotropic vectors consists precisely of all those linear functionals whose kernel is a splitting subspace of $E$.

*Proof.* For $y$ in $E$ we have $\ker d(y) = (ky)^{\perp}$ which is a closed subspace of $E$. Thus $\operatorname{im}(d) \subseteqq E'$. Next let $f$ be in $E'$. If $f$ is the zero functional then $f = d(0)$ and $f$ is in $\operatorname{im}(d)$. So assume $f$ is not identically zero. Then $\ker(f)$ is a hyperplane in $E$. Thus there is line $kw$ with $E = \ker(f) \oplus kw$. Now pick a nonzero vector $z$ in $\ker(f)^{\perp}$. Then

$$(0) = E^{\perp} = (\ker(f) \oplus kw)^{\perp} = \ker(f)^{\perp} \cap (kw)^{\perp}$$

so that $\Phi(w, z) \neq 0$. Let $y = ((\Phi(w, z))^{-1} f(w))^{\theta^{-1}} z$. Note $y$ is in $kz$ which is contained in $\ker(f)^{\perp}$. Thus $\Phi(w, y) = f(w)$.

Now let $x$ be any vector in $E$. Then there is a unique $x_1$ in $\ker(f)$ and $x_2$ in $kw$ such that $x = x_1 + x_2$. Then $f(x) = f(x_2)$ and $\Phi(x, y) = \Phi(x_2, y)$. But $x_2 = \lambda w$ so $f(x) = f(x_2) = \lambda f(w) = \lambda\Phi(w, y) = \Phi(\lambda w, y) = \Phi(x_2, y) = \Phi(x, y)$. Thus $f = d(y)$ and hence $\operatorname{im}(d) = E'$.

The fact that $d$ is a monomorphism follows from the nondegeneracy of $\Phi$.

If $y$ is anisotropic, then $y$ does not belong to $ky^{\perp}$ so $\ker d(y) = (ky)^{\perp}$ and $(ky)^{\perp} \oplus ky = E$. On the other hand if $\ker(f) \oplus \ker(f)^{\perp} = E$, then $\ker(f)$ is closed so there is a $y$ with $f = d(y)$. Since $(ky)^{\perp} \oplus ky = E$, $y$ is clearly anisotropic.

Note that the theorem above implies the usual Frechet-Riesz Representation theorem for real, complex, and quaternionic Hilbert spaces.

The corollaries below follow readily.

COROLLARY 3.2. *If $\Phi$ admits nonzero isotropic vectors, then there are closed subspaces of $E$ that are not splitting.*

COROLLARY 3.3. *The orthodual of $E$ is a total subspace of $E^*$.*

COROLLARY 3.4. *Let $M$ be a closed subspace of $E$ with $x$ a vector not in $M$. Then there is an orthocontinuous linear functional $f$ such that $f(x) \neq 0$, but $M \subseteqq \ker(f)$.*

**4. Adjoint.** Let $(E, \Phi)$ be a quadratic space. We shall imitate the von Neumann formulation of the notion of adjoint. Let $T$ be a relation on $E$ with graph $G(T)$. We say $T$ is a *closed relation* if $G(T) = G(T)^{\perp\perp}$ where $\perp$ is taken relative to $\Phi \oplus \Phi$ $E \oplus E$. Note a closed relation is necessarily a linear relation i.e. $T$ or $G(T)$ if you prefer, is a subspace of $E \oplus E$. The closure $\bar{T}$ of the relation $T$ is defined by $G(\bar{T}) = G(T)^{\perp\perp}$. Clearly $\bar{T}$ extends $T$. We also note that if $T$ is a closed linear relation, then $\ker(T)$ is a closed linear subspace of $E$.

Now define $U: E \times E \to E \times E$ by $U(x, y) = (-y, x)$. Then $U$ is an everywhere defined linear bijection with $U^{-1}(y, x) = (x, -y)$. Also note that $\Phi \oplus \Phi(Uz, w) = \Phi \oplus \Phi(z, U^{-1}w)$ and for $M \subseteqq E \times E$, we have $U(M^{\perp}) = U(M)^{\perp}$. For $T$ any relation on $E$, define $T^*$ a relation on $E$ by $G(T^*) = U(G(T))^{\perp}$. Call $T^*$ the *adjoint* of $T$. Note then that *every* linear operator has an adjoint. The question is whether or not the adjoint is single valued.

The usual definition of adjoint is given by demanding the existence of a linear operator $T^*$ for a given linear operator $T$, such that the identity $\Phi(Tx, y) = \Phi(x, T^*y)$ holds for all $x$ and $y$. It is interesting to note this formal identity persists. For if $T$ is a relation on $E$ with $(x, z)$ in $G(T)$ and $(y, w)$ in $G(T^*)$, then

$$\Phi(z, y) = \Phi(x, w) .$$

If we formally write $z = Tx$ and $w = T^*y$, we recover the previous equation.

It was brought to our attention that the next theorem was previously obtained by R. Arens [1] p. 16, Prop. 3.32. The Hilbert space origin of the idea goes back to J. von Neumann [12].

THEOREM 4.1. *Let $T$ be a relation on $E$. Then $T^*$ is single valued if and only if* $(\mathrm{dom}(T))^{\perp\perp} = E$.

In view of [1], we omit the proof.

It is interesting to note that the single valuedness of $T^*$ depends only on the nature of the domain of $T$ and not whether $T$ is single valued or even linear.

COROLLARY 4.2. (1) *Let $T$ be a linear relation on $E$. Then $T^*$ is single valued if and only if $T$ has an orthogonally dense domain;*

(2) *$T^*$ has dense domain if and only if $T^{**}$ is single valued;*

(3) *The closure of a linear operator is single valued exactly when its adjoint has a dense domain.*

Following S. S. Holland Jr., (to whom we are indebted for several ideas of this section), we shall use the term *CDD operator* to mean a closed domain dense linear operator.

COROLLARY 4.3. *The adjoint* $T^*$ *of a CDD operator* $T$ *is CDD and* $T = T^{**}$.

THEOREM 4.4. *Let* $T$ *be a CDD operator. Then* $T^*$ *satisfies* $\Phi(Tx, y) = \Phi(x, T^*y)$ *for all* $x$ *in* $\mathrm{dom}(T)$ *and* $y$ *in* $\mathrm{dom}(T^*)$. *Also any linear operator* $S$ *satisfying* $\Phi(Tx, y) = \Phi(x, Sy)$ *for all* $x$ *in* $\mathrm{dom}(T)$ *and* $y$ *in* $\mathrm{dom}(S)$ *is such that* $S \subseteqq T^*$. *If* $\mathrm{dom}(S) = \mathrm{dom}(T^*)$, *then* $S = T^*$.

*Proof.* Since $T$ is domain dense, $T^*$ is single valued and

$$\Phi(Tx, y) = \Phi(x, T^*y)$$

for all $x$ in $\mathrm{dom}(T)$ and $y$ in $\mathrm{dom}(T^*)$. If $\Phi(Tx, y) = \Phi(x, Sy)$ for all $x$ in $\mathrm{dom}(T)$ then $\Phi \oplus \Phi((y, Sy), (-Tx, x)) = -\Phi(y, Tx) + \Phi(Sy, x) = 0$ for all $x$ in $\mathrm{dom}(T)$ so that $(y, Sy)$ is in $U(G(T))^{\perp} = G(T^*)$. Thus $y$ is in $\mathrm{dom}(T^*)$ and $T^*y = Sy$. Thus $S \subseteqq T^*$.

In Hilbert space, a bounded linear has a topologically closed graph and conversely. We can prove that if $T$ is a domain dense linear operator on $E$ and $T^*$ is domain dense then $T$ has a $\perp$-closed graph. It would be more interesting to prove the following open question: *Algebraic Closed Graph Theorem* if $T$ is an everywhere defined closed linear operator then $T$ has an everywere defined adjoint. We conjecture this is not true in general but is true in the case that every closed subspace of our quadratic space is splitting.

5. **Orthocontinuity.** In Hilbert space, the continuous linear operators are of great interest. We shall show how to approach these algebraically.

Let $(E, \Phi)$ be a quadratic space with $T\colon E \to E$ linear. We say $T$ is *orthocontinuous* if for all subspaces $M$ of $E$ we have

$$T(M^{\perp\perp}) \subseteqq T(M)^{\perp\perp}.$$

PROPOSITION 5.1. *Let* $T\colon E \to E$ *be linear. Then the following statements are equivalent*

(1) $M = M^{\perp\perp}$ *implies* $T^{-1}(M) = (T^{-1}(M))^{\perp\perp}$
(2) $M$ *closed implies* $T^{-1}(M)$ *closed*
(3) $T(M^{\perp\perp}) \subseteqq T(M)^{\perp\perp}$
(4) $T^{-1}(N^{\perp\perp}) \supseteqq (T^{-1}(N))^{\perp\perp}$
(5) $T$ *is orthocontinuous*

The proof is easy and is omitted.

LEMMA 5.2. *Let* $T: E \rightarrow E$ *be an everywhere defined linear operator. Suppose* $\mathrm{dom}(T^*) = E$. *Then for any* $M = M^{\perp\perp}$ *we have* $T^{-1}(M) = (T^*(M^{\perp}))^{\perp}$. *In particular then* $\ker(T) = \mathrm{im}(T^*)^{\perp}$.

*Proof*. Let $M = M^{\perp\perp}$. Then $x$ is in $T^{-1}(M)$ if and only if $Tx$ is in $M = M^{\perp\perp}$ if and only if $\Phi(Tx, y) = 0$ for all $y$ in $M^{\perp}$ if and only if $\Phi(x, T^*y) = 0$ for all $y$ in $M^{\perp}$ if and only if $x$ is orthogonal to $T^*(M^{\perp})$.

Next we make a connection between the domain of the adjoint and the orthodual.

THEOREM 5.3. *If* $T$ *is an everywhere defined linear operator on* $E$, *then* $\mathrm{dom}(T^*)$ *comprises exactly those* $y$ *in* $E$ *for which the linear functional* $f_y(x) = \Phi(Tx, y)$ *is orthocontinuous.*

*Proof*. Since $T$ is domain dense, $T^*$ is single valued and

$$\Phi(Tx, y) = \Phi(x, T^*y)$$

for all $x$ in $E$ and all $y$ in $\mathrm{dom}(T^*)$. First let $y$ be in $\mathrm{dom}(T^*)$. Then $x$ is in $\ker(f_y)$ if and only if $f_y(x) = 0$ if and only if $\Phi(Tx, y) = 0$ if and only if $\Phi(x, T^*y) = 0$ if and only if $x$ is in $(kT^*y)^{\perp}$. Thus

$$\ker(f_y) = (kT^*y)^{\perp}$$

is closed.

Conversely, let $y$ be a vector such that $f_y$ is an orthocontinuous linear functional. Then by Frechet-Riesz, there is a unique vector $y^*$ such that $f_y(x) = \Phi(x, y^*)$ for all $x$ in $E$. That is, $\Phi(Tx, y) = \Phi(x, y^*)$ for all $x$ in $E$. Thus

$$\Phi \oplus \Phi((y, y^*), (-Tx, x)) = -\Phi(y, Tx) + \Phi(y^*, x) = 0$$

for all $x$ in $E$ so that $(y, y^*)$ is in $U(G(T))^{\perp} = G(T^*)$. This means $y$ is in $\mathrm{dom}(T^*)$.

We are now in a position to relate orthocontinuity to the adjoint. We first state a lemma whose proof will be omitted.

LEMMA 5.4. *Let* $T$ *be an everywhere defined linear operator on* $E$. *Define the linear functional* $f_y$ *by* $f_y(x) = \Phi(Tx, y)$ *for all* $x$ *in* $E$. *Then* $\ker(f_y) = T^{-1}((ky)^{\perp})$.

THEOREM 5.5. *Let* $T$ *be an everywhere defined linear operator on* $E$. *Then* $T$ *is orthocontinuous if and only if* $T^*$ *is everywhere* *defined.*

*Proof.* Let $T$ be orthocontinuous. Then $T^{-1}((ky)^{\perp})$ is closed for all $y$ in $E$. Thus by (5.3) and (5.4), $y$ is in $\operatorname{dom}(T^*)$.

Conversely if $\operatorname{dom}(T^*) = E$, then for

$$M = M^{\perp\perp},\ T^{-1}(M) = (T^*(M^{\perp}))^{\perp}$$

by (5.2) and this is closed so $T$ is orthocontinuous.

COROLLARY 5.6. *$T$ is orthocontinuous if and only if $T^{-1}((ky)^{\perp})$ is closed for all $y$ in $E$.*

We close by remarking that the algebra of bounded operators on Hilbert space is a well studied object. The algebraic analogue for a quadratic space is the *adjoint algebra*, $Ad(E, \Phi)$, of all linear operators on $E$ that have everywhere defined adjoints.

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UNIVERSITY OF FLORIDA

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# THE EQUATION $y'(t) = F(t, y(g(t)))$

MURIL L. ROBERTSON

**A functional differential equation, in general, is a relationship in which the rate of change of the state of the system at time $t$ depends on the state of the system at values of time, perhaps other than the present.**

**In this paper, sufficient conditions are given for $g$ so that the initial value problem $y'(t) = F(t, y(g(t)))$, $y(p) = q$, may be solved uniquely; where $F$ is both continuous into the Banach space $B$, and is Lipschitzean in the second position.**

1. DEFINITIONS. If $p$ is a real number and $I = \{I_1, I_2, \cdots\}$ is a collection of intervals so that $p \in I_1$ and $I_n \subseteqq I_{n+1}$ for each positive integer $n$, then $I$ is said to be a nest of intervals about $p$. Let $I_0 = \{p\}$ and $a_0 = b_0 = p$. Also, let $[a_n, b_n] = I_n$ for each nonnegative integer $n$. Let $I^*$ denote the union of all elements of $I$.

In general $B$ denotes a Banach space; and if $D$ is a real number set, let $C[D, B]$ denote the set of continuous functions from $D$ into $B$. Whenever $D$ is an interval, $C[D, B]$ is taken to be a Banach space with supremum norm $|\cdot|$.

If $g$ is a continuous function from $I^*$ into $I^*$ so that $g(I_n) \subseteqq I_n$ for each positive integer $n$, then $g$ is said to be an $I$-function. If $g$ is an $I$-function then for each positive integer $n$, define the following:

$$\begin{aligned} A_n &= \{x \in [a_n, a_{n-1}]: g(x) \notin I_{n-1}\} , \\ B_n &= \{x \in [b_{n-1}, b_n]: g(x) \notin I_{n-1}\}, \text{ and} \\ E_n(s) &= [p, g(s)] \cap (A_n \cup B_n), \text{ for each } s \in I_n . \end{aligned}$$

Let $\int_D h(s)ds$ denote the Lebesgue integral of $h$ over the subset $D$ of the domain of the Lebesgue integrable function $h$.

Let $F$ denote a continuous function from $I^* \times B$ into $B$ so that $||F(x, y) - F(x, z)|| \leqq M(x)\cdot||y - z||$ for all $x \in I^*$ and $y, z \in B$, where $M$ is Lebesgue integrable on each $I_n$. Furthermore, if $f$ is a continuous nonnegative valued function from $I^*$ to the reals, and $m$ is a positive integer, let $\int_p^x(M, f, g, m)$ denote

$$\left|\int_p^x M(s_1)\left|\int_p^{(g_1)} M(s_2)\right| \cdots \left|\int_p^{g(s_{m-1})} M(s_m)f(s_m)ds_m\right| \cdots |ds_2|ds_1\right| .$$

If $D$ is either $A_n$ or $B_n$, let $\int(M, f, D, m)$ denote

$$\int_D M(s_1)\int_{E_n(s_1)} M(s_2) \cdots \int_{E_n(s_{m-1})} M(s_m)f(s_m)ds_m \cdots ds_2ds_1 .$$

If $D$ is a subset of the domain of the function $h$, let $h|_D$ denote the restriction of $h$ to $D$. Also, let $f \circ g$ denote the composition of $f$ with $g$, whenever applicable; $f \circ g(x) = f(g(x))$.

## 2. Main results.

THEOREM A. *Suppose $I$ is a nest of intervals about $p, q \in B$, $g$ is an $I$-function, $k$ is a sequence of positive integers, and for each positive integer $n$, $\alpha_n = \int(M, 1, A_n, k(n)) < 1$ and $\beta_n = \int(M, 1, B_n, k(n)) < 1$. Then there is a unique function $y \in C[I^*, B]$ so that $y'(t) = F(t, y(g(t)))$ and $y(p) = q$, for all $t \in I^*$. [We say then that the initial value problem (IVP) has unique solution.]*

*Proof.* Since, $I_0 = \{p\}$, then certainly $y_0 = \{(p, q)\}$ is the unique function in $C[I_0, B]$ so that for all $t \in I_0$, $y_0(t) = q + \int_p^t F(s, y_0(g(s)))ds$.

Next, suppose $n$ is a nonnegative integer so that there is a unique function $y_n \in C[I_n, B]$ so that, for each $t \in I_n$, $y_n(t) = q + \int_p^t F(s, y_n(g(s)))ds$. The following is the construction of $y_{n+1}$. Let $D = \{f \in C[I_{n+1}, B]: f|_{I_n} = y_n\}$ and let $m = k(n+1)$. Then, if $f \in D$ and $t \in I_{n+1}$, let $T$ be so that $Tf(t) = q + \int_p^t F(s, f(g(s)))ds$. Then, certainly $T$ is from $D$ into $D$.

LEMMA 1. *If $f, h \in D$ and $t \in I_{n+1}$, then*

$\|T^m f(t) - T^m h(t)\| \leqq \int_p^t (M, \|f \circ g - h \circ g\|, g, m)$, *for each positive integer $m$.*

*Proof of Lemma* 1. (by induction on $m$) If $m = 1$,

$$\begin{aligned}
\|Tf(t) - Th(t)\| &= \left\|\int_p^t [F(s, f(g(s))) - F(s, h(g(s)))]ds\right\| \\
&\leqq \left|\int_p^t \|F(s, f(g(s))) - F(s, h(g(s)))\|ds\right| \\
&\leqq \left|\int_p^t M(s)\cdot\|f(g(s)) - h(g(s))\|ds\right| = \int_p^t (M, \|f \circ g - h \circ g\|, g, 1) .
\end{aligned}$$

Now, suppose the lemma holds for $m = r$. Then,

$$\begin{aligned}
&\|T^{r+1}f(t) - T^{r+1}h(t)\| \\
&\quad = \left\|\int_p^t [F(s, T^r f(g(s))) - F(s, T^r h(g(s)))]ds\right\| \\
&\quad \leqq \left|\int_p^t \|F(s, T^r f(g(s))) - F(s, T^r h(g(s)))\|ds\right| \\
&\quad \leqq \left|\int_p^t M(s)\cdot\|T^r f(g(s)) - T^r h(g(s))\|ds\right| \\
&\quad \leqq \left|\int_p^t M(s_1)\cdot\int_p^{g(s_1)} (M, \|f \circ g - h \circ g\|, g, r)ds_1\right| ,
\end{aligned}$$

by the induction hypothesis, but this equals $\int_p^t (M, \|f\circ g - h\circ g\|, g, r+1)$.

LEMMA 2. *If $N$ is a bounded, measurable function from $I_{n+1}$ to the reals so that $N(s) = 0$ whenever $s$ is in $I_{n+1}\backslash(A_{n+1} \cup B_{n+1})$, then*

$$\int_p^{a_{n+1}} (M, N, g, m) = \int (M, N, A_{n+1}, m) ,$$

*and*

$$\int_p^{b_{n+1}} (M, N, g, m) = \int (M, N, B_{n+1}, m) .$$

*Proof of Lemma* 2. (by induction on $m$) If $m = 1$, $\int_p^{a_{n+1}} (M, N, g, 1) = \left|\int_p^{a_{n+1}} M(s)N(s)ds\right| = \int_{A_{n+1}} M(s)N(s)ds = \int (M, N, A_{n+1}, 1)$, because $N$ is 0 at each point of $[p, a_{n+1}]\backslash A_{n+1}$. Suppose the lemma is true for $m = r$. Then, $\int_p^{a_{n+1}} (M, N, g, r+1) = \int_p^{a_{n+1}} (M, U, g, r)$, where $U(s) = \left|\int_p^{g(s)} M(t)N(t)dt\right|$, for all $s \in I_{n+1}$. If $s \in I_{n+1}\backslash(A_{n+1} \cup B_{n+1})$, $g(s) \in I_n$. Thus, $N$ is 0 on $[p, g(s)]$, and so $U(s) = 0$. Whence, $U$ satisfies the conditions for $N$ in the lemma. So, by the induction hypothesis, $\int_p^{a_{n+1}} (M, U, g, r) = \int (M, U, A_{n+1}, r) = \int (M, N, A_{n+1}, r+1)$, because $U(s) = \int_{E_{n+1}(s)} M(t)N(t)dt$. The proof of the second equality in the lemma is similar. Thus, Lemma 2 is proven.

Now, the two lemmas are applied. By Lemma 1, $\|T^m f(t) - T^m h(t)\| \leqq \int_p^t (M, \|f\circ g - h\circ g\|, g, m)$, for all $t \in I_m$, $\leqq \max\left\{\int_p^{a_{n+1}} (M, \|f\circ g - h\circ g\|, g, m), \int_p^{b_{n+1}} (M, \|f\circ g - h\circ g\|, g, m)\right\}$ which by Lemma 2 is $= \max\left\{\int (M, \|f\circ g - h\circ g\|, A_{n+1}, m), \int (M, \|f\circ g - h\circ g\|, B_{n+1}, m)\right\}$, because $\|f(g(s)) - h(g(s))\| = 0$ for all $s \in I_{n+1}\backslash(A_{n+1} \cup B_{n+1})$. Thus, $|T^m f - T^m h| \leqq \max\left\{\int (M, \|f\circ g - h\circ g\|, A_{n+1}, m), \int (M, \|f\circ g - h\circ g\|, B_{n+1}, m)\right\} \leqq \max\left\{\int (M, 1, A_{n+1}, m), \int (M, 1, B_{n+1}, m)\right\}\cdot|f - h|$. Thus, $T^m$ is a contraction map from the complete metric space $D$ into $D$. Thus $T^m$ has a unique fixed point $y_{n+1}$. It is a known result that this implies that $y_{n+1}$ is the unique fixed point of $T$. $[(Ty_{n+1}) = T(T^m(Ty_{n+1}) = T^m(Ty_{n+1})$, but only $y_{n+1}$ is so that $y_{n+1} = T^m y_{n+1}$. So $Ty_{n+1} = y_{n+1}$, and uniqueness is clear.]

Thus, $y_{n+1}(t) = Ty_{n+1}(t) = q + \int_p^t F(s, y_{n+1}(g(s)))ds$, for all $t \in I_{n+1}$, and is the unique such function. Hence, by inductive definition, for each positive integer $i$, there is a unique function $y_i \in C[I_i, B]$ so that for all $t \in I_i$, $y_i(t) = q + \int_p^t F(s, y_i(g(s)))ds$. Now, define $y \in$

$C[I^*, B]$ so that $y(t) = y_n(t)$, whenever $t \in I_n$. Since $m \leqq n$ implies $y_n|_{I_m} = y_m$, $y$ is well-defined, and $y(t) = q + \int_p^t F(s, y(g(s)))ds$, for all $t \in I^*$. Now, suppose $z(t) = q + \int_p^t F(s, z(g(s)))ds$, for all $t \in I^*$, and $z \in C[I^*, B]$. Then, if $n$ is a positive integer, and $t \in I_n$, $z|_{I_n}(t) = q + \int_p^t F(s, z|_{I_n}(g(s)))ds$. So, $z|_{I_n} = y_n = y|_{I_n}$ for each positive integer $n$. Thus, $z = y$.

COROLLARY 1. *Let $M$ be the constant 1 function, and let $k(n) = 2$, for all $n$. Suppose for each $n$, $\int_{A_n} \min\{|g(x) - a_{n-1}|, |g(x) - b_{n-1}|\}dx < 1$, and $\int_{B_n} \min\{|g(x) - a_{n-1}|, |g(x) - b_{n-1}|\}dx < 1$. Then, the IVP has a unique solution.* [See Figure 1. All the shaded area between each pair of vertical dashed lines is less than one.]

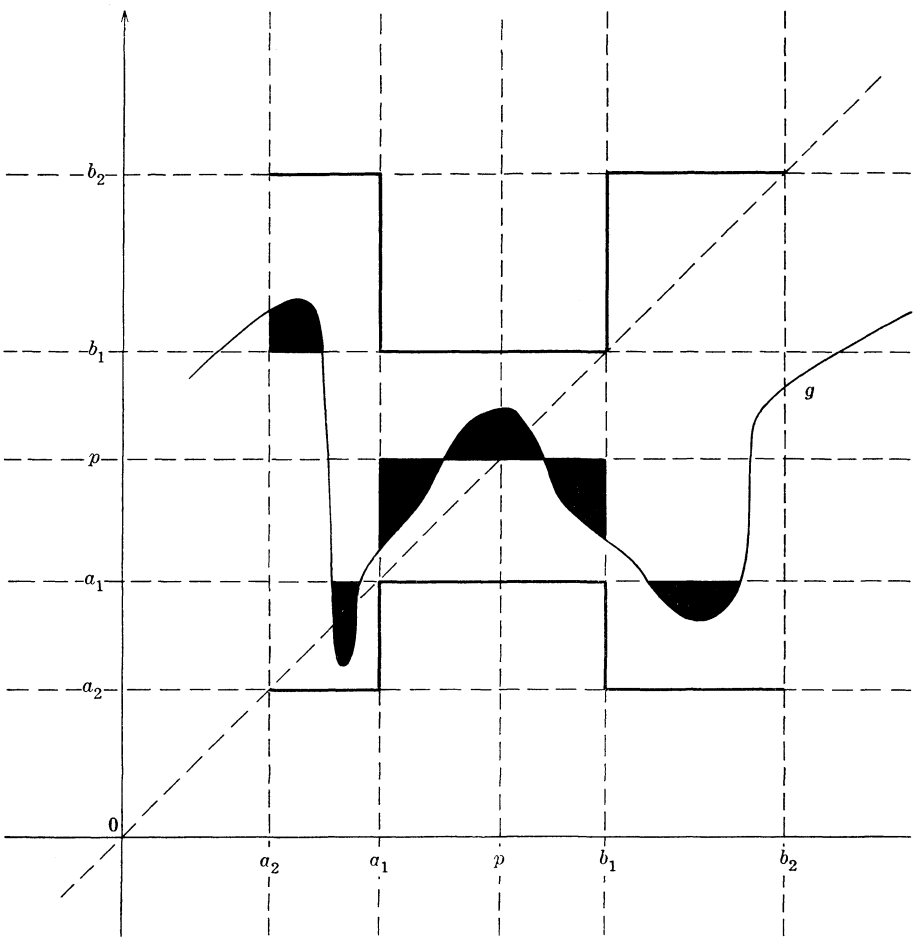


FIGURE 1

*Proof.* $\alpha_n = \int_{A_n} M(s_1) \int_{E_n(s_1)} M(s_2) ds_2 ds_1 = \int_{A_n} \int_{E_n(s_1)} ds_2 ds_1$. Now, $s_1 \in A_n$ implies

$$E_n(s_1) = \begin{cases} A_n \cap [p, g(s_1)], \text{ if } g(s_1) \in A_n, \text{ and} \\ B_n \cap [p, g(s_1)], \text{ if } g(s_1) \in B_n . \end{cases}$$

Thus, $E_n(s_1) \subseteqq [g(s_1), a_{n-1}]$ if $g(s_1) \in A_n$, and in this case, $|g(s_1) - a_{n-1}| \leqq |g(s_1) - b_{n-1}|$. Also, $E_n(s_1) \subseteqq [b_{n-1}, g(s_1)]$ if $g(s_1) \in B_n$, and in this case, $|g(s_1) - b_{n-1}| \leqq |g(s_1) - a_{n-1}|$. Thus, $E_n(s_1)$, which is certainly measurable, must have measure $\leqq \min\{|g(s_1) - a_{n-1}|, |g(s_1) - b_{n-1}|\}$. Hence, $\int_{A_n} \int_{E_n(s_1)} ds_2 ds_1 \leqq \int_{A_n} \min\{|g(s_1) - a_{n-1}|, |g(s_1) - b_{n-1}|\} ds_1$, because $\int_{E_n(s_1)} ds_2$ is the measure of $E_n(s_1)$. Thus, $\alpha_n < 1$, and similarly $\beta_n < 1$, for each positive integer $n$. Apply Theorem A.

COROLLARY 2. *Suppose $k(n) = 1$ for each $n$. Then, if $\int_{A_n} M < 1$ and $\int_{B_n} M < 1$, for each $n$, the IVP has unique solution.*

*Proof.* Immediate.

COROLLARY 3. *Suppose $M$ is the constant 1 function and $k(n) = 1$ for each $n$. Then if $\max\{b_n - b_{n-1}, a_{n-1} - a_n\} < 1$, for each $n$, the IVP has unique solution.*

*Proof.* $A_n \subseteqq [a_{n-1}, a_n]$ and $B_n \subseteqq [b_{n-1}, b_n]$ implies $\int_{A_n} 1 \leqq \int_{a_n}^{a_{n+1}} 1 = a_{n-1} - a_n$ and $\int_{B_n} 1 \leqq \int_{b_{n-1}}^{b_n} 1 = b_n - b_{n-1}$. Apply Corollary 2.

The following example illustrates the advantage of allowing $k(n)$ to assume integral values other than 1.

EXAMPLE. Let $F$ be so that $M = 1$ in the *IVP*—$y(p) = q$, $y'(t) = F(t, y(g(t)))$, where

$$g(x) = \begin{cases} 2x \qquad , \text{ if } x \in [0, p], \text{ and} \\ 4p - 2x, \text{ if } x \in [p, 2p] . \end{cases}$$

then it is straightforward to show that if $J$ is a subinterval of $[0, 2p]$ and $g(J) \subseteqq J$, then $J = [0, 2p]$. Thus, if $I$ is a nest of intervals about any point of $[0, 2p]$ and $I^* = [0, 2p]$, then $I_n = [0, 2p]$ for each positive integer $n$, if $g$ is to be an $I$-function. Thus, in order to apply Corollary 3, it seems necessary to require $p < 1$, in order to solve the *IVP*. However, if Theorem A is applied with $k(n) = m$ for all $n$, then Theorem B, which follows, shows that the condition $p < 2^{(m-1)/m}$ gives the best apparent bound for the size of $p$ in order to solve the *IVP*. Now, since $m$ is arbitrary, clearly, $p$ may be any positive number less than 2.

THEOREM B. *If $g$ is as in the above example, and for each positive integer $n$, $F_n(x) = \int_p^x (1, 1, g, n+1)$, then*

(1) *$F_n$ is symmetric about $p$. That is, for each $n$, $F_n(x) = F_n(2p - x)$, for all $x \in [0, p]$; and*

(2) *$F_n(x) + F_n(p - x) = p^{n+1}/2^n$, for each $n$, and for all $x \in [0, p/2]$.*

*Proof.* (induction on $n$) Suppose $n = 1$. Then, if $x \in [0, 2p]$, $F_1(x) = \left|\int_p^x |g(s) - p|ds\right|$, which is

$$F_1(x) = \begin{cases} p^2/2 - px + x^2, & \text{if } x \in [0, p/2]\ , \\ px - x^2, & \text{if } x \in [p/2, p]\ , \\ -2p^2 + 3px - x^2, & \text{if } x \in [p, 3p/2], \text{ and} \\ 5p^2/2 - 3px + x^2, & \text{if } x \in [3p/2, 2p]\ . \end{cases}$$

It is straightforward to show that $F_1$ satisfies the conditions (1) and (2) of the theorem. Now, suppose the theorem is true for the positive integer $k$. Then, for each $x \in [0, 2p]$, $F_{k+1}(x) = \left|\int_p^x F_k(g(s))ds\right|$. If $x \in [0, p]$, $F_{k+1}(2p - x) = \left|\int_p^{2p-x} F_k(g(s))ds\right|$. Thus, if $x \leqq s \leqq p$, $g(s) = 2s = 4p - 2(2p - s) = g(2p - s)$. So, $F_{k+1}(x) = \int_p^x F_k(g(s))ds = \int_{2p-x}^p F_k(g(2p - s))(-1)ds$, by change of variable, but this is $\int_p^{2p-x} F_k(g(2p - s))ds = \int_p^{2p-x} F_k(g(s))ds = F_{k+1}(2p - x)$. Thus, $F_{k+1}$ is symmetric about $p$.

Now, suppose $x \in [0, p/2]$. Then,

$$\begin{aligned} &F_{k+1}(x) + F_{k+1}(p - x) \\ &\quad = \int_x^p F_k(g(s))ds + \int_{p-x}^p F_k(g(s))ds \\ &\quad = \int_x^p F_x(2s)ds + \int_{p-x}^p F_k(2s)ds, \text{ because } g(s) = 2s \\ &\quad = \int_x^p F_k(2s)ds + \int_{p-x}^p F_k(2p - 2s)ds, \text{ because } g(z) = g(2p - z) \\ &\quad = \int_x^p F_k(2s)ds - \int_{2x}^0 (1/2)F_k(s)ds, \text{ by change of variable} \end{aligned}$$

$$
\begin{aligned}
&= \int_x^p F_k(2s)ds + (1/2)\int_0^{2x} F_k(s)ds \\
&= \int_x^p F_k(2s)ds + \int_0^x F_k(2s)ds, \text{ by change of variable} \\
&= \int_0^p F_k(2s)ds \\
&= (1/2)\int_0^{2p} F_k(s)ds, \text{ by change of variable} \\
&= \int_0^p F_k(s)ds, \text{ because } F_k \text{ is symmetric about } p \\
&= \int_0^{p/2} F_k(s)ds + \int_{p/2}^p F_k(s)ds \\
&= \int_0^{p/2} F_k(s)ds - \int_0^{p/2} F_k(p-s)\,(-1)ds, \text{ by change of variable} \\
&= \int_0^{p/2} \{F_k(s) + F_k(p-s)\}ds \\
&= \int_0^{p/2} \{p^{k+1}/2^k\}ds, \text{ by the induction hypothesis} \\
&= p^{k+2}/2^{k+1}.
\end{aligned}
$$

By Theorem B, $F_n(0) + F_n(p - 0) = p^{n+1}/2^n$. But, $F_n(p) = 0$, by definition of $F_n$, and thus $F_n(0) = p^{n+1}/2^n$. Also, $F_n(2p) = F_n(2p - 0) = F_n(0) = p^{n+1}/2^n$. Thus, if $p^{n+1}/2^n < 1$, then $\alpha_{n+1} \leqq F_n(0) = p^{n+1}/2^n < 1$, and $\beta_{n+1} \leqq F_n(2p) = p^{n+1}/2^n < 1$. Apply Theorem A.

3. **Applications.** The following is a generalization of a theorem by Anderson [1].

Let $F$ be a continuous real-valued function with domain $D$ of the plane $R \times R$ so that the partial derivative $F_2$ is continuous on $D$ and $(0, b) \in D$. Let $h'$ and $k$ be so that if $|x| \leqq h'$ and $|y - b| \leqq k$, then $(x, y) \in D$. Let $K = \sup\{|F(x, y)|: |x| \leqq h' \text{ and } |y - b| \leqq k\}$, $h = \min\{k', k/K\}$, and $M = \sup\{|F_2(x, y)|: |x| \leqq h \text{ and } |y - b| \leqq k\}$.

THEOREM C. *Suppose there are intervals* $I_1 \subseteqq I_2 \subseteqq \cdots \subseteqq I_m = [-h, h]$ *so that max* $\{b_n - b_{n-1}, a_{n-1} - a_n\} \cdot M < 1$ *for each integer in* $[1, m]$, *and so that* $0 \in I_1$. *Let* $I_j = I_m$ *for each* $j \geqq m$. *Then, if* $g$ *is an I-function, there is a unique function* $y$ *so that* $y(0) = b$ *and* $y'(t) = F(t, y(g(t)))$, *for all* $t \in [-h, h]$.

*Proof.* Let $E = \{(x, y): |x| \leqq h, |y - b| \leqq k\}$, and let $G$ be an extension of $F|_E$ so that

$$
G(x, y) = \begin{cases} F(x, b - k), & \text{if } y \leqq b - k, \text{ and} \\ F(x, b + k), & \text{if } y \geqq b + k\,. \end{cases}
$$

By continuity of $F_2$ and the mean value theorem, it follows that $F$ is Lipschitzean in the second position with constant $M$. It follows, also, that $G$ has the same Lipschitz constant $M$. Then, by Corollary 2, there is a unique function $y \in C[I^*, B] = C[[-h, h], R]$ so that $y'(t) = G(t, y(g(t)))$, $y(0) = b$, for all $t \in [-h, h]$. Equivalently, $y(t) = b + \int_0^t G(s, y(g(s)))ds$, for all $|t| \leqq h$. Thus, $|y(t) - b| = \left|\int_0^t G(s, y(g(s)))ds\right| \leqq h \cdot \sup\{|G(s, y(g(s)))|: |s| \leqq h\}$, and since the range of $G$ is a subset of the range of $F|_E$, we have that this is $\leqq h \cdot \sup\{|F(x, v)|: |x| \leqq h, |v - b| \leqq k\} = h \cdot K \leqq k$, by definition of $h$. Thus, $G(x, y(g(x))) = F(x, y(g(x)))$, for all $|x| \leqq h$. So, $y'(t) = F(t, y(g(t)))$, $y(0) = b$, for all $t \in [-h, h]$.

The following is a generalization of a theorem by Kuller [3].

THEOREM D. *Suppose only that $g$ is a continuous function with connected, real domain $E$ so that $g$ is not the identity, but $g o g$ is the identity. Then, if $M = 1$ and $q \in B$, there is a segment $Q$ about the unique fixed point $p'$ of $g$ so that if $p \in Q \cap E$, the IVP has unique solution.*

*Proof.* Kuller proves that $g$ has a unique fixed point $p'$ and that $g$ is strictly decreasing. Let $0 < k < 1/2$. Let $\beta_0 = p$ and let $\beta$ be a nondecreasing sequence of reals so that $\beta_i - \beta_{i-1} < k$, for each positive integer $i$, and so that $\beta$ converges to the right boundary of $E$, which may be $+\infty$. Then, for each positive integer $i$, let $\{\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in_i}\}$ be so that $g(\beta_i) = \alpha_{in_i} \geqq \cdots \geqq \alpha_{i2} \geqq \alpha_{i1} = g(\beta_{i-1})$ and also so that $\alpha_{ij} - \alpha_{i,j+1} < k$, for all $j$. Then, $\{[\alpha_{ij}, g(\alpha_{ij})]: i \geqq 1$ and $1 \leqq j \leqq n_i\}$ is a monotonic collection of intervals, each containing $p$. Let $I_1 = [\alpha_{11}, g(\alpha_{11})]$. Suppose $I_m$ has been defined to be $[\alpha_{ij}, g(\alpha_{ij})]$. Then, let $g(\alpha_{ij})$

$$I_{m+1} = \begin{cases} [\alpha_{i,j+1}, g(\alpha_{i,j+1})], \text{ if } j < n_i, \text{ and} \\ [\alpha_{i+1,2}, g(\alpha_{i+1,2})], \text{ if } j = n_i \,. \end{cases}$$

Relabel $I_n$ to be $[a_n, b_n]$. Then, $\max\{a_{n-1} - a_n, b_n - b_{n-1}\} < 1$, for each positive integer $n$. Let $Q = (a_1, b_1)$. Then apply Corollary 3.

Kuller required differentiability of $g$ in order to solve $y' = y \circ g$, $y(p') = q$, where $p'$ is the unique fixed point of $g$.

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# CONTINUA IN WHICH ONLY SEMI-APOSYNDETIC SUBCONTINUA SEPARATE

LELAND E. ROGERS

**E. J. Vought has characterized hereditarily locally connected compact metric continua as those which are hereditarily aposyndetic, and (subsequently) as those which are aposyndetic and have only aposyndetic separating subcontinua. Also, Vought characterized hereditarily locally connected, cyclically connected compact metric continua as those having no cut point and separated only by aposyndetic subcontinua. In this paper it is shown that similar characterizations can be obtained when a larger class of subcontinua are allowed to separate, namely those which are semi-aposyndetic.**

A *continuum* is a nondegenerate closed connected set. If $x$ and $y$ are points of the continuum $M$, we say that $M$ is *aposyndetic at $x$ with respect to $y$* if there exists a subcontinuum $H \subset M - \{y\}$ containing $x$ in its interior. The continuum $M$ is *aposyndetic at $x$* if $M$ is aposyndetic at $x$ with respect to each point of $M - \{x\}$. If $M$ is aposyndetic at each point $x \in M$, then we say that $M$ is *aposyndetic*. If $x$ and $y$ are points of a continuum $M$, then $M$ is *semi-aposyndetic at $\{x, y\}$* if $M$ is aposyndetic at one (at least) of $x$ and $y$ with respect to the other. If $M$ is semi-aposyndetic at each 2-point subset, then we say that $M$ is *semi-aposyndetic*. Thus every aposyndetic continuum must be semi-aposyndetic. But the converse does not hold, indeed, $M$ may be aposyndetic at *none* of its points yet still be semi-aposyndetic, as shown in the example below. A set $D$ *separates* $M$ if $M - D$ is not connected, and a point $z$ *cuts* $M$ if there exist points $x, y \in M - \{z\}$ such that every subcontinuum of $M$ containing both $x$ and $y$ also contains $z$. A continuum $M$ is *cyclically connected* if each pair of points of $M$ are contained in a simple closed curve in $M$. A property (e.g., locally connected, aposyndetic, or semi-aposyndetic) of a continuum $M$ is *hereditary* if each subcontinuum of $M$ has that property.

The notion of semi-aposyndesis has recently been shown to be useful in the study of $n$-mutual aposyndesis in the Cartesian products of continua [8]. Also, C. L. Hagopian has a number of results concerning semi-aposyndetic plane continua [2; 3; 4], the most interesting being that non-separating semi-aposyndetic plane continua are arcwise-connected [3]. That semi-aposyndesis is weaker than aposyndesis is evident: the cone over any regular Hausdorff space $S$ is semi-aposyndetic [8, p. 240] but clearly not always aposyndetic.

EXAMPLE. *A compact planar semi-aposyndetic continuum which* *is aposyndetic at none of its points.* Let $K$ be a cone over the Cantor set $C$ (built in $[0, 1]$), i.e. $[0, 1] \times C$ with $\{0\} \times C$ identified. Let $B$ denote the copy of $[0, 1] \times \{0\}$ in $K$. Assume that $K$ is situated in the plane so that $B$ coincides with the line segment $\{(x, \sqrt{3}/6) \mid -1/2 \leqq x \leqq 1/2\}$, with the order on $B$ agreeing with that of $L$ from $(-1/2, \sqrt{3}/6)$ to $(1/2, \sqrt{3}/6)$. Let $f$ and $g$ denote the rotation maps of 120° and 240° respectively. Finally, let $M = K \cup f(K) \cup g(K)$, with $B \cup f(B) \cup g(B)$ forming a triangle and the rest of $M$ outside this triangle. It is clear that $M$ has the required properties.

Vought [10, p. 96] showed that hereditary aposyndesis and hereditary local connectedness are equivalent. Since the cone over the Cantor set is hereditarily semi-aposyndetic, it is clear that his result does not hold when hereditary aposyndesis is replaced by hereditary semi-aposyndesis. However, in the event that the continuum is aposyndetic, such a substitution does work. It should be noted that the proofs of Theorems 2, 3, and 4 are patterned in general after those of Vought's in [9].

First we extract a result from [8, p. 242]:

LEMMA 1. *Let $M$ be a compact metric semi-aposyndetic continuum. If $M$ is irreducible between two points, then $M$ is an arc.*

Another useful and well-known result is

LEMMA 2. *Let $x$ be a point of a compact metric continuum $M$ such that $M$ is aposyndetic at each point of $M - \{x\}$ with respect to $x$. Then $x$ cuts in $M$ if and only if $x$ separates in $M$.*

THEOREM 1. *Let $M$ be a compact metric continuum. Then $M$ is hereditarily locally connected if and only if $M$ is aposyndetic and hereditarily semi-aposyndetic.*

*Proof.* Suppose that $M$ is not hereditarily locally connected. Then [11, p. 18] there exist disjoint subcontinua $C_1, C_2 \cdots$ converging to a subcontinuum $C$ disjoint from each $C_i$. Let $x$ and $y$ be distinct points of $C$. Let $x_i, y_i \in C_i$ (for each $i$) such that $x = \lim x_i$ and $y = \lim y_i$. For each $i$, let $A_i$ be an irreducible subcontinuum of $C_i$ from $x_i$ to $y_i$. Then by Lemma 1, each $A_i$ is an arc. Let $z \in \lim A_i - \{x, y\}$ [taking a subsequence, if necessary]. By the aposyndesis of $M$, there exist subcontinua $H$ and $K$ in $M - \{z\}$ such that $x \in H^\circ$ and $y \in K^\circ$ (for any set $S$, $S^\circ$ denotes the interior of $S$). We may assume that each $A_i$ meets $H \cup K$ and that no $A_i$ is contained in $H \cup K$. Select $z_i \in A_i - (H \cup K)$ [for each $i$] such that $z = \lim z_i$. Let $A_i'$ be the subarc of $A_i$

which is the closure of the $z_i$-component of $A_i - (H \cup K)$. Let $A' = \lim A'_i$ [taking a subsequence, if necessary]. Let $w \in A' - (H \cup K \cup \{z\})$, and let $w_i \in A'_i$ (for each $i$) such that $w = \lim w_i$. Let $p_i$ and $q_i$ denote the endpoints of $A'_i$. We may assume that $w_i$ precedes $z_i$ in the order that $A'_i$ has from $p_i$ to $q_i$. For each $i$, let $D_i$ be the subarc of $A'_i$ defined by $D_i = [p_i, z_i]$ for odd $i$, and $D_i = [w_i, q_i]$ for even $i$. Finally, let $B$ denote the continuum $C \cup H \cup K \cup (\cup D_i)$. By hypothesis, $B$ must be semi-aposyndetic. However, it is easily seen that $B$ is aposyndetic at neither of $w$ and $z$ with respect to the other. This contradiction concludes the proof of the theorem.

Bing [1, p. 499] showed that for compact metric continua in which no subcontinuum separates, aposyndesis at a point implied local connectedness at that point. Vought [9, p. 258] allowed aposyndetic subcontinua to separate and obtained the same conclusion. When semi-aposyndetic subcontinua are allowed to separate, we show that if $M$ is both aposyndetic and semi-locally-connected at $x$, then $M$ is connected im kleinen at $x$, but not necessarily locally connected at $x$. Whether the "semi-locally-connected at $x$" is actually necessary is unknown to the author. (Clearly semi-locally-connected at $x$ without aposyndetic at $x$ is not sufficient, because of the cone over the Cantor set.) First we prove a useful lemma.

LEMMA 3. *Suppose $B$ is a subcontinuum of the compact metric continuum $M$, $x$ is a point of $M - B$, and $A$ is a subcontinuum of $M$ irreducible from $x$ to $B$. If $A \cup B$ is semi-aposyndetic, then $A$ is an arc.*

*Proof.* By Lemma 1, we need only show that $A$ is semi-aposyndetic. Suppose there exist distinct points $w, z \in A \cap B$. Since $A \cup B$ is semi-aposyndetic, there exists a subcontinuum $H$ of $A \cup B$ such that, say, $w \in H^\circ$ and $z \notin H$. It $x \in H$ then any subcontinuum of $H$ irreducible from $x$ to $B$ would contradict the irreducibility of $A$. Thus $x \notin H$. If $A - H$ is connected, then $\mathrm{Cl}(A - H)$ is a continuum missing $w$ but containing $x$ and $z$. This contradiction implies that $A - H = E \cup F$, separated, with $x \in E$. The continuum $H \cup E$ contains both $x$ and $w$. Thus any subcontinuum of $H \cup E$ irreducible from $x$ to $B$ would contradict the irreducibility of $A$. Thus $A \cap B$ consists of only a single point $w$.

Suppose that $y, z \in A$ such that $A$ is not semi-aposyndetic at $\{y, z\}$. By the semi-aposyndesis of $A \cup B$, there is a subcontinuum $H$ of $A \cup B$ such that, say, $y \in H^\circ$ (relative to $A \cup B$) and $z \notin H$. By the choice of $y$ and $z$, it follows that $H \not\subset A$. Then $H - \{w\} = E \cup F$, separated, with $y \in E$. Hence $E \cup \{w\}$ is a subcontinuum of $A$ containing $y$ in its interior (relative to $A$) and missing $z$. This contradiction completes the proof.

THEOREM 2. *Let $M$ be a compact metric continuum in which only semi-aposyndetic subcontinua separate. If $M$ is both aposyndetic at $x$ and semi-locally-connected at $x$, then $M$ is connected im kleinen at $x$.*

*Proof.* Suppose $M$ is not connected im kleinen at $x$. Then [11, p. 18] there exists an open set $U$ containing $x$, and a sequence $C_1, C_2, \cdots$ of closures of distinct components of $U$ such that $x \in C = \lim C_i$, and $C \cap C_i = \phi$ (for each $i$).

We may assume that $x$ is a non-separating point of $M$, since if $K$ is a component of $M - \{x\}$, then $x$ is a non-separating point of of $K \cup \{x\}$, and we would need only show that each $K \cup \{x\}$ is connected im kleinen at $x$ in order to complete the proof.

Since $M$ is semi-locally-connected at $x$, $M$ is aposyndetic at each point of $M - \{x\}$ with respect to $x$. Hence $M - U$ can be covered by a collection of subcontinua missing $x$, and by compactness, a finite number of these cover $M - U$. Then since $x$ does not separate, by Lemma 2 we have that $x$ does not cut. Hence the union of this finite collection of subcontinua is contained in a subcontinuum missing $x$. Thus we may assume that $M - U$ is connected.

We first note that if $B$ is any subcontinuum of $C_i$ irreducible from $x_i$ to Bd $U$ [Bd denotes boundary], then $B \cup (M - U)$ is a separating subcontinuum of $M$ and hence is semi-aposyndetic. Thus by Lemma 3, each such continuum $B$ is an arc. Now for each $i$, let $p_i, q_i \in C_i - U$ [$p_i$ and $q_i$ possibly the same point] such that there are arcs $T_i$ and $S_i$ in $(C_i \cap U) \cup \{p_i\}$ and $(C_i \cap U) \cup \{q_i\}$ respectively irreducible from $x_i$ to $p_i$ and $q_i$ respectively. Let $p = \lim p_i$ and $q = \lim q_i$ (taking a subsequence of $\{C_i\}_{i=1}^{\infty}$ if necessary). If $p = q$ for each possible choice of sequences $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$, then $M$ would not be aposyndetic at $x$ with respect to $p$. Hence there are sequences $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$ such that $p \neq q$. For each $i$, let $A_i$ be an arc from $p_i$ to $q_i$ contained in $T_i \cup S_i$; hence $A_i - U = \{p_i, q_i\}$. Let $A = \lim A_i$ (taking a subsequence, if necessary), let $w$ and $z$ be distinct points of $A$, and let $w_i, z_i \in A_i - \{p_i, q_i\}$ (for each $i$) such that $w = \lim w_i$ and $z = \lim z_i$. We may assume that for each $i$, $w_i$ precedes $z_i$ in the order that $A_i$ has from $p_i$ to $q_i$. For each $i$, let $D_i$ be the subarc of $A_i$ defined by $D_i = [p_i, z_i]$ for odd $i$, and $D_i = [w_i, q_i]$ for even $i$. Finally, let $B$ denote the continuum $(M - U) \cup A \cup (\cup D_i)$. Then $B$ is not semi-aposyndetic at $\{w, z\}$ but it does separate $M$. This contradiction establishes the theorem.

A well-known example (see Figure 3-9 of [5, p. 113]) of a continuum which is connected im kleinen at $x$ but not locally connected at $x$ satisfies the hypotheses of Theorem 2 and hence shows that the conclusion cannot be improved to "locally connected" as in the cases of Bing's and Vought's results.

THEOREM 3. *A compact metric continuum $M$ is hereditarily locally connected if and only if $M$ is aposyndetic and each separating subcontinuum is semi-aposyndetic.*

*Proof.* Using Theorems 1 and 2 (and the fact that a continuum is locally connected if it is connected im kleinen at each point), the proof of Theorem 3 is essentially the same as Vought's proof [9, p. 259].

The final result is a "semi-aposyndetic version" of Vought's Theorem 3 of [9, p. 260], which generalizes Bing's result [1, p. 504] that a compact metric continuum in which no point cuts and no subcontinuum separates must be a simple closed curve.

We first prove two lemmas.

LEMMA 4. *Suppose that no point cuts in the compact metric continuum $M$, $x$ is a point of the open set $U \subset M$,* Bd $U$ *is nondegenerate, and each subcontinuum of $M$ irreducible from $x$ to* Bd $U$ *is an arc. Then for each $\varepsilon > 0$, there exists an arc $A$ in* Cl $U$ *with end points in* Bd $U$ *such that the distance from $x$ to $A$ is less than $\varepsilon$.*

*Proof.* We shall assume that each arc $S$ irreducible from a point $p$ of $U$ to Bd $U$ is ordered from $p$ to Bd $U$. Furthermore, for $a, b \in S$, $S[a, b]$ denotes the closed interval of $S$ from $a$ to $b$; open and half-open interval notation denote analogous subsets of $S$.

Let $T$ be an arc irreducible from $x$ to Bd $U$, and let $b$ be the point of $T \cap \text{Bd } U$. Let $Q$ be the set of all points $y \in T$ such that there exists an arc $S$ containing $y$ and irreducible between two points of Bd $U$. Since no point cuts, there exists an arc $S'$ containing $x$ and intersecting Bd $U - \{b\}$ but missing $b$. Then in $T \cup S'$ there is an arc which contains a point of $T - \{b\}$ and is irreducible between $b$ and some other point of Bd $U$. Hence $Q \neq \varnothing$. Let $q = \text{glb } Q$. We need only show that $q = x$.

Assume that $q \neq x$. Since $q$ does not cut $x$ from Bd $U$, there exists an arc $D$ from $x$ to Bd $U$ missing $q$. Since $q = \text{glb } Q$, $D \cap T(q, b] \neq \varnothing$. Let $y$ be the first point (with respect to the order on $D$) of $D \cap T(q, b]$. Let $z$ be the last point (w.r.t. $D$) of $D[x, y] \cap T[x, q]$. We may assume that $D = T[x, z] \cup D[z, y] \cup T[y, b]$.

Since $q$ is either in $Q$ or a limit point of $Q$, there exists a point $w \in T(z, y) \cap Q$ (possibly $w = q$). Thus there are arcs $A$ and $B$ each from $w$ to Bd $U$ such that $A \cap B = \{w\}$. We may assume that $w$

precedes all other points of $(A \cup B) \cap T$[w.r.t. $T$]. If $D \cap B = \emptyset$, then $z \in Q$ because of the arc $B \cup T[z, w] \cup D[z, b]$. But since this contradicts the fact that $q = \text{glb}\, Q$, we have that $D \cap B \neq \emptyset$. Let $v$ denote the first point (w.r.t. $D$) of $D \cap B$. If $A \cap D[z, v] = \emptyset$, then $z \in Q$ because of the continuum $A \cup T[z, w] \cup D[z, v] \cup B[v, b']$ where $b'$ is the point of $B \cap \text{Bd}\, U$. This contradiction implies that $A \cap D[z, v] \neq \emptyset$. Let $p$ be the first point (w.r.t. $D$) of $A \cap D[z, v]$ and let $a$ be the point of $A \cap \text{Bd}\, U$. Then $A[p, a] \cup D[z, p] \cup T[z, w] \cup B$ shows that $z \in Q$. This contradiction implies that $q = x$ and the proof is complete.

LEMMA 5. *Suppose that $M$ is a compact metric continuum in which no point cuts and only semi-aposyndetic subcontinua separate. If $M$ is semi-aposyndetic at $\{x, y\}$, then $M$ is aposyndetic at $x$ with respect to $y$.*

*Proof.* Assume that $M$ is not aposyndetic at $x$ with respect to $y$. By semi-aposyndesis, there exists a subcontinuum $B \subset M - \{x\}$ such that $y \in B^\circ$. Let $C, C_1, C_2, \cdots$ be the closures of distinct components of $M - B$ such that $x \in \lim C_i \subset C$. Using Lemmas 3 and 4, we can construct (for each $i$) points $p_i$ and $q_i$ in $B \cap C_i$ and an arc $A_i$ irreducible from $p_i$ to $q_i$ in $C_i$ such that $A_i \cap B = \{p_i, q\}$ and $\lim A_i$ is non-degenerate [taking a subsequence, if necessary]. Let $A = \lim A_i$ and select distinct points $w, z \in A$. Let $w_i, z_i \in A_i - \{p_i, q_i\}$ (for each $i$) such that $w = \lim w_i$ and $z = \lim z_i$. We may assume that $w_i$ precedes $z_i$ in the order that $A_i$ has from $p_i$ to $q_i$. Let $D_i$ be the subarc of $A_i$ defined by $D_i = [p_i, z_i]$ for odd $i$, and $D_i = [w_i, q_i]$ for even $i$. Then $(\cup D_i) \cup A \cup B$ is a subcontinuum which separates $M$ but which is not semi-aposyndetic at $\{w, z\}$. This contradiction concludes the proof of the lemma.

THEOREM 4. *A compact metric continuum $M$ is hereditarily locally connected and cyclically connected if and only if no point cuts in $M$ and only semi-aposyndetic subcontinua separate $M$.*

*Proof.* Since the necessity is obvious, we consider the sufficiency. Using Theorem 3, Lemma 2, and [7, p. 138], it is clear that we need only show that $M$ is aposyndetic.

Suppose that $x$ and $u$ are points of $M$ such that $M$ is not aposyndetic at $x$ with respect to $u$. Since no point cuts in $M, M$ is both aposyndetic and semi-locally-connected on a dense $G_\delta$-subset $Z$ of $M$ [6, p. 412]. By Theorem 2, $M$ is connected im kleinen at each point of $Z$. Let $y, z \in Z - \{x, u\}$, and let $H$ and $K$ be disjoint subcontinua in $M - \{x, u\}$ such that $y \in H^\circ$ and $z \in K^\circ$.

Suppose that $M - (H \cup K)$ is connected. Then the continuum $\mathrm{Cl}\,[M - (H \cup K)]$ is semi-aposyndetic since it separates $y$ from $z$. Hence $M$ is semi-aposyndetic at $\{x, u\}$. By Lemma 5, $M$ is aposyndetic at $x$ with respect to $u$. This contradiction implies that $M - (H \cup K)$ is not connected.

Thus $M - (H \cup K) = D \cup E$, separated. One of $H \cup D \cup K$ and $H \cup E \cup K$ must be a continuum. We shall show that the other is also. Let $H \cup D \cup K$ be a continuum and suppose that $H \cup E \cup K = P \cup Q$, separated subcontinua, with $H \subset P$ and $K \subset Q$.

The continuum $H \cup D \cup K$ is not irreducible about $H \cup K$, or else points in $D$ will cut $P$ from $Q$. Let $W$ be a proper subcontinuum of $H \cup D \cup K$ containing $H \cup K$. Suppose $P \neq H$ and $Q \neq K$. Then the three continua $H \cup D \cup K$, $P \cup W$, and $Q \cup W$ each separate $M$ and hence are semi-aposyndetic. Also each of $x$ and $u$ is in the interior of one of them. Thus their union, namely $M$, is semi-aposyndetic at $\{x, u\}$. Then by Lemma 5, $M$ is aposyndetic at $x$ with respect to $u$. Thus it cannot be the case that $P \neq H$ and $Q \neq K$. We assume, without loss of generality, that $P = H$. Then $Q = K \cup E$.

In order to show that $x \in D$, we suppose that this is not the case, i.e., that $x \in E$. The continuum $Q$ is not irreducible about $K \cup \{x\}$, or else $x$ will be cut (in $M$) from $K$ by any point of $E - \{x\}$. Let $T$ be a proper subcontinuum of $Q$ containing both $x$ and $K$. In order to show that $Q - T$ is connected, we suppose that $Q - T = T_1 \cup T_2$, separated. Then $T \cup T_1$ and $T \cup T_2$ are separating, hence semi-aposyndetic, subcontinua. Assume that $u \notin T$, so that $u \in T_1$, say. Then $T \cup T_1$ is aposyndetic at either (1) $u$ with respect to $x$, or (2) $x$ with respect to $u$. In the first case, it would follow immediately that $M$ is aposyndetic at $u$ with respect to $x$, and by Lemma 5 we would have a contradiction. In the second case, $M$ would be aposyndetic at $x$ with respect to $u$ because of the continuum which is the union of $T_2$, $T$, and the subcontinuum of $T \cup T_1$ missing $u$ and containing $x$ in its interior (relative to $T \cup T_1$). This contradiction implies that $u \in T$. Each of $T \cup T_1$ and $T \cup T_2$ are semi-aposyndetic at $\{x, u\}$. Without loss of generality, we may assume that there is a subcontinuum $S_1$ of $T \cup T_1$ such that $x \in S_1^\circ$ (relative to $T \cup T_1$) and $u \notin S_1$. Now $T \cup T_2$ cannot be aposyndetic at $x$ with respect to $u$ since it would follow that $M$ also is aposyndetic at $x$ with respect to $u$. Thus there is a subcontinuum $S_2$ of $T \cup T_2$ such that $u \in S_2^\circ$ (relative to $T \cup T_2$) and $x \notin S_2$. The continuum $T \cup S_1$ separates $T \cup T_1$ into sets $A_1$ and $B_1$ (otherwise $S_2 \cup \mathrm{Cl}\,(T_1 - S_1)$ would be a continuum with $u$ in its interior and missing $x$, and by Lemma 5 we would arrive at a contradiction). Similarly $T \cup S_2$ separates $T \cup T_2$ into sets $A_2$ and $B_2$. Then $T \cup S_1 \cup S_2 \cup A_1 \cup A_2$ is a continuum. Since it separates $M$, it must be semi-aposyndetic. Thus it contains a subcontinuum $S_3$ which, say, misses

$x$ and contains $u$ in its relative interior. In a similar manner, $T \cup S_1 \cup S_2 \cup B_1 \cup B_2$ is a semi-aposyndetic subcontinuum of $M$. If it contains a continuum missing $x$ and containing $u$ in its relative interior, then the union of that continuum with $S_3$ will miss $x$ and contain $u$ in its interior (relative to $M$) and by Lemma 5, we would arrive at a contradiction. So there must be a subcontinuum $S_4$ missing $u$ and containing $x$ in its interior (relative to $T \cup S_1 \cup S_2 \cup B_1 \cup B_2$). Again in a similar manner, $T \cup S_1 \cup S_2 \cup B_1 \cup A_2$ is a continuum which separates $M$ and hence is semi-aposyndetic. In case this continuum is aposyndetic at $x$ with respect to $u$, then it follows that $M$ is also. Thus there is a subcontinuum $S_5$ which misses $x$ and contains $u$ in its relative interior. Then $S_3 \cup S_5$ is a continuum missing $x$ and containing $u$ in its interior (relative to $M$) and by Lemma 5, $M$ is aposyndetic at $x$ with respect to $u$. This contradiction implies that $Q - T$ is connected. The dense $G_\delta$-set $Z$ intersects $Q - T$, so the continuum $\mathrm{Cl}\,(Q - T)$ is decomposable and hence can be written as the union of two proper subcontinua $X$ and $Y$. Suppose $X$ does not intersect $T$. Then $x$ is in the interior of the continuum $Y \cup T$ that separates $M$. It follows that $M$ is semi-aposyndetic at $\{x, u\}$. Then by Lemma 5, we arrive at a contradiction. Thus both $X$ and $Y$ must intersect $T$. Each of the continua $X \cup T$ and $Y \cup T$ separate $M$ and hence are semi-aposyndetic. Using an argument similar to the one above (which involved $T \cup T_1$ and $T \cup T_2$), we arrive at a contradiction.

Since the assumption that $x \in E$ has led to a contradiction, it must be that $x \in D$. The set $D$ cannot be connected, or else, $\mathrm{Cl}\,D$ is semi-aposyndetic since it separates $M$, and by Lemma 5 we would have a contradiction. Thus $D = D_1 \cup D_2$, separated, with $x \in D_1$. Let $A$ denote the $x$-component of $D_1 \cup H \cup K$. Since $D_1 \cup H \cup K$ has at most two components, $x \in A^\circ$. If $K \subset A$, then $A$ is a continuum which separates $D_2$ from $E$, and hence is semi-aposyndetic. Then by Lemma 5, we would arrive at a contradiction. Thus we suppose that $K \cap A = \phi$. Then $A$ meets $H$, and $\mathrm{Cl}\,D_2$ meets both $H$ and $K$. Let $D' = D_2 \cup E$ and $E' = D_1$. Then $H \cup K \cup D'$ is connected while $H \cup K \cup E'$ is not. However, earlier (the portion of the proof which preceded this paragraph) we showed that $x$ could not lie in such a part of a separation of $M - (H \cup K)$. This contradiction implies that the original supposition that $H \cup E \cup K$ is not connected is false. Hence both $H \cup D \cup K$ and $H \cup E \cup K$ are continua.

Suppose both $H \cup D \cup K$ and $H \cup E \cup K$ are irreducible about $H \cup K$. Since $M$ has no cut points, no point of $D$ cuts any other point of $D$ from $H \cap K$ in $H \cup D \cup K$. Assume that $H$ cuts a point $d$ of $D$ from $K$ in $H \cup D \cup K$. Since no point cuts in $M$ and $H \cap \mathrm{Cl}\,D$ cuts the point $d$ from $K$ in $M$, then $H \cap \mathrm{Cl}\,D$ must contain more than one point. If $H \cap \mathrm{Cl}\,D \cap \mathrm{Cl}\,E \neq \phi$, then $\mathrm{Cl}\,D \cup \mathrm{Cl}\,E$ is a separating, hence semi-aposyndetic, subcontinuum, and by Lemma 5 we have a contradiction. Thus $H \cap \mathrm{Cl}\, D \cap \mathrm{Cl}\, E = \phi$. Consequently, $\mathrm{Cl}\, H^\circ \cap \mathrm{Cl}\, D \neq \phi$, or else the continuum $H \cup D \cup K$ would be the union of two separated sets $\mathrm{Cl}\, H^\circ \cup (H \cap \mathrm{Cl}\, E)$ and $K \cup \mathrm{Cl}\, D$. Next, using Lemma 5 and the fact that the continuum $\mathrm{Cl}\, D \cup K \cup \mathrm{Cl}\, E$ is the complement of $H^\circ$, it follows that $H^\circ$ is connected. Similarly, $K^\circ$ is connected. Suppose $\mathrm{Cl}\, H^\circ$ contains a proper subcontinuum $R$ which intersects both $H \cap \mathrm{Cl}\, D$ and $H \cap \mathrm{Cl}\, E$. Then the continuum $\mathrm{Cl}\, D \cup R \cup \mathrm{Cl}\, E$ is semi-aposyndetic since it separates $H^\circ - R$ from $K^\circ$, and by Lemma 5 we reach a contradiction. Thus $\mathrm{Cl}\, H^\circ$ is irreducible from $H \cap \mathrm{Cl}\, D$ to $H \cap \mathrm{Cl}\, E$. Similarly $\mathrm{Cl}\, K^\circ$ is irreducible from $K \cap \mathrm{Cl}\, D$ to $K \cap \mathrm{Cl}\, E$. It follows that $\mathrm{Cl}\, K^\circ \cup \mathrm{Cl}\, D$ is irreducible from $H \cap \mathrm{Cl}\, D$ to $K \cap \mathrm{Cl}\, E$. Note that $\mathrm{Cl}\, H^\circ$ and $\mathrm{Cl}\, K^\circ \cup \mathrm{Cl}\, D$ are the only two subcontinua of $M$ irreducible from $H \cap \mathrm{Cl}\, D$ to $\mathrm{Cl}\, E$. Let $a \in \mathrm{Cl}\, H^\circ \cap \mathrm{Cl}\, D$ and let $b \in H \cap \mathrm{Cl}\, D - \{a\}$. Since no point cuts in $M$, there exists a continuum $R$ which contains $b$, intersects $\mathrm{Cl}\, E$, and misses the point $a$. Then $R$ must contain one of the two continua $\mathrm{Cl}\, H^\circ$ and $\mathrm{Cl}\, K^\circ \cup \mathrm{Cl}\, D$, each of which contains the point $a$. Since $a \notin R$, we have a contradiction.

Using a similar argument for the case of $K$ cutting $b$ in $D$ from $H$ in $H \cup D \cup K$, we have that neither $H$ nor $K$ cuts the other from any point of $D$ in $H \cup D \cup K$. Thus the upper semi-continuous decomposition whose elements are points of $D$ together with the two sets $H$ and $K$ is an arc [1, p. 501]. Similarly, $H \cup E \cup K$ can be decomposed into an arc. Then $M$ is aposyndetic at each point of $D \cup E$, hence at $x$. This contradiction implies that one of $H \cup D \cup K$ and $H \cup E \cup K$ is not irreducible about $H \cup K$.

Let $N$ be a proper subcontinuum of $H \cup D \cup K$ irreducible about $H \cup K$. Since the $G_\delta$-set $Z$ is dense, there exist points $p$ and $q$ in $D - (N \cup \{x, u\})$ and $E - \{x, u\}$ respectively at which $M$ is connected im kleinen. Thus there exist subcontinua $P$ and $Q$ such that $P \in P^\circ \subset P \subset D - (N \cup \{x, u\})$ and $q \in Q^\circ \subset Q \subset E - \{x, u\}$. As was shown above (with $M - (H \cup K)$), we have that $M - (P \cup Q) = S \cup T$, separated, such that $P \cup S \cup Q$ and $P \cup T \cup Q$ are continua. We may assume that $N \subset S$. Thus the continuum $P \cup T \cup Q$ misses $N$ (hence $H \cup K$) and therefore is contained in $D \cup E$. But since $p \in D$ and $q \in E$, the continuum $P \cup T \cup Q$ intersects both parts of the separation $D \cup E$. This impossibility implies, contrary to our initial assumption, that $M$ is aposyndetic at $x$. Thus the proof is complete.

Just as in [9, p. 262], an easy application of Theorem 4 yields the following result due to Bing [1, p. 504]:

COROLLARY. *Every compact metric continuum in which no point cuts and no subcontinuum separates is a simple closed curve.*

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# BI-INVARIANT PSEUDO-LOCAL OPERATORS ON LIE GROUPS

L. PREISS ROTHSCHILD

**Let $G$ be a connected Lie group whose Lie algebra is not the semi-direct sum of a compact algebra and a solvable algebra. It is shown that any bi-invariant pseudo-local operator on $G$ is the sum of an invariant differential operator and an operator with smooth kernel.**

**1. Introduction.** We consider a class of operators on Lie groups, satisfying a weak local property. Roughly, a *pseudo-local operator* on a manifold $M$ is a continuous linear operator, $P$, on the space of compactly supported functions on $M$, which extends to an operator $P'$ on the space of compactly supported distributions on $M$, such that $P'$ preserves singular support. It has been shown by Kohn and Nirenberg [3] that any pseudo-differential operator is pseudo-local. Stekaer [6] has proved that any bi-invariant pseudo-local operator on a complex semisimple Lie group is the sum of an invariant differential operator and an operator with smooth kernel. The proof of this theorem reduces to verifying that every smooth, invariant function on the Lie algebra of $G$ minus the origin can be extended smoothly over the origin. Our main result is the verification of this hypothesis for a large class of Lie groups, proving the above theorem for these groups. For a given Lie group, this theorem implies that the class of bi-invariant differential operators on that group can be substantially extended only by considering operators which do not satisfy local properties.

After the original version of this paper had been submitted, the author learned that these results have been extended by Anders Melin [8] to include any Lie algebra which is not the direct sum of a compact algebra and an abelian one.[1] Independently, the author had extended the results to include the nilpotent case.

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**2. Definitions and notation.** Let $G$ be a Lie group and $C^\infty(G)$, (resp. $C_0^\infty(G)$), the space of smooth functions (resp. smooth functions with compact support) on $G$. The dual of $C^\infty(G)$, which is the space of compactly supported distributions on $G$, will be denoted $\mathscr{E}'(G)$, while the dual of $C^\infty(G)$, the space of distributions on $G$, will be denoted $\mathscr{D}'(X)$.

1. The author is indebted to Sigurdur Helgason for informing her of Melin's work.

For $u \in \mathscr{E}'(X)$ we define the singular support of $u$, denoted sing supp $u$, as $\{x \in X \mid u \notin C^\infty(U)$ for any neighborhood $U$ of $x\}$.

A continuous linear operator $P: C_0^\infty(X) \to C^\infty(X)$ is called a *pseudo-local operator* if it extends to a continuous operator $P': \mathscr{E}'(X) \to \mathscr{D}'(X)$ such that $P'$ preserves singular support; that is,

$$\text{sing supp } P'u \subseteqq \text{sing supp } u \qquad \text{for } u \in \mathscr{E}'(X).$$

We now assume that there is a Lie group $G$ which operates differentiably on $X$. That is, there is a differentiable map $z$

$$z: G \times X \to X$$

such that $z(ab, x) = z(a, z(b, x))$ for all $a, b \in G$, and all $x \in X$. If $f \in C_0^\infty(X)$ we define ${}_af$, the *left translate of* $f$ *by* $a \in G$ as

$${}_af(X) = f(z(a, x)) \qquad \text{for } x \in X\,.$$

If $X = G$, then the *right translate of* $f$ *by* $a \in G$ is defined by

$$f_a(b) = f(ba) \qquad \text{for } a \in G\,.$$

We call the pseudo-local operator $P$ *left invariant* (resp. *right invariant*) if

$$P({}_af) = {}_a(Pf) \quad f \in C_0^\infty(X) \quad (\text{resp. } P(f_a) = (Pf)_a)\,.$$

If $G = X$ is a Lie group, $P$ is called *bi-invariant* if it is both left and right invariant.

Let $\mathfrak{g}$ be the Lie algebra of $G$, and let 1 denote the identity in $G$. $G$ acts a group of automorphisms on $\mathfrak{g}$ via the adjoint representation, $Ad$. For any $a \in G$, $x \in g$, we write $a.\,x$ for $Ad(a)x$. A function $f$ on $\mathfrak{g}$ or $\mathfrak{g} - \{0\}$ is called *invariant* if it is constant on the orbits of $G$ on $\mathfrak{g}$. A function on $G$ or $G - \{1\}$ is called *invariant* if it is constant on the conjugacy classes of $G$. If $\mathscr{F}$ is a family of functions, $\mathscr{F}^G$ will denote the subset of invariant functions.

**3. Pseudo-local operators on Lie groups.** Our main result is the following.

THEOREM. *Let $G$ be a connected Lie group and $\mathfrak{g}$ its Lie algebra. If $\mathfrak{g}$ is not the semi-direct sum of a compact algebra and a solvable one, then every bi-invariant pseudo-local operator on $G$ is the sum of an invariant differential operator and an operator with smooth kernel.*

This theorem has been proved by Stekaer-Hansen in the special case where $\mathfrak{g}$ is complex reductive, non-abelian, using the following reduction to a criterion involvling invariant functions on $\mathfrak{g}$.

PROPOSITION 1 (*Stetkaer*). *Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. If the restriction map $r: (C^\infty(\mathfrak{g}))^G \to (C^\infty(\mathfrak{g} - \{0\})^G$ is surjective, then every bi-invariant pseudo-local operator on $G$ is the sum of an invariant differential operator and an operator with smooth kernel.*

For the proof of this proposition see Stetkaer [6].

We shall refer to the condition on $\mathfrak{g}$ in the proposition as *Stetkaer's hypothesis.*

Stetkaer's verification of this hypothesis for the case where $\mathfrak{g}$ is complex reductive uses a result of Kostant ([4] Theorem 7). Kostant's theorem implies the existence of a hyperplane $\mathfrak{v} \subset \mathfrak{g} - \{0\}$ and a smooth map $t: \mathfrak{g} \to \mathfrak{v}$ satisfying the following conditions.

(i) There is a dense subset $\mathfrak{r} \subset \mathfrak{g}$ such that for any $x \in \mathfrak{r}$, there exists a unique $x' \in \mathfrak{v}$ with $a.x = x'$ for some $a \in G$.

(ii) For any $x \in \mathfrak{r}$, $t(x) = x'$. In particular, if $y \in \mathfrak{v} \cap \mathfrak{r}$, then $t(y) = y$. Conditions (i) and (ii) above show that any invariant function $f$ on $\mathfrak{g} - \{0\}$ is completely determined by its values on $\mathfrak{v}$. Since $\mathfrak{v} \subset \mathfrak{g} - \{0\}$, the function $f \circ t$ is defined and smooth on all of $\mathfrak{g}$. Therefore $f \circ t$ is the desired extension of $f$ since in agrees with $f$ on $\mathfrak{g} - \{0\}$.

Since Kostant's result does not extend even to real reductive Lie groups, we shall use a substantially different approach in our proof.

4. **Proof of the main theorem.** We shall verify Stetkaer's hypothesis in the case where $\mathfrak{g}$ is not the semi-direct sum of a compact Lie algebra and a solvable Lie algebra. If $f$ is an invariant function which is smooth on $\mathfrak{g} - \{0\}$, it will be shown by explicit computation that all partial derivatives of $f$ can be extended continuously over 0. We shall define a one parameter subgroup $\{a_t\}_{t \in R}$ of $G$ and show that for a suitable basis of $\mathfrak{g}$ the transformation of the partial derivatives with respect to this basis can be easily computed (Lemma 3). Invariance of $f$ under the action of this one-parameter group is sufficient to prove the theorem, since the action of $\{a_t\}_{t \in R}$ pushes "most" small elements in $\mathfrak{g} - \{0\}$ to the unit sphere.

Let $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ be a Levi decomposition of $\mathfrak{g}$, with $\mathfrak{g}_1$ semisimple, and $\mathfrak{g}_2$ solvable. By assumption, $\mathfrak{g}_1$ is not compact. From the structure theory of semisimple algebras, it is well known that $\mathfrak{g}_1$ contains a subalgebra $\mathfrak{u}$, where $\mathfrak{u}$ is isomorphic to $\mathfrak{sl}(2, \boldsymbol{R})$, the Lie algebra of the real special linear group. (For the proof of this, as well as the details of the representation theory of $\mathfrak{u}$, to be used later, see

Serre (5), Chapitre IV and VI) or Helgason ([1] Chapter VI].) For any $x \in \mathfrak{g}$ let $ad\ x$ be the endomorphism defined by $ad\ x(y) = [x, y]$ for all $y \in \mathfrak{g}$; i.e. $ad$ is the adjoint representation. If $\mathfrak{u}$ is any such subalgebra, let $x \in \mathfrak{u}$ be the inverse image of the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ under a fixed isomorphism of $\mathfrak{u}$ with $\mathfrak{sl}(2, \boldsymbol{R})$. From the representation theory of three-dimensional simple Lie algebras there is a vector sum decomposition

$$\mathfrak{g} = \mathfrak{g}^{(0)} + \sum_{j=1}^{k} \mathfrak{g}^{(r_j)} + \sum_{j=1}^{k} \mathfrak{g}^{(-r_j)} ,$$

where $\mathfrak{g}^{(r_j)} = \{y \in \mathfrak{g} \mid [x, y] = r_j y\}$, $\mathfrak{g}^{(-r_j)} = \{y \in \mathfrak{g} \mid [x, y] = -r_j y\}$, $\mathfrak{g}^{(0)} = \{y \in \mathfrak{g} \mid [x, y] = 0\}$ where the $r_j$ are all positive integers.

$$\text{Let } \mathfrak{g}^{+} = \sum_{j=1}^{k} \mathfrak{g}^{(r_j)} \text{ and } \mathfrak{g}^{-} = \sum_{j=1}^{k} \mathfrak{g}^{(-r_j)} .$$

We make the convention that $r_{-j} = -r_j$. Let $x_{0_1}, x_{0_2}, \cdots, x_{0_{p(0)}}$ be a basis for $\mathfrak{g}^{(0)}$, and for each $j$, positive and negative, let $x_{j_1} x_{j_2}, \cdots x_{j_{p(j)}}$ be a basis for $\mathfrak{g}^{(r_j)}$. Give $\mathfrak{g}$ the metric for which the above basis is orthonormal. We write $|y|$ for the length of an element $y \in \mathfrak{g}$. Any $y \in \mathfrak{g}$ has a unique decomposition $y = y_+ + y_0 + y_-$, with $y_+ \in \mathfrak{g}^+$, $y_0 \in \mathfrak{g}^{(0)}$ and $y_- \in \mathfrak{g}^-$. Then $|y|^2 = |y_+|^2 + |y_0|^2 + |y_-|^2$.

Let $\boldsymbol{D}$ be the family of all partial derivatives for the given basis. We write $D_{n_j}$ for the partial derivative

$$\frac{\partial^{n_{j_1}}}{\partial x_{j_1}} \frac{\partial^{n_{j_2}}}{\partial x_{j_2}} \cdots \frac{\partial^{n_{j_m}}}{\partial x_{j_m}}$$

where $n_j = (n_{j_1}, n_{j_2}, \cdots, n_{j_m})$, with $n_{j_i} \geqq 0$ for all $i$, for all $j$, positive, negative or zero. Any $D \in \boldsymbol{D}$ can be written $D_{n_{-k}} \cdots D_{n_k}$. The *order* of $D_{n_j}$, $O(D_{n_j})$, is defined by

$$O(D_{n_j}) = \sum_{i=1}^{m} n_{j_i} .$$

Then the order of $D$ is defined by

$$O(D) = \sum_{j=-k}^{k} O(D_{n_j}) .$$

The *height of* $D$, $h(D)$ is defined by

$$h(D) = \sum_{j=-k}^{h} r_j\, O(D_{n_j}) .$$

For any real $t$, let $a_t = \exp tx$, where $\exp\colon \mathfrak{g} \to G$ is the exponential map. Then $\{a_t\}_{t \in \boldsymbol{R}}$ is a one-parameter subgroup; we shall need only the invariance of $f$ under $\{a_t\}$.

Then if $y \in \mathfrak{g}^{(r_j)}$, $a_t \cdot y = e^{tr_j}y$, where $a_t \cdot y$ denotes the adjoint action of $a_t$ on the element $y \in \mathfrak{g}$. The following lemma shows how the partial derivatives transform under the action of $a_t$.

LEMMA 2. *$Df(a_t \cdot y) = e^{-th(D)}Df(y)$ for $y \in \mathfrak{g}$, $y \neq 0$, for any $D \in \boldsymbol{D}$.*

*Proof.* We prove the formula by induction on the order of $D$. Suppose first that $D = \partial/\partial x_{j_s}$. Then

$$\begin{aligned}\frac{\partial f}{\partial x_{j_s}}(a_t \cdot y) &= \lim_{\varepsilon \to 0} \frac{f(a_t \cdot y + \varepsilon x_{j_s}) - f(a_t \cdot y)}{\varepsilon}, \\ &= \lim_{\varepsilon \to 0} \frac{f(y + \varepsilon a_t^{-1} \cdot x_{j_s}) - f(y)}{\varepsilon},\end{aligned}$$

by the invariance of $f$,

$$\begin{aligned}&= \lim_{\varepsilon \to 0} \frac{f(y + \varepsilon e^{-tr_j} x_{j_s}) - f(y)}{\varepsilon} \\ &= e^{-tr_j} \frac{\partial f}{\partial x_{j_s}}(y) = e^{-\mathrm{th}(D)}Df(y).\end{aligned}$$

Now assume the lemma is true whenever $O(\bar{D}) < k$. If $O(D) = k$, then $D = (\partial/\partial x_{j_l})\bar{D}$, where $O(\bar{D}) = k - 1$.

$$\begin{aligned}Df(a_t \cdot y) &= \lim_{\varepsilon \to 0} \frac{\bar{D}f(a_t \cdot y + \varepsilon x_{j_l}) - \bar{D}f(a_t y)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} e^{-th(\bar{D})} \frac{(\bar{D}f(y + \varepsilon a_t^{-1} \cdot x_{j_l}) - \bar{D}f(y))}{\varepsilon} \\ &= e^{-th(\bar{D})} \lim_{\epsilon \to 0} \frac{\bar{D}f(y + \varepsilon e^{-tr} x_{j_l}) - \bar{D}f(y)}{\varepsilon} \\ &= e^{-th(\bar{D})} e^{-tr_j} \frac{\partial}{\partial x_{j_l}} Df(y) \\ &= e^{-th(D)}D,\end{aligned}$$

which proves Lemma 2.

LEMMA 3. *Let $y \in \mathfrak{g} - \{0\}$ and $\delta > 0$ such that $|y| < \delta$. Then for $D \in \boldsymbol{D}$ and any $\varepsilon > 0$, there exists $y' \in \mathfrak{g}$, $|y'| < \delta$, such that*

$$y'_+ \neq 0 \text{ and } y'_- \neq 0 \tag{1}$$

*and*

$$|Df(y) - Df(y')| < \varepsilon. \tag{2}$$

*Proof.* Since $Df$ is continuous at $y$, there exists a neighborhood

$V$ of $y$ such that $z \in V$ implies $|Df(y) - Df(z)| < \varepsilon$. The intersection of $V$ with the ball of radius $\delta$ around the origin is again a neighborhood, $V'$, of $y$. Since $|y|^2 = |y_+|^2 + |y_0|^2 + |y_-|^2$, if either $y_+$ or $y_-$ is 0, we may choose $z'_+ \in \mathfrak{g}^+$ and $z'_- \in \mathfrak{g}^-$ sufficiently small so that $y' = y + z'_+ + z'_-$ is still in $V'$ which proves the lemma.

We now define a compact neighborhood $U$ of width 1/2 around the unit sphere, i.e.,

$$U = \{y \in g \mid 1/2 \leqq |y| \leqq 3/2\} .$$

Since $U$ is compact, for any $D \in \boldsymbol{D}$, $\varepsilon > 0$, there exists $\delta_D > 0$ such that

$$|y_1 - y_2| < \delta_D \Longrightarrow |Df(y_1) - Df(y_2)| < \varepsilon \text{ for any } y_1, y_2 \in U .$$

LEMMA 4. *Let $D \in \boldsymbol{D}$ with $h(D) \neq 0$. Then for any $\varepsilon > 0$, there exists a neighborhood $S_D$ of 0 in $\mathfrak{g}$ such that if $y \in S_D - \{0\}$, then $|Df(y)| < \varepsilon$.*

*Proof.* Let $M = \max_{z \in U} |Df(z)|$, Then for any $t \in \boldsymbol{R}$, $z \in U$ $|Df(a_t \cdot z)| \leqq |e^{-th(D)}M|$, by Lemma 2. We shall assume, to minimize notation, that $h(D) > 0$. The proof for $h(D) < 0$ is similar. Choose $t_1$ satisfying $|e^{-t_1 h(D)}M| < \varepsilon$. Then $|e^{-th(D)}M| < \varepsilon$ for all $t > t_1$. Now let $r = \max_{j=1,2\cdots k} r_j$, and let $\delta_D = \min(1/2, e^{-t_1}/r)$. We define $S_D$ as the sphere of radius $\delta_D$, and we shall show that this satisfies our condition. For suppose $y = y_+ + y_0 + y_- \in S_D$. By Lemma 3 it suffices to assume that $y_+ \neq 0$ and $y_- \neq 0$. Since $|y|^2 = |y_+|^2 + |y_0|^2 + |y_-|^2$, we have $|y_-| < \delta_D$. Since $y_- \neq 0$, there exists $t$ such that $|a_{-t} \cdot y| = 1$. Then

$$|a_{-t} \cdot y - a_{-t} \cdot y_-| = |a_{-t} \cdot (y_0 + y_+)| \leqq |y_0 + y_+| < \delta_D \leqq \frac{1}{2} .$$

So that $a_{-t} \cdot y \in U$. But

$$1 = |a_{-t} \cdot y_-| \leqq |r| \; |e^t| \; |y_-| ,$$

so that

$$e^t \geqq 1/r|y_-| > 1/r(e^{-t_1}/r) = e^{t_1} ,$$

which proves that $t > t_1$. Therefore, since $a_{-t} \cdot y \in U$, it follows from the definition of $t_1$ that $|Df(a_t \cdot (a_{-t} \cdot y))| < \varepsilon$. Since $a_t \cdot (a_{-t} \cdot y) = y$, this proves Lemma 4.

LEMMA 5. *For any $D \in \boldsymbol{D}$ with $h(D) = 0$ and any $\varepsilon > 0$, there exists a neighborhood $S_D$ of 0 in $\mathfrak{g}$ such that if $y, y' \in S_D - \{0\}$, then $|Df(y) - Df(y')| < \varepsilon$.*

*Proof of Lemma* 5. Choose $\delta$, $0 < \delta \leqq 1$, such that for any $z, z' \in U$,

$$|z - z'| < \delta \text{ implies } |Df(z) - Df(z')| < \frac{\varepsilon}{5},$$

and let $S_D$ be the ball of radius $\delta/2$ around 0. Now let $y, y' \in S_D$ be arbitrary. We will show that $|Df(y) - Df(y')| < \varepsilon$.

We write

$$y = y_+ + y_0 + y_-$$

and

$$y' = y'_+ + y'_0 + y'_-$$

as before. By Lemma 3 we may assume that $y_+, y_-, y'_+ y'_-$ are all nonzero. We show first

$$(3) \qquad |Df(y_0 + y_+) - Df(y)| < \frac{\varepsilon}{5} \text{ and } |Df(y'_0 + y'_+) - Df(y')| < \frac{\varepsilon}{5}.$$

For this, choose $t > 0$ such that $|a_t \cdot y_+| = 1$. As in the proof of Lemma 4, $a_t \cdot y$ and $a_t \cdot (y_0 + y_+) \in U$. Then $|a_t \cdot y - a_t \cdot (y_0 + y_t)| = |a_t \cdot y_-| < |y_-| < \delta$. By the choice of $\delta$, we have $|Df(a_t \cdot y) - Df(a_t \cdot (y_0 + y_+)) < \varepsilon/5$. Since $h(D) = 0$, $Df$ is invariant under $a_t$, so that the first inequality of (3) holds. The proof of the second is the same.

By continuity of $Df$ at $y_0 + y_+$ and $y'_0 + y'_+$, we may choose $q_- \in \mathfrak{g}^-$ with $|q_-|$ sufficiently small so that

$$(4) \quad |Df(\bar{y}) - Df(y_0 + y_+)| < \varepsilon/5 \text{ and } |Df(\bar{y}') - Df(y'_0 + y'_+)| < \varepsilon/5,$$

where $\bar{y} = q_- + y_0 + y_+$ and $\bar{y}' = q_- + y'_0 + y'_+$. Now choose $s > 0$ such that $|a_{-s} \cdot q_-| = 1$, which is possible since $q_- \neq 0$. Then $a_{-s} \cdot \bar{y} \in U$ and $a_{-s} \cdot \bar{y}' \in U$. We shall show

$$(5) \qquad |Df(\bar{y}) - Df(\bar{y}')| < \frac{\varepsilon}{5}.$$

Indeed,

$$\begin{aligned} |a_{-s} \cdot \bar{y} - a_{-s} \cdot \bar{y}'| &= |a_{-s} \cdot (y_0 + y_+) - a_{-s} \cdot (y'_0 + y'_+)| \\ &\leqq |a_{-s} \cdot (y_0 + y_+)| + |a_{-s}(y'_0 + y'_+)| \\ &\leqq |y_0 + y_+| + |y'_0 + y'_+| \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta, \end{aligned}$$

which proves $|Df(a_{-s} \cdot \bar{y}) - Df(a_{-s} \cdot \bar{y}')| < \varepsilon/5$. Then (5) follows immediately since $Df$ is invariant under $a_{-s}$.

To complete the proof of Lemma 5, note that

$$\begin{aligned} |Df(y) - Df(y')| &\leqq |Df(y) + Df(y_0 + y_+)| + |Df(y') - Df(y'_0 + y'_+)| \\ &\quad + |Df(y_0 + y_+) - Df(\bar{y})| + |Df(y'_0 + y'_+) \\ &\quad - Df(\bar{y}')| + |Df(\bar{y}) - D(\bar{y}')| \\ &< \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} \text{ by (3), (4) and (5)} = \varepsilon\,. \end{aligned}$$

We may now complete the proof of the theorem. Lemmas 4 and 5 show that for any $D \in \boldsymbol{D}$, the function $Df$ can be extended continuously over 0.

We shall assume the following, which can be proved using elementary calculus.

(6) *If $h(x)$ is a function on $\boldsymbol{R}$ such that $dh/dx$ exists and is continuous off* 0, *then $h$ is differentiable if the function $dh/dx$ can be extended continuously over* 0.

By (6) and induction, it follows that $Df$ exists and is continuous on all of $\mathfrak{g}$ for any $D \in \boldsymbol{D}$.

This finishes the proof of the theorem.

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TUFTS UNIVERSITY
PRESENT ADDRESS: COLUMBIA UNIVERSITY

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# THE FIXED POINT PROPERTY FOR ARCWISE CONNECTED SPACES: A CORRECTION

R. E. SMITHSON AND L. E. WARD, JR.

**Several years ago the second author stated a fixed point theorem for a class of arcwise connected spaces which includes the dendroids as well as certain nonunicoherent continua. Subsequently the first author detected a flaw in the proof. The present collaboration has produced a correct proof. Since the theorem has not been subsumed in the literature of the intervening years and since other authors have alluded to it, it seems desirable to publish the new proof.**

For recent references to the theorem, see [1], [4] and [7]. The original, erroneous argument can be found in [5]. (The error (p. 1277) occurs in the assertion that $S' = \bigcup(S'_\gamma\}$ is connected, and hence that $\mathscr{N}$ has a maximal member.)

In the present exposition a few changes have been made in terminology. In what follows an *arc* is a compact connected Hausdorff space with exactly two non-cutpoints. A space $X$ is *arcwise connected* if for each two elements $x$ and $y$ of $X$ with $x \neq y$, there exists an arc $[x, y]$ contained in $X$. It is convenient to write $[x, x] = \{x\}$ and $[x, y) = (y, x] = [x, y] - \{y\}$. A *circle* is the union of two arcs which meet only in their endpoints. We write $\square$ to denote the empty set. If $e \in X$ then an *e-ray* is the union of a maximal nest of arcs $[e, x]$. If $R$ is an $e$-ray then

$$K_R = \bigcap\{\overline{R - [e, x)}\colon [e, x] \subset R\} ,$$

where the bar denotes closure. If $X$ is not compact then it may be that $K_R$ is empty, but in the compact case this cannot occur.

THEOREM. *If $X$ is an arcwise connected Hausdorff space which contains no circle, if $e \in X$ and if $f\colon X \to X$ is continuous, then $f$ has a fixed point or there exists an e-ray $R$ such that $f(K_R) \subset K_R$.*

COROLLARY. *If $X$ is an arcwise connected Hausdorff space which contains no circle and if there exists $e \in X$ such that $K_R$ has the fixed point property for each e-ray $R$, then $X$ has the fixed point property.*

Before embarking on the proof of the theorem, some subsidiary results will be helpful.

LEMMA 1. *If $X$ is a Hausdorff space, $A$ is an arc and $f\colon A \to$*

*$X$ is continuous, then $f(A)$ is arcwise connected.*

Since $A$ is locally connected and compact it follows that $f(A)$ is locally connected. In contrast to the case where $A$ is separable, the arcwise connectivity of $f(A)$ is not immediate [3]. A proof of Lemma 1 can be found in the thesis of J. K. Harris [2]; it is a modification of an argument first used by J. L. Kelley (see, for example, [6; p. 39].) We give a sketch of that argument.

If $x$ and $y$ are elements of $f(A)$, then there exists a closed subset $F$ of $A$ which is minimal with respect to $\{x, y\} \subset f(F)$ and $f(a) = f(b)$ whenever $a$ and $b$ are the endpoints of a complementary interval of $A - F$. It follows from this minimality that $f(F)$ is connected and that $x$ and $y$ are the only non-cutpoints of $f(F)$. Therefore $f(F)$ is an arc, and so $f(A)$ is arcwise connected.

*For the remainder of this paper $X$ is an arcwise connected Hausdorff space which contains no circle and $e \in X$.* In particular, if $x$ and $y$ are distinct elements of $X$ then the arc $[x, y]$ is unique. Consequently the relation $x \leqq y$ if and only if $x \in [e, y]$ is a partial order. As usual, if $x \leqq y$ and $x \neq y$ we write $x < y$.

Of course each arc in $X$ has a natural order which does not necessarily agree with the partial order $\leqq$. If $a$ and $b$ are elements of $X$ and if $p$ precedes $q$ in the natural order on $[a, b]$, we write $[a, p, q, b]$.

LEMMA 2. *If $a$, $b$ and $c$ are elements of $X$ such that $a < b$ and $a \not\leq c$, then $a \in [b, c]$.*

*Proof.* If $b \leqq c$ then by transitivity the hypothesis that $a \not\leqq c$ is contradicted. Therefore, by the uniqueness of arcs there exists $d \neq b$ such that $[e, b] \cap [e, c] = [e, d]$. Moreover,

$$a \in [e, b] - [e, d] \subset [d, b] \subset [d, b] \cup [d, c] = [b, c] .$$

LEMMA 3. *Let $f\colon X \to X$ be continuous and suppose $x$ and $t$ are elements of $X$ such that $x < t < f(x)$, $t < f(t)$ and $f(x) \not\leqq f(t)$. Then there exists $y \in (x, t]$ such that $f(y) \in [f(x), f(t)]$ and $f(y) \leqq f(x)$.*

*Proof.* By the uniqueness of arcs there exists $z \in X$ such that $[z, f(x)] = [e, f(x)] \cap [f(t), f(x)] \subset [f(t), f(x)]$, and therefore by Lemma 1, $[z, f(x)] \subset f([x, t])$. Because $f(x) \not\leqq f(t)$ and $z \leqq f(t)$ it follows that $z \neq f(x)$. Consequently there exists $y \in (x, t]$ such that $z = f(y)$.

LEMMA 4. *If $f\colon X \to X$ is continuous and if $p$ and $q$ are elements of $X$ such that $[f(p), p, q, f(q)]$, then there exists $x \in [p, q]$ such that*

$x = f(x)$.

*Proof.* By a straightforward maximality argument there exists $[x, y] \subset [p, q]$ which is minimal relative to $[f(x), x, y, f(y)]$. If $f(x) \neq x$ then $x = f(x_1)$ where $x_1 \in (x, y]$ so that $[x_1, y]$ contradicts the minimality of $[x, y]$. Therefore $f(x) = x$.

A subset $C$ of $X$ is called a *chain* if it is simply ordered with respect to the partial order $\leqq$.

LEMMA 5. *If $x \in X$ such that $x \nleqq f(x)$ and if there exists $t_1 \in X$ such that $t_1 \leqq f(t_1) \leqq x$, then $f$ has a fixed point.*

*Proof.* Let $T$ be a subset of $X$ which is maximal with respect to $T \cup f(T) \subset [e, x]$ and $t \leqq f(t)$ for all $t \in T$. Since $T \subset [e, x]$, there is a least upper bound $t_0$ of $T$. We will show that $t_0 = f(t_0)$.

Suppose first that $t_0 \nleqq f(t_0)$ and $f(t_0) \nleqq t_0$. Then there exist disjoint open sets $U$ and $V$ such that $t_0 \in V$, $f(V) \subset U$ and $U \cap [e, t_0] = \square = V \cap [e, f(t_0)]$. If $t \in T$ is chosen so that $[t, t_0] \subset V$, then $[f(t), f(t_0)] \subset f([t, t_0]) \subset U$ since, by Lemma 1, $f([t, t_0])$ is arcwise connected. Since $t < f(t)$ and $t \nleqq f(t_0)$, it follows from Lemma 2 that $t \in [f(t), f(t_0)] \subset U$, and this contradicts our assumption that $U$ and $V$ are disjoint. Therefore, either $f(t_0) \leqq t_0$ or $t_0 \leqq f(t_0)$.

If $f(t_0) < t_0$ then there exist disjoint open sets 0 and $W$ such that $t_0 \in 0$ and $f(0) \subset W$. If $y \in T$ is chosen so that $[y, t_0] \subset 0$, then $[f(y), f(t_0)] \subset W$ and, since $f(t_0) < y \leqq f(y)$, it follows that $y \in W$. Again this is a contradiction and therefore $t_0 \leqq f(t_0)$.

If $t_0 < f(t_0)$ then there are disjoint open sets $U'$ and $V'$ such that $t_0 \in V'$, $f(V') \subset U'$ and $U' \cap [e, t_0] = \square$. If $s \in [t_0, x]$ is chosen so that $[t_0, s] \subset V'$, then $s < f(t_0)$ and hence $[f(t_0), f(s)] \subset U'$. By Lemma 3 there exists $z \in (t_0, s]$ such that $f(z) \in [f(t_0), f(s)]$ and $f(z) \leqq f(t_0)$. Since $z < f(z) \leqq f(t_0) \leqq x$, the maximality of the set $T$ is contradicted. Therefore $t_0 = f(t_0)$.

*Proof of the theorem.* Let $\mathscr{S}$ denote the family of all subsets $S$ of $X$ such that $S \cup f(S)$ is a chain and $t \leqq f(t)$ for each $t \in S$. Clearly $\{e\} \in \mathscr{S}$, so by Zorn's Lemma $\mathscr{S}$ has a maximal member $S_0$.

Suppose $S_0 \cup f(S_0) \subset [e, x]$ for some $x \in X$. If $x \nleqq f(x)$ then $f$ must have a fixed point by Lemma 5. If $x \leqq f(x)$ for each $x$ such that $S_0 \cup f(S_0) \subset [e, x]$ then by maximality both $x$ and $f(x)$ are members of $S_0$ and hence $x = f(x)$.

Therefore we may assume that $S_0 \cup f(S_0)$ is cofinal in some ray $R$. It follows readily that $S_0$ is cofinal in $R$. We will show that if

$f(K_R) - K_R \neq \square$ then $f$ has a fixed point. Choose $y \in K_R$ such that $f(y) \in X - K_R$; then there is a generalized sequence $x_n$ (i.e., a function whose domain is some ordinal number) in $R$ such that $x_n < x_{n+1}$ and $x_n \to y$. Since $S_0$ is cofinal in $R$, the sequence $x_n$ can be so chosen that there exists $y_n \in S_0 \cap [x_n, x_{n+1}]$, for each $n$.

If there exists $n_1$ such that $x_{n_1} \notin [e, f(x_{n_1})]$ then $[f(y_{n_1}), y_{n_1}, x_{n_1}, f(x_{n_1})]$, so that by Lemma 4, $f$ has a fixed point. Consequently we may assume $x_n \leqq f(x_n)$ for each $n$. Moreover, since $f(y) \notin K_R$ we may assume $f(x_n) \notin R$, for each $n$.

If there exists $n_2$ such that $f(x_{n_2}) \leqq f(f(x_{n_2}))$ then we may find $m$ such that $y_m \notin [e, f(f(x_{n_2}))]$ and therefore $[f(y_m), y_m, f(x_{n_2}), f(f(x_{n_2}))]$. Again, $f$ has a fixed point by Lemma 4. Hence we may assume that $x_n < f(x_n) \nleqq f(f(x_n))$. But then the hypotheses of Lemma 5 are satisfied.

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UNIVERSITY OF WYOMING
AND
UNIVERSITY OF OREGON

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# ZEROS OF SUMS OF SERIES WITH HADAMARD GAPS

L. R. SONS

**If $f$ is a function of the complex variable $z$ in the unit disc and the power series expansion for $f$ about zero can be expressed as a finite sum of series with Hadamard gaps, then $f(z)$ assumes every finite value infinitely often provided the coefficients in the power series expansion of $f$ do not tend to zero and the average value of $(\log^{+}1/|f(re^{i\theta})|)^{p}$ does not grow too rapidly as $r \to 1^{-}$ for some $p > 1$.**

**1. Introduction and statement of results.** Let $f$ be a function analytic in $|z| < 1$ for which

$$f(z) = c_0 + \sum_{k=1}^{\infty} c_k z^{n_k} \tag{1}$$

where $\{n_k\}$ is a sequence of positive integers for which

$$\frac{n_{k+1}}{n_k} \geqq q > 1 , \qquad (k \geqq 1) . \tag{2}$$

The series in (1) is said to have Hadamard gaps.

If $q$ is greater than about 100, G. and M. Weiss [9] proved $f(z)$ assumes every finite value infinitely often provided

$$\sum_{k=0}^{\infty} |c_k| = \infty . \tag{3}$$

If $q > 1$, W. H. J. Fuchs [2] showed $f(z)$ assumes every finite value infinitely often provided

$$\limsup_{k\to\infty} |c_k| > 0 . \tag{4}$$

In [3] Fuchs has extended his result to show that $f$ assumes every finite value infinitely often in each sector

$$S = \{z \,|\, \alpha < \arg z < \beta \quad \text{and} \quad |z| < 1\}$$

where $\alpha$ and $\beta$ are fixed real numbers.

The original result of Fuchs may also be extended as follows:

TEOREM 1. *Let $\{n_k\}$ be a sequence of positive integers for which (2) holds. Let $l$ be a fixed positive integer, and let $n_k^{(i)}$ for $i=1, 2, \cdots, l$ be integers for which*

$$n_{k-1} < n_k^{(l)} < n_k^{(l-1)} < \cdots < n_k^{(1)} < n_k .$$

*Suppose $f$ is a function analytic in $|z| < 1$ for which*

$$(5)\qquad \begin{aligned} f(z) &= a_0 + \sum_{k=1}^{\infty} (a_{n_k^{(l)}} z^{n_k^{(l)}} + a_{n_k^{(l-1)}} z^{n_k^{(l-1)}} + \cdots + a_{n_k^{(1)}} z^{n_k^{(1)}} + a_{n_k} z^{n_k}) \,, \\ &= \sum_{k=0}^{\infty} c_k z^k \,. \end{aligned}$$

*Suppose* (4) *holds and for some $p > 1$ there exists a constant $C$ with $0 < C < +\infty$ such that*

$$(6)\qquad \frac{1}{2\pi}\int_0^{2\pi}(\log^+ 1/|f(re^{i\theta})|)^p d\theta \leqq C\Big(\frac{1}{2\pi}\int_0^{2\pi}(\log^+ 1/|f(re^{i\theta})|)d\theta\Big)^p$$

*for a sequence of values of $r$ approaching one. Then $f(z)$ assumes every finite value infinitely often in $|z| < 1$.*

Two immediate corollaries of Theorem 1 are:

COROLLARY 1. *Assume the hypothesis of Theorem 1 with $n_k^{(\alpha)} = n_k - \alpha$ for $k = 1, 2, 3, \cdots$ and $0 < \alpha \leqq l$. Then $f(z)$ assumes every finite value infinitely often in $|z| < 1$.*

COROLLARY 2. *Let $f$ be a function analytic in $|z| < 1$ for which*

$$f(z) = \sum_{k=0}^{\infty} c_k z^k = f_0(z) + f_1(z) + \cdots + f_t(z)$$

*where for each $i$, $f_i(z)$ has a power series expansion about zero with Hadamard gaps. If* (4) *holds and for some $p > 1$ there exists a constant $C$ with $0 < C < +\infty$ such that* (6) *holds for a sequence of values of $r$ approaching one, then $f(z)$ assumes every finite value infinitely often in $|z| < 1$.*

Corollary 1 is a special case of Theorem 1 and extends a result of $C$. Pommerenke [6] who showed that functions of the type of Corollary 1 without the assumption (6) must assume every value at least once. (G. Schmeisser [7] has recently extended the method in [6] to show the Pommerenke-type series assume every value infinitely often). Corollary 2 follows from Theorem 1 by noticing that $f(z)$ can be rewritten, if necessary, to be in the form (5).

For functions of the form (5) for which

$$\limsup_{r\to 1^-} \frac{\log M(r)}{-\log(1-r)} > 2(2+l)$$

where $M(r)$ denotes the maximum modulus of $f(z)$ on $|z| = r$, we remark in [8] that

$$\limsup_{r\to 1-} \frac{n(r)}{-\log(1-r)} > 0 \,,$$

where $n(r)$ denotes the number of zeros of $f$ in $|z| \leqq r$. It seems probable that functions of the type of Theorem 1 also assume every finite value infinitely often in each sector

$$\{z \mid \alpha < \arg z < \beta \quad \text{and} \quad |z| < 1\}$$

where $\alpha$ and $\beta$ are fixed real numbers. It has been shown by P. Erdos and A. Renyi [1] that if $\{n_k\}$ is an increasing sequence of natural numbers satisfying

$$\liminf_{(k-j)\to\infty} (n_k - n_j)^{1/(k-j)} = 1 ,$$

then, for any sequence $\{\omega_k\}$ of natural numbers for which

$$\lim_{k\to\infty} \omega_k = +\infty$$

there exists a sequence $\{m_k\}$ of natural numbers such that

$$0 \leqq m_k - n_k < \omega_k$$

and a function $g$, analytic in $|z| < 1$ with the power series expansion

$$g(z) = \sum_{k=0}^{\infty} b_k z^{m_k} ,$$

where the $b_k$ are positive, such that $g(z)$ is unbounded in $|z| < 1$, but bounded in the domain $|z| < 1, |\arg z| > \varepsilon$, for any $\varepsilon > 0$.

If $f$ is an analytic function in $|z| < 1$, D. Gaier and W. Meyer-Konig [5] have defined the radius $R_\varphi$ defined by $z = re^{i\varphi}, 0 \leqq r < 1$, *singular* for $f$ if $f(z)$ is unbounded in any sector $|z| < 1, \varphi - \varepsilon < \arg z < \varphi + \varepsilon$ with $\varepsilon > 0$. They showed that if $f$ is unbounded in $|z| < 1$ and the power series expansion for $f$ about zero has Hadamard gaps, then every radius is singular for $f$. We have

THEOREM 2. *Suppose $f$ is a function which is analytic in $|z| < 1$ and has the power series expansion* (5). *Suppose*

$$\limsup_{r\to 1^-} \{\max |c_k| r^k\} = \infty .$$

*Then each radius $R_\varphi (0 \leqq \varphi < 2\pi)$ is singular for $f$.*

In section two the necessary lemmas are stated and the theorems are proved, while section three contains the proof of the essential lemma which enables us to use the idea of G. H. Hardy and J. E. Littlewood of accentuating the dominance of the largest term in the series (5) by repeated differentiation (c.f. Fuchs [2]).

## 2. Proofs of the Theorems.

We need three lemmas:

LEMMA 1 (Fuchs [2]). *Let $g$ be a function analytic in $|z| < R$. If, for some positive integer $p$*

$$|g^{(p)}(z)| \leqq M \qquad (|z| < R)\ ,$$

*and*

$$|g^{(p)}(0)| \geqq A > 0\ ,$$

*then $g(z)$ assumes in $|z| < R$ every value $w$ lying in the disc*

$$|w - g(0)| < KR^p A^{p+1} M^{-p}$$

*where $K$ is a positive number depending only on $p$.*

LEMMA 2 (Gaier [4]). *Let $E$ be a closed subset of $\{z \mid |z| = 1\}$ and assume that $E$ has measure $2\pi\gamma$ where $0 < \gamma < 1$. If $p$ is a polynomial with $N$ terms, then*

$$\max_{|z|=1} |p(z)| \leqq C_N(\gamma) \cdot \max_{z \in E} |p(z)|$$

*where*

$$\log C_N(\gamma) = \Big(\frac{\gamma}{1-\gamma}\gamma^{-N} - \frac{1}{1-\gamma}\Big)\log 3\ . \tag{7}$$

LEMMA 3. *Assume the hypothesis of Theorem 1. Let $p$, $\nu$, and $\alpha$ be positive integers where $0 \leqq \alpha \leqq l$. For $k = 1, 2, 3, \cdots$ let $n_k^{(0)} = n_k$. Define*

$$S_0 = \exp\{-p/n_\nu^{(\alpha)}\}\ , \quad S_1 = \exp\Big\{-\frac{P}{2n_\nu^{(\alpha)}}\Big(1 + \frac{\log q}{q-1}\Big)\Big\}\ ,$$

*and*

$$W_{k,(\beta)} = n_k^{(\beta)}(n_k^{(\beta)} - 1)(n_k^{(\beta)} - 2) \cdots (n_k^{(\beta)} - p + 1)S^{n_k^{(\beta)}}$$

*where $S_0 < S < S_1$; $0 \leqq \beta \leqq l$; and $k = 1, 2, 3 \cdots$. Then for a fixed $\gamma$ with $0 < \gamma < 1$ there exists an integer $p_0$ depending on $q$ and $l$ such that for $p > p_0$ and $\nu > \nu_0(p)$*

$$\sum_{k \neq \nu-1, \nu, \nu+1} (W_{k,(l)} + W_{k,(l-1)} + \cdots + W_{k,(0)}) < \frac{1}{4}(C_{3l+3}(\gamma))^{-1} W_{\nu,(\alpha)}\ ,$$

*where $C_{3l+3}(\gamma)$ is defined by (7).*

*Proof of Theorem* 1. We note that it suffices to show $f(z)$ assumes zero infinitely often in $|z| < 1$. So suppose $f(z)$ is zero only a finite number of times in $|z| < 1$ and denote these zeros by $z_1, z_2, \cdots, z_j$. Then $N(r, 1/f) = O(1)$, and it follows from the first fundamental theorem of Nevanlinna theory that

$$m(r, f) = m(r, 1/f) + O(1) . \tag{8}$$

For $0 < r < 1$,

$$|f(re^{i\theta})| > \lambda_q\Big(\sum_{k=0}^{\infty} |c_k|^2 r^{2k}\Big)^{1/2} > 0$$

on a set of $\theta$ of measure not less than $\mu_q > 0$ [10, p. 216[1]], so

$$m(r, f) \to \infty \tag{9}$$

as $r$ approaches one by condition (4).

For a value $r'$ for which (6) holds, let $\mathscr{E}(r')$ denote the set of $\theta$ in $[0, 2\pi]$ at which

$$\log^+ \frac{1}{|f(r'e^{i\theta})|} > \frac{1}{2} m(r', 1/f) . \tag{10}$$

Denote the measure of $\mathscr{E}(r')$ by $|\mathscr{E}(r')|$. Then using Hölder's inequality and (6)

$$\begin{aligned}\pi m(r', 1/f) &\leqq \int_{\mathscr{E}(r')} \log^+ \frac{1}{|f(r'e^{i\theta})|} d\theta \\ &\leqq \Big(\int_{\mathscr{E}(r')} \Big(\log^+ \frac{1}{|f(r'e^{i\theta})|}\Big)^p d\theta\Big)^{1/p} |\mathscr{E}(r')|^{1/q} \\ &\leqq (2\pi C)^{1/p} m(r', 1/f) |\mathscr{E}(r')|^{1/q} .\end{aligned}$$

Thus,

$$(\pi/(2\pi C)^{1/p})^q \leqq |\mathscr{E}(r')| .$$

Define $\gamma$ by

$$2\pi\gamma = (\pi/(2\pi C)^{1/p})^q .$$

Let $\rho = \max_{1 \leqq k \leqq j} |z_k|$, and let

$$U = \limsup_{k \to \infty} |c_k| .$$

If $U < \infty$, let $N$ be the least integer such that

$$|c_k| < \frac{3}{2} U$$

for $k > N$. If $U = \infty$, let $N = 0$.

Define

$$\mu(r) = \sup_{k > N} |c_k| r^k , \qquad (0 \leqq r < 1) .$$

Let $V = V(r)$ be the largest integer such that

[1] Theorem 8.20 on page 215 easily extends to finite sums of series with Hadamard gaps, and so Theorem 8.25 on page 216 does also.

$$|c_V| r^V > \frac{1}{2}\mu(r) .$$

If $U = \infty$, we see

$$|c_V| r^V > 1 , \qquad (r > r_0) ,$$

and also $V(r) \to \infty$ as $r \to 1^-$. If $U < \infty$, we see

$$|c_V| r^V > \frac{1}{3} U$$

and again $V(r) \to \infty$ as $r \to 1^-$ since there are infinitely many integers $k$ with

$$|c_k| > \frac{3}{4} U > \frac{1}{2}\mu(r) , \qquad (r < 1) .$$

Using the notation of Lemmas 1 and 3, we choose $p \geqq \max(N, p_0)$ and choose $r$ so close to one that

$$\rho < rS_0 = r \exp\{-p/n_\nu^{(\alpha)}\}$$

where $n_\nu^{(\alpha)} = V(r)$, $n_{\nu-1}^{(l)} > 2p$, and $\nu > \nu_0(p)$. We may assume $r(S_0 S_1)^{1/2}$ is a value $r'$ for which (6) holds. Let

$$T(z) = \sum_{k=\nu-1}^{\nu+1} (a_{n_k^{(l)}} z^{n_k^{(l)}} + \cdots + a_{n_k} z^{n_k}) .$$

By Lemma 3

$$\sum_{k \neq \nu-1, \nu, \nu+1} \left( \sum_{i=0}^{l} |a_{n_k^{(i)}}| r^{n_k^{(i)}} W_{k,(i)} \right) \leqq \mu(r) \sum_{k \neq \nu-1, \nu, \nu+1} (W_{k,(l)} + \cdots + W_{k,(0)}) ,$$
$$\leqq 2 |a_{n_\nu^{(\alpha)}}| r^{n_\nu^{(\alpha)}} \cdot \frac{1}{4} (C_{3l+3}(\gamma))^{-1} W_{\nu,(\alpha)} .$$

Hence,

$$f^{(p)}(rSe^{i\theta}) = T^{(p)}(rSe^{i\theta}) + E(rSe^{i\theta})$$

where

$$|E(rSe^{i\theta})| < \frac{1}{2} (C_{3l+3}(\gamma))^{-1} n_\nu^{(\alpha)} (n_\nu^{(\alpha)} - 1) \cdots (n_\nu^{(\alpha)} - p + 1) |a_{n_\nu^{(\alpha)}}| (rS)^{n_\nu^{(\alpha)} - p} .$$

Consequently,

$$|f^{(p)}(rSe^{i\theta})| \leqq C_1(p, l)(n_\nu^{(\alpha)})^p |a_{n_\nu^{(\alpha)}}| (rS_1)^{n_\nu^{(\alpha)}} ,$$

and using Lemma 2 on the polynomial $T^{(p)}(rSe^{i\theta})$ where $rS$ is a value $r'$ for which (6) is valid we find

$$|f^{(p)}(rSe^{i\theta})| \geqq C_2(p, l, \gamma)(n_\nu^{(\alpha)})^p |a_{n_\nu^{(\alpha)}}| (rS_0)^{n_\nu^{(\alpha)}}$$

for values of $\theta$ in $\mathscr{E}(rs)$.

Therefore if $r' = r(S_0S_1)^{1/2}$ is a value for which (6) holds, then we may apply Lemma 1 to

$$g(\zeta) = f(r(S_0S_1)^{1/2}e^{\zeta+i\theta})$$

with

$$R = \frac{1}{4}\left(1 + \frac{\log q}{q-1}\right)\frac{p}{n_\nu^{(\alpha)}}$$

where $p$ and $\nu$ satisfy the hypotheses of Lemma 3. Then

$$\begin{aligned} R^pA^{p+1}M^{-p} &> C_3(p, q, l, \gamma)\,|a_{n_\nu^{(\alpha)}}|\,(rS_0)^{n_\nu^{(\alpha)}}(S_0/S_1)^{pn_\nu^{(\alpha)}}\,, \\ &> C_4(p, q, l, \gamma)\mu(r)\,, \\ &> C_5(p, q, l, \gamma, U)\,. \end{aligned}$$

Thus $f(z)$ takes every value $w$ in the disc

$$|w - f(r(S_0S_1)^{1/2}e^{i\theta})| < C_6(p, q, l, \gamma, U)\,.$$

But by (10) we note that

$$|f(r(S_0S_1)^{1/2}e^{i\theta})| < \exp\left(-\frac{1}{2}m(r(S_0S_1)^{1/2}), 1/f)\right)\,,$$

and because of (8) and (9) we conclude that when $r$ is near enough to one

$$|f(r(S_0S_1)^{1/2}e^{i\theta})| < C_6(p, q, l, \gamma, U)\,.$$

Thus $f(z)$ will assume the value zero at points arbitrarily near $|z| = 1$ which contradicts our earlier assumption and proves the theorem.

*Proof of Theorem* 2. Suppose there is some radius $R_\varphi$ which is not singular for $f$, and so there exists an $\varepsilon > 0$ such that $|f(z)|$ is bounded in the sector $\mathscr{S} = \{z \mid \varphi - \varepsilon < \arg z < \varphi + \varepsilon, |z| < 1\}$. Then for each complex number $a$, $f(z) - a$ is also bounded in $\mathscr{S}$. Thus taking $2\pi\gamma = 2\varepsilon$, the argument of Theorem 1 shows $f$ assumes $a$ infinitely often in $\mathscr{S}$. Since $a$ is arbitrary, $|f|$ is unbounded in $\mathscr{S}$, and therefore $R_\varphi$ is singular for $f$.

3. *Proof of Lemma* 3. If $n_k^{(\beta)} < p$, then $W_{k,(\beta)} = 0$. Turning to $p \leqq n_k^{(\beta)} < n_\nu^{(\alpha)}$, we first observe that for fixed $\beta$ with $0 \leqq \beta \leqq l$,

$$\frac{n_{k+2}^{(\beta)}}{n_k^{(\beta)}} \geqq q > 1\,, \qquad (k = 1, 2, 3, \cdots)\,.$$

Assume $W_{k+2,(\beta)} \neq 0$ and $n_{k+2}^{(\beta)} \leqq n_\nu^{(\alpha)}$. Then

$$(11)\quad \begin{aligned} \frac{W_{k,(\beta)}}{W_{k+2,(\beta)}} &\leqq \left(\frac{n_k^{(\beta)}}{n_{k+2}^{(\beta)}}\right)^p (S_0)^{-n_{k+2}^{(\beta)}+n_k^{(\beta)}} \leqq \left(\frac{n_k^{(\beta)}}{n_{k+2}^{(\beta)}}\right)^p \exp\left\{p\left(1-\frac{n_k^{(\beta)}}{n_{k+2}^{(\beta)}}\right)\right\}, \\ &< \sup_{0<t<1/q} \exp\{p(1-t+\log t)\} \\ &\leqq \exp\{p(1-q)-\log q\}. \end{aligned}$$

Hence the right-hand side of (11) is less than $A^{-1}=(1+16(l+1)C_{3l+3}(\gamma))^{-1}$ provided

$$p > \frac{\log A}{\log q - 1 + q^{-1}} = p_1 .$$

Proceeding in a similar manner, we may also show

$$\frac{W_{\nu-3,(\beta)}}{W_{\nu,(\alpha)}} \leqq \frac{1}{A} \quad \text{and} \quad \frac{W_{\nu-2,(\beta)}}{W_{\nu,(\alpha)}} \leqq \frac{1}{A}$$

when $p > p_1$. Consequently for $W_{k,(\beta)} \neq 0$ where $k \leqq \nu - 3$ and $k+2n=\nu-1$, we see

$$(12)\quad \frac{W_{k,(\beta)}}{W_{k+2,(\beta)}} \cdot \frac{W_{k+2,(\beta)}}{W_{k+4,(\beta)}} \cdots \frac{W_{\nu-5,(\beta)}}{W_{\nu-3,(\beta)}} \frac{W_{\nu-3,(\beta)}}{W_{\nu,(\alpha)}} \leqq \left(\frac{1}{A}\right)^n, \quad (0 \leqq \beta \leqq l),$$

and for $W_{k,(\beta)} \neq 0$ where $k \leqq \nu - 2$ and $k + 2n = \nu$ we see

$$(13)\quad \frac{W_{k,(\beta)}}{W_{k+2,(\beta)}} \cdot \frac{W_{k+2,(\beta)}}{W_{k+4,(\beta)}} \cdots \frac{W_{\nu-4,(\beta)}}{W_{\nu-2,(\beta)}} \cdot \frac{W_{\nu-2,(\beta)}}{W_{\nu,(\alpha)}} \leqq \left(\frac{1}{A}\right)^n, \quad (0 \leqq \beta \leqq l).$$

Using (12) and (13) provided $p > p_1$, we get

$$(14)\quad \begin{aligned} \sum_{k<\nu-1} (W_{k,(l)} + W_{k,(l-1)} + \cdots + W_{k,(0)}) &< \frac{2(l+1)}{A} \sum_{n=0}^{\infty} \left(\frac{1}{A}\right)^n W_{\nu,(\alpha)}, \\ &\leqq \frac{2(l+1)}{A-1} W_{\nu,(\alpha)}. \end{aligned}$$

Now for $k \geqq \nu > \nu_0(p)$ and $x$ any integer with $0 < x \leqq p - 1$, we have simultaneously

$$\frac{n_{k+2}^{(\beta)} - x}{n_k^{(\beta)} - x} < 2^{1/p}\left(\frac{n_{k+2}^{(\beta)}}{n_k^{(\beta)}}\right), \quad \frac{n_{\nu+2}^{(\beta)} - x}{n_\nu^{(\alpha)} - x} < 2^{1/p}\left(\frac{n_{\nu+2}^{(\beta)}}{n_\nu^{(\alpha)}}\right),$$

and

$$\frac{n_{\nu+3}^{(\beta)} - x}{n_\nu^{(\alpha)} - x} < 2^{1/p}\left(\frac{n_{\nu+3}^{(\beta)}}{n_\nu^{(\alpha)}}\right).$$

Then when $n_k^{(\beta)} \geqq n_\nu^{(\alpha)}$,

$$\begin{aligned} \frac{W_{k+2,(\beta)}}{W_{k,(\beta)}} &< 2\left(\frac{n_{k+2}^{(\beta)}}{n_k^{(\beta)}}\right)^p S^{n_{k+2}^{(\beta)}-n_k^{(\beta)}}, \\ &< 2t^p(S_1^{n_k^{(\beta)}})^{(t-1)}, \end{aligned}$$

$$\leqq 2t^p\left\{\exp\left(-\frac{1}{2}\left(1+\frac{\log q}{q-1}\right)\right)\right\}^{p(t-1)} = \theta(t)$$

where $t = n_{k+2}^{(\beta)}/n_k^{(\beta)} \geqq q$. For $t \geqq q$, $\theta(t) \leqq \theta(q)$, so

$$\frac{W_{k+2,(\beta)}}{W_{k,(\beta)}} \leqq 2\exp\left\{-\frac{1}{2}p(q-1-\log q)\right\}, \tag{15}$$

when $n_k^{(\beta)} \geqq n_\nu^{(\alpha)}$. Hence the right-hand side of (15) is less than $1/A$ provided

$$p > \frac{2\log(2A)}{q-1-\log q} = p_2 .$$

Proceeding in a similar manner we may also show

$$\frac{W_{\nu+2,(\beta)}}{W_{\nu,(\alpha)}} \leqq \frac{1}{A} \quad \text{and} \quad \frac{W_{\nu+3,(\beta)}}{W_{\nu,(\alpha)}} \leqq \frac{1}{A},$$

when $p > p_2$. Consequently for $k \geqq \nu + 2 \geqq \nu_0(p) + 2$ and $k = \nu + 2n$, we see

$$\frac{W_{k,(\beta)}}{W_{k-2,(\beta)}} \cdot \frac{W_{k-2,(\beta)}}{W_{k-4,(\beta)}} \cdots \frac{W_{\nu+4,(\beta)}}{W_{\nu+2,(\beta)}} \frac{W_{\nu+2,(\beta)}}{W_{\nu,(\alpha)}} \leqq \left(\frac{1}{A}\right)^n, \quad (0 \leqq \beta \leqq l), \tag{16}$$

and for $k \geqq \nu + 3 > \nu_0(p) + 3$ and $k = \nu + 2n + 1$, we see

$$\frac{W_{k,(\beta)}}{W_{k-2,(\beta)}} \cdot \frac{W_{k-2,(\beta)}}{W_{k-4,(\beta)}} \cdots \frac{W_{\nu+5,(\beta)}}{W_{\nu+3,(\beta)}} \frac{W_{\nu+3,(\beta)}}{W_{\nu,(\alpha)}} \leqq \left(\frac{1}{A}\right)^n, \quad (0 \leqq \beta \leqq l). \tag{17}$$

Using (16) and (17) provided $p > p_2$, we get

$$\begin{aligned} \sum_{k>\nu+1} (W_{k,(l)} + W_{k,(l-1)} + \cdots + W_{k,(0)}) &< \frac{2(l+1)}{A} \sum_{n=0}^{\infty} \left(\frac{1}{A}\right)^n W_{\nu,(\alpha)}, \\ &\leqq \frac{2(l+1)}{A-1} W_{\nu,(\alpha)} . \end{aligned} \tag{18}$$

Combining (14) and (18) we now have the lemma provided $p_0$ is the maximum of $p_1$ and $p_2$ (and remembering that $A = 1 + 16(l+1)C_{3l+3}(\gamma)$).

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IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY
AND
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# INTERPOLATION SETS FOR UNIFORM ALGEBRAS

ARNE STRAY

**Let $A$ be a uniform algebra on a compact Hausdorff space $X$ and let $E \subset X$ be a closed subset which is a $G_\delta$. Denote by $B_E$ all functions on $X \backslash E$ which are uniform limits on compact subsets of $X \backslash E$ of bounded sequences from $A$.**

**It is proved that a relatively closed subset $S$ of $X \backslash E$ is an interpolation set and an intersection of peak sets for $B_E$ if and only if each compact subset of $S$ has the same property w. r. t. $A$. In some special cases the interpolation sets for $B_E$ are characterized in a similar way. A method for constructing infinite interpolation sets for $A$ and $B_E$ whenever $x \in E$ is a peak point for $A$ in the closure of $X \backslash \{x\}$, is presented.**

With $X$ as above let $S \subset X$ be a topological subspace. Then $C_b(S)$ denotes all bounded continuous complexvalued functions on $S$ and we put $||f|| = \sup \{|f(x)|: x \in S\}$ if $f \in C_b(S)$.

A subset $S$ of $X \backslash E$ closed in the relative topology is called an interpolation set for $B_E$ if any $f \in C_b(S)$ has an extension to $X \backslash E$ which belongs to $B_E$. If there exists $f \in B_E$ such that $f = 1$ on $S$ and $|f| < 1$ on $(X \backslash E) \backslash S$, we call $S$ a peak set for $B_E$. If $S$ has both this properties it is called a peak interpolation set for $B_E$. Peak and interpolation sets for $A$ are defined in the same way.

It is easy to see that $B_E$ is a Banach algebra with the norm $N(f) = \inf \{\sup_n ||f_n||: \{f_n\} \subset A, f_n \to f$ uniformly on compact subsets of $X \backslash E\}$. It is an interesting problem in itself when this norm coincides with sup norm on $X \backslash E$.

In case $X = \{z: |z| \leqq 1\}$ and $A$ is the classical disc algebra of all continuous functions on $X$ which are analytic in $D = \{z: |z| < 1\}$ the interpolation sets for $B_E$ (where $E$ is a closed subset of $\partial X$) are characterized by that $S \cap \partial X$ has zero linear measure and that $S \cap D$ is an interpolation set for $H^\infty(D)$, the algebra of all bounded analytic functions on $D$. This result was obtained in [8] by E. A. Heard and J. H. Wells.

Their work has been generalized in different ways. Various authors have considered more general subsets $E$ of $\{z: |z| \leqq 1\}$ and more general algebras of analytic functions. ([2], [3], [4], [6], [9] and [10]).

In this note we wish to generalize the results of Heard and Wells to the setting of uniform algebras. We start with an extension of Theorem 2 in [8].

THEOREM 1. *Let $S \subset X \backslash E$ be closed in the relative topology. Assume $X$ is the maximal ideal space of $A$. The following statements are equivalent*:

( i ) *Given $g \in C_b(S)$, $\varepsilon > 0$ and an open set $U \supset S$, there exists $f \in B_E$ such that $f = g$ on $S$, $\|f\| = \|g\|$, $|f| < \varepsilon$ on $(X \backslash E) \backslash U$ and $N(f) \leqq \|g\|(1 + \varepsilon)$.*

(ii) *There exists a constant $M$ such that if $g \in C_b(S)$, $\varepsilon > 0$ and $U \supset S$ is open we can find $f \in B_E$ such that $f = g$ on $S$, $|f| < \varepsilon$ on $(X \backslash E) \backslash U$ and $N(f) \leqq M\|g\|$.*

(iii) *Each compact subset of $S$ is an interpolation set and an intersection of peak sets for $A$.*

*Proof.* That (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Choose $g \in C(K)$ with $\|g\| = 1$.

Let $K \subset S$ be compact, $U$ and $W$ open sets such that $K \subset W \subset \bar{W} \subset U \subset \bar{U} \subset X \backslash E$ and choose $\varepsilon > 0$. By hypothesis there exists $g_1 \in B_E$ equal to $g$ on $K$ such that $|g_1| < \varepsilon/2$ on $\bar{U} \backslash W$ and $N(g_1) \leqq M$.

Hence we can find $g_2 \in A$ with $\|g_2\| \leqq M$, $|g - g_2| < \varepsilon$ on $K$, $|g_2| < \varepsilon$ on $\bar{U} \backslash W$ and $\|g_2\| \leqq M$. By ([8], Lemma 2) applied to the restriction map $B_E \rightarrow C(K)$ we get that any $g \in C(K)$ we get that any $g \in C(K)$ has an extension $f$ to $X$ such that $f \in A$, $\|f\| \leqq M/(1 - \varepsilon)$ and $|f| < \varepsilon/(1 - \varepsilon)$ on $\bar{U}/W$. Essentially by Bishops "1/4 − 3/4-Theorem" (See [5], Th. 11.1 p. 52) we can use what is proved until now to find a compact set $K_1$ and $f_1 \in A$ such that $f_1 = 1$ on $K_1$, $|f_1| < 1$ on $U \backslash K_1$ and $K \subset K_1 \subset W$. By "Rossis Local Peak Set Theorem" ([5], p. 91) $K_1$ is a peak set for $A$ and (iii) is proved.

It remains to prove (iii) $\Rightarrow$ (i). We only indicate how to modify our proof of Lemma 2.1 in [10] to apply to the present situation. As in that lemma we construct a sequence $\{f_n\}_{n=1}^{\infty} \subset A$ with the properties listed there. Let $t \in \langle 0, 1 \rangle$. The sum $\sum_1^{\infty} f_n = f \in B_E$ and the proof of Lemma 2.1 gives (i) if we can show that $N(f) \leqq 1 + t$. This is obtained by constructing $\{f_n\}$ such that $\|f_n + f_{n+1}\| \leqq 1 + 1/2 \cdot t$ for $n = 0, 1, \cdots$.

This can be obtained if when constructing $f_{n+1}$ we arrange it so that $|f_n + f_{n+1}| = |f_n| + |f_{n+1}|$ on $K_{n+1} \cup K_{n+2}$ ($K_{n+1}, K_{n+2}$ as in [10]) and then if needed, modify $f_{n+1}$ to $h \cdot f_{n+1}$ where $h \in A$ equals $1 = \|h\|$ on $K_{n+1} \cup K_{n+2} \cup K_{n+3}$, is small where $|f_n + f_{n+1}|$ may be large and has a small imaginary part.

We now state a lemma which is due to A. M. Davie:

LEMMA 1. *There exists a sequence $\{Q_k\}_{k=1}^{\infty}$ of polynomials with the following properties*:

(1) $\sum_1^n Q_k(z) \rightarrow 1$ *uniformly on compact subset of* $\{z: |z| < 1\}$

(2) $Q_k(1) = 0$ *for* $k = 1, 2, \cdots$ *and* $\sum_1^{\infty} |Q_k(z)| \leqq 3$ *if* $|z| \leqq 1$.

For a construction of $\{Q_k\}$ see the proof of Theorem 2.4 in [1]. We now have:

THEOREM 2. *Let $E$ be a peak set for $A$ and let $S \subset X\backslash E$ be closed in the relative topology. The following statements are equivalent*:

(i) *$S$ is an interpolation set for $B_E$.*

(ii) *There exists $M > 0$ such that if $K \subset S$ is compact and $g \in C(K)$ we can find $f \in A$ equal to $g$ an $K$ and with $||f|| \leqq M||g||$.*

*Proof.* (ii) follows from (i) as in the first part of the proof that (ii) $\Rightarrow$ (iii) in Theorem 1. For the converse an argument used by Davie in [1] works: Choose $h \in A$ peaking on $E$ and put $E_k = S \cap \{x: |Q_k \circ h(x)| \geqq \varepsilon \cdot h^{-k}\}$ where $\varepsilon > 0$ is given in advance. Let $g \in C_b(S)$ with $||g|| = 1$. Choose by hypothesis $g_k \in A$ equal to $g$ on $E_k$ with $||g_k|| \leqq M$ and put $G = \sum_{k=1}^{\infty} (Q_k \circ h) \cdot g_k$. Then by Lemma 1 $G \in B_E$, $||G|| \leqq 3M$ and if $x \in S$ we have

$$|G(x) - g(x)| = \left|\sum_1^\infty (g_k(x) - g(x))Q_k \circ f(x)\right|$$
$$\leqq \sum_1^\infty \in 2^{-k} = \varepsilon \, .$$

By Lemma 2 in [8] (i) follows.

The hypothesis that $E$ is a peak set for $A$ seems unnecessary, but we needed it to apply Lemma 1. It would be of interest to get some examples where Theorem 2 holds without assuming $E$ to be a peak set.

A case which deserves investigation is when $A$ is an algebra of generalized analytic functions ([5], Ch VII) viewed as a uniform algebra on its maximal ideal space. Then $B_E$ is very easy to describe whenever $E$ is a closed subset of the Šilov boundary of $A$. In particular the norm $N(f)$ coincides with sup norm on $X\backslash E$ in this case.

We want to give two examples where a more detailed description of the interpolation sets for $B_E$ can be given.

(a) Let $U \subset \boldsymbol{C}^n$ be a strictly pseudoconvex domain with $C^2$ boundary and let $X$ be the closure of $U$. Let $A$ be the algebra $A(U) = \{f \in C(X): f|_U$ is analytic$\}$.

In this case Theorem 2 is valid if $E$ is any closed subset $\partial U$ and the interpolation set $S$ can then also be characterized by the following:

(I): Each compact subset of $S \cap \partial U$ is a peak interpolation set for $A$,

and

(II): $S \cap U$ is an interpolation set for $H^\infty(U)$, the algebra of all bounded analytic functions in $U$.

For a proof of this note that (i) $\Rightarrow$ (ii) in Theorem 2 holds whenever $E$ is a closed $G_\delta$. That (ii) $\Rightarrow$ (II) is a simple normal family argument and I also follows from (ii) by a result of N. H. Varopoulos [11] and since each $x \in \partial U$ is a peak point for $A(U)$ in this special case.

To obtain (i) from (I) and (II) one can argue as in the proof of Theorem 2.2 in [10]. To use that proof one needs an approximation result similar to Theorem 2.1 in [10]. This nontrivial result is contained in a recent work of R. M. Range [9].

(b) Assume $A$ is a Dirichlet algebra on its Šilov boundary $Y$.

Let $E$ be a peak interpolation set for $A$ and let $S \subset X \backslash E$ be closed in the relative topology and assume $S \backslash Y$ countable. Then one can prove that $S$ is an interpolation set for $B_E$ if each compact subset of $S \cap Y$ is an interpolation set for $A$ and if for some constant $C$ the following result holds: If $P$ is a nontrivial Gleason part for $A$ and $S \cap P = z_1, z_2, \cdots$ and $\alpha_1, \alpha_2, \cdots$ are numbers such that $|\alpha_k| \leqq 1$ for $k = 1, 2, \cdots$ there exists $f \in H^\infty(P)$ such that $f(z_k) = \alpha_k$ for $k = 1, 2, \cdots$ and $|f| \leqq C$ on $P$. (For the necessary definitions see [5] on page 34, 142 and 161).

Using this hypothesis and the Wermer-Glicksberg decomposition ([5], Thm. 7.11, p. 45) we can prove that $S \cup E$ is an interpolation set for $A$. This is done in the same way as Glicksberg proves Theorem 4.1 in [7]. But then $S$ is an interpolation set for $B_E$ by Theorem 2.

In [8] Heard and Wells described an explicit method for constructing infinite interpolation sets for $B_{\{x\}}$ if $x \in X$ is a non-isolated peak point for $A$. Their method didn't depend on Carlesons characterization of the interpolating sequences for $H^\infty(D)$.

We indicate here how the polynomials $\{Q_k\}$ can be used for a similar construction avoiding an unnecessary hypothesis about connectedness which Heard and Wells assumed. ([8], Theorem 3).

THEOREM 3. *Let $x \in X$ be a peak point for $A$ and $P \subset X \backslash \{x\}$ a set which contains $x$ in its closure. Then an infinite interpolation set for $B_{\{x\}}$ contained in $P$ can be constructed.*

*Proof.* Choose $\varepsilon > 0$ and $f \in A$ peaking at $x$. For $k = 1, 2, \cdots$ choose numbers $n_k$ and $m_k$ such that $n_k < m_k < n_{k+1}$ and put $H_k = \sum_{n_k}^{m_k} Q_j \circ f$. Using Lemma 1 it is easy to see that we can arrange it such that the sets $E_k = \{x: |H_k(x)| \geqq \varepsilon 2^{-k}\}$ and

$$B_k = P \cap \{x: |H_k(x) - 1| < \varepsilon 2^{-k}\}$$

are nonempty for $k = 1, 2, \cdots$ and that $E_i \cap E_j = \varnothing$ if $i \neq j$.

If we choose $x_k \in B_k$ for $k = 1, 2, \cdots$ then $S = \{x_k\}_{k=1}^{\infty}$ is an interpolation set for $B_{\{x\}}$. For if $g \in C_b(S)$ and we put $G = \sum_1^\infty g(x_k)H_k$ then $G \in B_{\{x\}}$, $||G|| \leqq 3||g||$ by Lemma 1 and $|G - g| < \varepsilon||g||$ on $S$.

Comments on Theorem 2:

We want to point out that the hypothesis that $E$ be a peak set cannot be omitted. If $A$ is any uniform algebra for which there exists an infinite interpolation set $F$ not meeting the Šilov boundary, one obtains a counterexample by taking $E$ to be a limit point of $F$ and $S = F \backslash E$. For an example of such an algebera $A$ we refer to Theorem 2.8. in [1]. On the other hand A. M. Davie has recently proved (private communication) that in case $A$ is the algebra $R(X)$ and $X$ is a compact plane set, Theorem 2 is valid without assuming $E$ to be a peak set.

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# APPLICATIONS OF RANDOM FOURIER SERIES OVER COMPACT GROUPS TO FOURIER MULTIPLIERS

ALESSANDRO FIGÀ-TALAMANCA AND J. F. PRICE

**The Fourier series of a function on a compact group can be "randomized" by operating on each of the Fourier coefficients by independent random unitary operators. In this paper the theory of random Fourier series is used to prove several new results for a type of Rudin-Shapiro sequence and for Fourier multipliers. Thus in §2 it is shown in effect that $\mathfrak{M}(L^p, L^q) \subseteq \mathfrak{M}(L^2, L^2)$ for all $p, q \in [1, \infty]$ except for the pair $(p, q) = (\infty, 1)$, while in §3 the theory of random Fourier series is used to construct a type of Rudin-Shapiro sequence. This sequence is then used in §4 to obtain, for compact groups in one case, and compact Lie groups in another, slightly more restricted versions of several known families of strict inclusions for Fourier multipliers over compact Abelian groups.**

**1. Notation and preliminaries.** Throughout this paper we suppose that $G$ is a compact group (always Hausdorff) with normalized Haar measure $\lambda_G$ and that $\Gamma$ is the set of equivalence classes of continuous irreducible unitary representations of $G$. The spaces of $p$-integrable functions, continuous functions and Radon measures over $G$ will be denoted by $L^p(G)$, $C(G)$ and $M(G)$ [or $L^p$, $C$ and $M$] respectively, while their respective norms will be denoted by $\|\cdot\|_p$, $\|\cdot\|_\infty$ and $\|\cdot\|_M$. We will identify each function with the measure which it generates.

If $\mu \in M(G)$, then $\mu$ is uniquely represented by the Fourier series

$$\mu \sim \sum_{\gamma \in \Gamma} d(\gamma) tr[\hat{\mu}(D_\gamma) D_\gamma(\cdot)] ,$$

where: $D_\gamma$ is a representative (which we assume to be fixed throughout the sequel) of the class $\gamma \in \Gamma$; $d(\gamma)$ is the (finite) dimension of $\gamma$; $tr$ denotes the usual trace; and $\hat{\mu}$ is the Fourier transform of $\mu$ with respect to $\{D_\gamma : \gamma \in \Gamma\}$, that is

$$\hat{\mu}(D_\gamma) = \int_G D_\gamma(x)^* d\mu(x) ,$$

for each $\gamma \in \Gamma$, $D_\gamma(x)^*$ denoting the Hilbert adjoint of $D_\gamma(x)$.

Let $H_\gamma$ denote the Hibert space of dimension $d(\gamma)$ corresponding to the representation $D_\gamma$, and let $\mathfrak{E}$ denote the set consisting of functions $W$ on $\Gamma$ such that $W(\gamma)$ is an endomorphism of $H_\gamma$ for each $\gamma$. We can now define the "randomizing group" for $G$. Let $\mathscr{G}$ denote the product group $\prod_{\gamma \in \Gamma} \mathscr{U}(H_\gamma)$, where $\mathscr{U}(H_\gamma)$ is the compact group of unitary endomorphisms of $H_\gamma$. Clearly $\mathscr{G}$ may thought of as a subset of $\mathfrak{E}$. Whenever $\mu \in M(G)$ and $U \in \mathscr{G}$ we denote the series

$$\sum_{\gamma \in \Gamma} d(\gamma) tr[\hat{\mu}(D_\gamma)\, U(\gamma) D_\gamma]$$

by $\mu^U$. The following two results are basic to this paper.

THEOREM 1.1. *Suppose that $\mathscr{G}$ is equipped with its Haar measure and that $\mu \in M(G)$ has the property that $\mu^U$ represents a measure for every $U$ in a subset of $\mathscr{G}$ with positive measure; then $\mu$ is in $L^2(G)$.*

THEOREM 1.2. *Suppose that $f \in L^2(G)$. Then $f^U$ is the Fourier series of a function in $\bigcap_{1 \leqq p < \infty} L^p(G)$ for almost every $U$ in $\mathscr{G}$, where $\mathscr{G}$ is equipped with its Haar measure.*

The above two theorems are due to Figà-Talamanca and Rider (see [4, (36.18)] and [2] or [4, (36.5)]).

MULTIPLIERS 1.3. If $A$ and $B$ are any two spaces selected from $L^p(G)$, $1 \leqq p \leqq \infty$, $C(G)$ and $M(G)$, we define $\mathfrak{M}(A, B)$ to be the set of $W \in \mathfrak{E}$ such that

$$\sum_{\gamma \in \Gamma} d(\gamma) tr[w(\gamma)\hat{\mu}(D_\gamma) D_\gamma]$$

is the Fourier series of an element in $B$ (we will denote this element by $T_W\mu$) whenever $\mu$ belongs to $A$. Clearly the operator $\mu \mapsto T_W\mu_a$ is linear, while its continuity is an immediate consequence of the closed graph theorem. Thus we define a norm on $\mathfrak{M}(A, B)$ as the usual operator norm on the set $\{T_W\colon W \in \mathfrak{M}(A, B)\}$, and we denote this set by $M(A, B)$.

## 2. Multipliers and pseudomeasures.

Let $\mathfrak{E}_\infty$ denote the subset of $\mathfrak{E}$ consisting of elements $W$ such that

$$\| W \|_\infty = \sup\{\| W(\gamma) \| \colon \gamma \in \Gamma\} < \infty ,$$

where $\| W(\gamma) \|$ denotes the usual operator norm for endomorphisms of $H_\gamma$. Whenever $G$ has the property that $\sup\{d(\gamma)\colon \gamma \in \Gamma\}$ is finite [for example, if $G$ is Abelian] and $A$, $B$ are selected from $L^p(G)$, $C(G)$ and $M(G)$, then it is banal to show that

$$\mathfrak{M}(A, B) \subseteqq \mathfrak{E}_\infty , \tag{2.1}$$

(see [4, Theorem (35.4), part IV]) and hence that each $T \in M(A, B)$ may be written in the form $T\colon f \mapsto f * \mu$, where $\mu$ is a pseudomeasure (see [6, §2.2]).

The inclusion (2.1) is known to be valid for some pairs $A$, $B$ over an unrestricted compact group (see, for example, the table on pp. 410–411 of Hewitt and Ross [4]) and in this section we extend its validity to some further pairs, thus completing five squares of Hewitt and Ross's table.

THEOREM 2.1. *Suppose that* $(A, B)$ *is one of the pairs* $(L^p, L^q)$, $(L^q, L^1)$, $(L^q, M)$, $(L^\infty, L^p)$ *or* $(C, L^p)$ *where* $1 < p < 2 < q < \infty$; *then*

$$\mathfrak{M}(A, B) = \mathfrak{C}_\infty \,. \tag{2.2}$$

REMARKS 2.2. (1) Four cases remain open: $(L^\infty, L^1)$, $(L^\infty, M)$, $(C, L^1)$ and $(C, M)$. We were not able to decide whether (2.1), and hence (2.2), is generally true for these cases. It is straightforward to show that $\mathfrak{M}(L^\infty, M) = \mathfrak{M}(C, M) = \mathfrak{M}(C, L^1)$. Also whenever $S \subseteqq \Gamma$ has the property that $\sup\{d(\gamma) \colon \gamma \in \Gamma\} = \infty$, it is not true that there exists $W \in \mathfrak{C} \backslash \mathfrak{C}_\infty$ with supp $W \subset S$ such that $W \in \mathfrak{M}(C, L^1)$ (cf. Theorem (35.4), part V, of Hewitt and Ross [4]). For example, when $S$ is a $\Lambda(p)$ set for some $p > 1$, Theorem 2.1 above applies to show that whenever $W \in \mathfrak{M}(C, L^1)$ has the property that supp $W \subseteqq S$, then $W \in \mathfrak{C}_\infty$; examples are known of sets $S$ which are $\Lambda(p)$ for all $p > 1$ and yet $\sup\{d(\gamma) \colon \gamma \in S\} = \infty$ (see Remark 10 of [2] or (37.11) (a) of [4]).

(2) There can be no analogue of Theorem 2.1 for non-compact locally compact Abelian groups. For example, if $G$ is a non-compact $LCA$ group and $1 \leqq p < q \leqq \infty$, then there exists a multiplier operator from $L^p(G)$ into $L^q(G)$ which cannot be written as convolution with a pseudomeasure; see Larsen [5, Theorem 5.5.5].

*Proof of* 2.1. By inspection of Table (36.20) of [4], it is clear that to prove equality in (2.2) we need only show that $\mathfrak{M}(A, B) \subseteqq \mathfrak{C}_\infty$. Suppose that $1 < p < 2 < q < \infty$ and that $W \in \mathfrak{M}(L^q, M)$, that is, that $W\hat{f} \in \hat{M}$ for all $f \in L^q$. Since $2 < q < \infty$, whenever $f \in L^q$, then

$$\hat{f}U \colon \gamma \longmapsto \hat{f}(D_\gamma)\, U(\gamma)$$

is the Fourier transform of an $L^q$ function for a set of $U$ in $\mathscr{G}$ of measure 1 (Theorem 1.2). In this case $W\hat{f}U$ is the Fourier transform of a measure for all such $U$ and so, by Theorem 1.1, $W\hat{f}$ must be the Fourier transform of an $L^2$ function. Thus $W \in \mathfrak{M}(L^q, L^2)$ and since it is know that $\mathfrak{M}(L^q, L^2) = \mathfrak{M}_\infty[4]$, we have proved (2.2) for the pairs $(L^q, L^p)$, $(L^q, L^2)$ and $(L^q, M)$.

If $\mathfrak{F}$ is a subset of $\mathfrak{C}$, write $\mathfrak{F}^* = \{W^* \colon W \in \mathfrak{F}\}$, where $W^*$ is defined by $\gamma \mapsto W(\gamma)^*$. Since we have just seen that $\mathfrak{M}(L^q, M) = \mathfrak{C}_\infty$ and since it is obvious that $(\mathfrak{C}_\infty)^* = \mathfrak{C}_\infty$, the proof of (2.2) can be completed by showing

$$\mathfrak{M}(C, L^{q'}) \subseteqq \mathfrak{M}(L^q, M)^* . \tag{2.3}$$

However (2.3) is a simple consequence of the theory of adjoint operators. For if $W \in \mathfrak{M}(C, L^{q'})$ we can define $T_W^*: L^q \to M$ by

$$\int_G \bar{g} d(T_W^* f) = \int_G (T_W g)^- f d\lambda_G$$

for $f \in L^q$, $g \in C$. Thus, whenever $f \in L^q$ and $g$ is a trigonometric polynomial,

$$\begin{aligned}
&\textstyle\sum_\gamma d(\gamma) tr[\hat{g}(D_\gamma)^* (T_W^* f)^\wedge (D_\gamma)] \\
&= \int_G \bar{g} d(T_W^* f) = \int_G (T_W g)^- f d\lambda_G \\
&= \textstyle\sum_\gamma d(\gamma) tr[(T_W g)^\wedge (D_\gamma)^* \hat{f}(D_\gamma)] \\
&= \textstyle\sum_\gamma d(\gamma) tr[\hat{g}(D_\gamma)^* W^*(\gamma) \hat{f}(D_\gamma)] .
\end{aligned}$$

Thus $(T_W^* f)^\wedge (D_\gamma) = W^*(\gamma)\hat{f}(D_\gamma)$ for all $f$ in $L^q$ showing that $W^* \in \mathfrak{M}(L^q, M)$, from which follows the required validity of (2.3).

We now look at the inclusion relation opposite to (2.1). The following simple proposition will describe exactly the cases when we have

$$\mathfrak{E}_\infty \subseteqq \mathfrak{M}(L^p, L^q) . \tag{2.4}$$

PROPOSITION 2.3. *Suppose that $G$ is infinite; then the inclusion* (2.4) *is valid if and only if $q \leqq 2 \leqq p$.*

*Proof.* (i) Il $q \leqq 2 \leqq p$, then $L^q \supseteqq L^2 \supseteqq L^p$ and so $\mathfrak{M}(L^p, L^q) \supseteqq \mathfrak{M}(L^2, L^2)$. However $\mathfrak{M}(L^2, L^2) = \mathfrak{E}_\infty$ and so (2.4) is satisfied.

(ii) On the other hand, suppose that $p < 2$ and that (2.4) is valid. Then certainly $\mathscr{G} \subseteqq \mathfrak{M}(L^p, L^q)$ and a straightforward application of Theorem 1.1 implies that $L^p \subseteqq L^2$, an absurdity when $G$ is infinite compact.

(iii) Finally we have the case $2 < q \leqq \infty$ and $2 \leqq p \leqq \infty$. If we also suppose $q \neq \infty$, then

$$\mathfrak{M}(L^p, L^q) \subseteqq \mathfrak{M}(C, L^q) \subseteqq \mathfrak{M}(L^{q'}, M)^*$$

by (2.3), and the proof proceeds as in paragraph (ii). The case $q = \infty$ follows easily from the inclusions.

$$\mathfrak{M}(L^p, L^\infty) \subseteqq \mathfrak{M}(L^\infty, L^\infty) = M(G)^\wedge .$$

**3. Rudin-Shapiro sequences.** Let $G$ be a compact group and $t$ any number in $(2, \infty]$. By a Rudin-Shapiro sequence of type $t$ (briefly, a $t$-*RS*-sequence) we shall mean a sequence $(h_n)_{n \in N}$, where $N = \{1, 2, \cdots\}$, of functions in $L^t(G)$ with the properties

$$\begin{cases} \inf \|h_n\|_2 > 0 , & \sup \|h_n\|_t < \infty , \\ \lim \|\hat{h}_n\|_\infty = 0 . \end{cases} \tag{3.1}$$

(Recall that by $\|\hat{h}_n\|_\infty$ we mean $\sup\{\|\hat{h}_n(D_\gamma)\| : \gamma \in \Gamma\}$.)

When $t = \infty$ the above definition is essentially that of the Rudin-Shapiro sequences discussed, for example, in Gaudry [3] (where it is shown that $\infty$-$RS$-sequences exist for all non-discrete locally compact Abelian groups) and in Edwards and Price [1, §5.4 and §§A.1–A.4] (where further sufficient conditions are given for the existence of $\infty$-$RS$-sequences). In this section we show that $t$-$RS$-sequences, $t < \infty$, exist for all infinite compact groups. However, we would point out that the proof is completely existential in nature. Similarly to [1, §5.4], it is easy to see that if $(h_n)$ satisfies (3.1), then we can construct a sequence $(k_n)$ from $(h_n)$ with the properties

$$\begin{cases} \|k_n\|_{t'} \geqq B_1 n \\ \|k_n\|_s \leqq B_2^{1/s} n & (t' \leqq s \leqq t) \\ \|\hat{k}_n\|_\infty \leqq 2^{-n} , \end{cases} \tag{3.2}$$

where $B_1$ and $B_2$ are strictly positive numbers independent of $n$.

LEMMA 3.1. (a) *Let $G$ be an infinite compact group and let $t \in (2, \infty)$. Then there exists a Rudin-Shapiro sequence $(h_n)$ of type $t$. Without loss of generality we can take $(h_n)$ with $\|h_n\|_2 = 1$ for all $n \in N$.*

(b) *Moreover, if $G$ is also a Lie group, then there exists a second $t$-$RS$-sequence, $(h_n^*)$ say, with $\|h_n\|_2 = \|h_n^*\|_2 (= 1)$, $h_n * h_n^* = h_n^* * h_n$, and a positive nonzero number $\rho$ independent of $n$ such that*

$$\rho^{1+1/p} \|\hat{h}_n\|_\infty^{2/p} \leqq \|h_n^* * h_n\|_p \leqq \|\hat{h}_n\|_\infty^{2/p} , \tag{i}$$

*for all $n \in N$, and $1 \leqq p \leqq 2$.*

REMARK 3.2. When $G$ is the circle group (the simplest compact Lie group) the original Rudin-Shapiro sequence $(\phi_n)$ consists of trigonometric polynomials such that $\hat{\phi}_n$ takes only the values $\pm 1$ on its support $[0, 2^n]$. One might suspect that in this case Lemma 3.1 (b) would be satisfied by taking $h_n = h_n^* = \phi_n/\|\phi_n\|_2$. Certainly (i) is satisfied (with $\rho = 1$) but however (ii) is not since $\|\hat{h}_n\|_\infty^2 = \|\phi_n\|_2^{-2}$, whereas

$$\|h_n^* * h_n\|_1 = \left\| \sum_{m=0}^{2^n} e^{imx} \right\|_1 / \|\phi_n\|_2^{-2} \sim \log 2^n \|\phi_n\|_2^{-2} .$$

This difference is not essential: by convolving the $n$th term of the classical Rudin-Shapiro sequence with the Fejér kernel of order $2^n$ one obtains sequences which, after normalizing, satisfy part (b) of the lemma. This depends on the fact that for $p > 1$ Fejér kernel and the Dirichlet kernel have essentially the same $L^p$ norms. For our purposes Rudin-Shapiro sequences based on Fejér type kernels are more convenient.

*Proof of* 3.1. Let $(U_n)$ be a contracting sequence of open, nonvoid, symmetric, central (that is, stable under inner automorphisms of $G$) sets in infinite, compact $G$ with the property that $\lim_n \lambda_G(U_n) = 0$. [When $G$ is also a Lie group we learn from (44.29) of [4] that there exists a number $k > 0$ such that the $U_n's$ may be selected to also satisfy

$$\text{(3.3)} \qquad \begin{cases} \lambda(U_n) \leqq k\lambda(U_{2n}) \\ U_{2n}U_{2n} \subseteqq U_n] \ . \end{cases}$$

Define $\chi_n$ to be the characteristic function of $U_n$. Since each $U_n$ is central, the Fourier series of each $\chi_n$ has the form

$$\chi_n \sim \sum_{\gamma \in \Gamma} d(\gamma)\hat{\chi}_n(D_\gamma)tr[D_\gamma] \ ,$$

where the $\hat{\chi}_n(D_\gamma)$ are complex numbers. By the proof of Theorem 4 of [2] (which is Theorem 1.2 above), there exists a number $B(t)$, independent of $n$, and a subset $\mathscr{U}_n$ of $\mathscr{G}$ with measure 1 such that

$$\text{(3.4)} \qquad ||\chi_n^W||_t \leqq B(t)\,||\chi_n||_2$$

for all $W$ in $\mathscr{U}_n$. Since $G$ is compact, the measure of $\mathscr{U}_n^{-1} = \mathscr{U}_n^*$ is also 1 so that $\mathscr{U}_n$ and $\mathscr{U}_n^*$ have a nonvoid intersection. Thus corresponding to each $n$ we can, and will, choose $W_n$ in $\mathscr{U}_n \cap \mathscr{U}_n^*$.

Let $h_n = \lambda(U_n)^{-1/2}\chi_n^{W_n}$ and $h_n^* = \lambda(U_n)^{-1/2}\chi_n^{W_n^*}$. Then

$$||h_n||_2 = 1,\ \sup_n ||h_n|_t \leqq B(t)$$

and

$$\begin{aligned} ||\hat{h}_n||_\infty &= \lambda(U_n)^{-1/2}\sup_\gamma ||\hat{\chi}_n(D_\gamma)\,W_n(\gamma)|| \\ &= \lambda(U_n)^{-1/2}||\hat{\chi}_n||_\infty \leqq \lambda(U_n)^{-1/2}||\chi_n||_1 \\ &= \lambda_G(U_n)^{1/2} \ . \end{aligned}$$

Thus $(h_n)$ is a $t$-$RS$-sequence, and so is $(h_n^*)$ by similar reasoning.

Clearly $||h_n||_2 = ||h_n^*||_2 = 1$ and $h_n^* * h_n = h_n * h_n^*$ (since both convolutions have $\lambda(U_n)^{-1}(\chi_n)^2$ as their Fourier transforms), so that if $G$ is a Lie group we have only to prove (b) of 3.1. The right-hand inequality for $p = 2$ is a trivial consequence of the fact that the norm of the operators $f \mapsto f * h_n$ from $L^2$ into $L^2$ is $\|\hat{h}_n\|_\infty$. To prove the left-hand inequality first note that $\|h_n^* * h_n\|_p = \lambda(U_n)^{-1}\|\chi_n * \chi_n\|_p$.

Suppose that the sequence $(U_n)$ is selected with the extra properties (3.3). Whenever $x \in U_{2n}$,

$$\begin{aligned}\chi_n * \chi_n(x) &= \int_{U_n} \chi_n(y^{-1}x)d\lambda(y) \\ &\geqq \int_{U_{2n}} \chi_n(y^{-1}x)d\lambda(y) \geqq \lambda_G(U_{2n}) .\end{aligned}$$

because if $y \in U_{2n}$ and $x \in U_{2n}$, then $y^{-1}x \in U_n$. Therefore

$$\begin{aligned}\int_G |\chi_n * \chi_n|^p d\lambda &\geqq \int_{U_{2n}} |\chi_n * \chi_n|^p d\lambda \\ &\geqq \lambda(U_{2n})\lambda(U_{2n})^p \geqq k^{p-1}\lambda_G(U_n)^{p+1} .\end{aligned}$$

Thus

$$\|h_n^* * h_n\|_p \geqq \lambda(U_n)^{-1}k^{-(p+1)/p}\lambda(U_n)^{(p+1)/p} \geqq k^{-(1+1/p)}\|\hat{h}_n\|_\infty^{2p}$$

(since $\|\hat{h}_n\|_\infty \leqq \|h_n\|_1 = (U_n)^{1/2}$) as required for (i), where $\rho = k^{-1}$.

To complete the proof of 3.1 (b) we establish the following straightforward string of inequalities:

$$\begin{aligned}\|h_n^* * h_n\|_1 &= \lambda(U_n)^{-1}\|\chi_n^* * \chi_n\|_1 \\ &= \lambda(U_n)^{-1}\|\chi_n\|_1^2 = \lambda(U_n)^{-1}\Big(\int_G \chi_n d\lambda\Big)^2 \\ &= \lambda(U_n)^{-1}\|\hat{\chi}_n(I)\|^2 \leqq \lambda(U_n)^{-1}\|\hat{\chi}_n\|_\infty^2 \\ &= \lambda(U_n)^{-1}\|\hat{\chi}_n W_n\|_\infty^2 = \|\hat{h}_n\|_\infty^2 ,\end{aligned}$$

4. **Strict inclusions for $\mathfrak{M}(L^p, L^q)$.** In this section we use the existence of Rudin-Shapiro sequences of type $t$, $t < \infty$, to prove several strict inclusions for the spaces $\mathfrak{M}(L^p, L^q)$. In particular, our results will imply:
and then use interpolation.

4.1. If $p, q$ and $r$ belong to $[1, \infty]$ and satisfy $1/p - 1/q \leqq 1 - 1/r$, then

$$\|g*f\|_q \leqq \|g\|_p\|f\|_r ,$$

whence we have, by considering the operators $g \mapsto g * f$,

$$L^r(G)^\wedge \subseteqq \mathfrak{M}(L^p, L^q) \tag{4.1}$$

(where $L^r(G)^\wedge$ denotes the subset of $\mathfrak{C}$ consisting of Fourier transforms of functions in $L^r(G)$). If furthermore $1 < p \leqq q < \infty$, $p \neq q'$ and $1 < r \leqq \infty$, Theorem 4.3 below shows that the inclusion in (4.1) is strict whenever $G$ is infinite.

4.2. If $G$ is a compact group, then

$$\mathfrak{M}(L^{p_1}, L^{q_1}) \subseteqq \mathfrak{M}(L^p, L^p) \tag{4.2}$$

whenever $p_1 \leqq p \leqq q_1$. If furthermore $G$ is an infinite compact Lie group and $2 < p < q_1$, then Theorem 4.4 below will show that inclusion (4.2) is strict.

The above two results are essentially extensions to compact groups or compact Lie groups of results in Gaudry [3] and Edwards and Price [1, §5] for locally compact Abelian groups; in fact we follow the broad outlines of the proofs used in [3].

THEOREM 4.3. *Let $G$ be an infinite compact group and let $r$ belong to $(1, \infty]$. Then whenever $1 < p \leqq q < \infty$ and $p \neq q'$, there exist elements in $\mathfrak{M}(L^p, L^q)$ which are not in $L^r(G)^\wedge$.*

*Proof.* Suppose that the hypotheses of the theorem are satisfied and that furthermore $\mathfrak{M}(L^p, L^q) \subseteqq L^r(G)^\wedge$. By the closed graph theorem this imbedding is continuous so there exists a number $K$ such that for every function in $L^u(G)$, with $1 - 1/u = 1/p - 1/q$ (see (4.1)), we have

$$||f||_r \leqq K||T_f||_{p,q} , \tag{4.3}$$

where $||T_f||_{p,q}$ denotes the norm of the multiplier operator $g \mapsto g * f$ from $L^p$ into $L^q$. We will show that (4.3) is impossible.

There are two cases.

*Case* 1. $1/p + 1/q < 1$. In this case an application of the Riesz-Thorin convexity theorem yields immediately that

$$||T_f||_{p,q} \leqq ||T_f||_{2,2}^{\alpha} ||T_f||_{s',\infty}^{1-\alpha}$$

where $1/p = \alpha/2 + (1 - \alpha)/s'$ and $1/q = \alpha/2$. Since $||T_f||_{2,2} = ||\hat{f}||_\infty$ and $||T_f||_{s',\infty} = ||f||_s$, we have

$$||T_f||_{p,q} \leqq ||\hat{f}||_\infty^{\alpha} ||f||_s^{1-\alpha} \tag{4.4}$$

with $\alpha = 2/q \neq 0$ and $1/s = q(1 - 1/q - 1/q)/(q - 2) \neq 0$. Put $t = \max\{u, s, r'\}$; then $t \neq \infty$ and from §3 we know that there exists a sequence $(k_n)$ of $L^t$ functions satisfying (3.2). Substituting in (4.4) yields

$$||T_{k_n}||_{p,q} \leqq \text{const. } 2^{-\alpha n} n^{1-\alpha} \tag{4.5}$$

which tends to zero as $n$ tends to infinity since $\alpha \neq 0$.

On the other hand

$$||k_n||_r \geqq ||k_n||_{t'} \geqq B_1 n . \tag{4.6}$$

Inequalities (4.5) and (4.6) together contradict (4.3) when $1/p + 1/q < 1$.

*Case* 2. $1/p + 1/q > 1$. A similar application of the convexity theorem yields

$$\begin{aligned} \| T_f \|_{p\,q} &\leqq \| T_f \|_{2,2}^{\alpha} \| T_f \|_{1,s}^{1-\alpha} \\ &= \| \hat{f} \|_{\infty}^{\alpha} \| f \|_{s}^{1-\alpha} \end{aligned}$$

where $1/p = \alpha/2 + 1 - \alpha$, $1/q = \alpha/2 + (1-\alpha)/s$ in which case $\alpha = 2p' \neq 0$ and $1/s = p(1/q + 1/p - 1)/(2-p) \neq 0$. Inequality (4.3) may be contradicted in a manner similar to that of Case 1 by using a sequence satisfying (3.2), again with $t = \max\{u, s, r'\}$.

THEOREM 4.4. *Suppose that $G$ is an infinite compact Lie group and that $p_0, q_0, p_1, q_1 (\varepsilon[1, \infty])$ have the properties that $p_0 \leqq q_0$, $1/p_0 + 1/q_0 < 1$, $p_1 < \infty$ and $q_1 \geqq 2$. If furthermore $q_1 > q_0$, then there exist elements in $\mathfrak{M}(L^{p_0}, L^{q_0})$ which are not in $\mathfrak{M}(L^{p_1}, L^{q_1})$.*

*This result remains valid when $\mathfrak{M}(L^{p_0}, L^{q_0})$ is replaced by $\mathfrak{M}(L^{q_0'}, L^{p_0'})$ and/or $\mathfrak{M}(L^{p_1}, L^{q_1})$ is replaced by $\mathfrak{M}(L^{q_1'}, L^{p_1'})$.*

*Proof.* Suppose that $G$ and $p_0, q_0, p_1, q_1$ satisfy the hypotheses of the theorem. By arguing as in the proof of Theorem 4.3 it is clear that the result may be proved by finding a sequence $(h_n)$ of functions such that

$$\min\{\| T_{h_n} \|_{p_1,q_1}, \| T_{h_n} \|_{q_1',p_1'}\} / \max\{\| T_{h_n} \|_{p_0,q_0}, \| T_{h_n} \|_{q_0',p_0'}\} \longrightarrow \infty \tag{4.7}$$

as $n \longrightarrow \infty$.

Let $(h_n)$ and $(h_n^*)$ denote a pair of $t$-$RS$-sequences satisfying Lemma 3.1 (b) with $t$ equal to the maximum of $p_1$, $(q_0 - 2)/q_0(1 - 1/q_0 - 1/p_0)$ and $(2 - q_0')/q_0'(1 - 1/q_0 - 1/p_0)$. Then, by proceeding as in the proof of Theorem 4.1, we have

$$\max(\| T_{h_n} \|_{p_0,q_0}, \| T_{h_n} \|_{q_1',p_0'}) \leqq \text{const.} \| \hat{h}_n \|_{\infty}^{2/q_0} . \tag{4.8}$$

On the other hand we have, by the definition of the norms,

$$\| T_{h_n} \|_{p_1,q_1} \geqq \| h_n^* * h_n \|_{q_1} / \| h_n^* \|_{p_1}$$

and

$$\| T_{i_n} \|_{q_1',p_1'} = \| T'_{h_n} \|_{p_1,q_1} \geqq \| h_n * h_n^* \|_{q_1} / \| h_n^* \|_{p_1} ,$$

where $T'_{h_n}$ is the operator $f \mapsto h_n * f$ (see the discussion in 5.3 of [1]). Thus

$$\min(\| T_{h_n} \|_{p_1,q_1}, \| T_{h_n} \|_{q_1',p_1'}) \geqq \| h_n^* * h_n \|_{q_1} / \| h_n^* \|_{p_1} . \tag{4.9}$$

Now it is easily shown that if $g \in L^{q_1}$ with $q_1 \geqq 2$, then

$$||g||_2^2 \leqq ||g||_1^\beta ||g||_{q_1}^{(1-\beta)q_1} \text{ where } 2 = \beta + (1-\beta)\cdot q_1 ,$$

and so

$$(4.10) \qquad ||h_n^* * h_n||_{q_1} \geqq ||h_n^* * h_n||_2^{2/q_1'} ||h_n^* * h_n||_1^{-1+2/q_1} .$$

Applying 3.1 (b), (4.10) and the definition of a $t$-$RS$-sequence yields

$$\begin{aligned} \min\,(||T_{h_n}||_{p_1,q_1}||T_{h_n}||_{q_1',p_1'}) &\geqq (\rho^{3/2}||\hat{h}_n||_\infty)^{2/p_1'}(\rho||\hat{h}_n||_\infty)^{-2+4/q_1}||h_n^*||_n^{-1} \\ &\geqq A||\hat{h}_n||_\infty^{2/q_1} , \end{aligned}$$

where $A$ is a non zero positive mumber. This inequality combines with (4.8) to show that (4.7) is satisfied whenever $q_1 > q_0$ since $||\hat{h}_n||_\infty \to 0$ as $n \to \infty$.

COROLLARY 4.5. *Theorem* 4.4 (*and hence also* 4.2) *remains valid for compact group $G$ which have a closed normal subgroup $G_0$ such that $G/G_0$ is an infinite* (*compact*) *Lie group.*

REMARKS 4.6. (i) We do not know whether Theorem 4.4 remains valid for all compact groups or, for that matter, whether part (b) of Theorem 3.1 remains valid in the general case. We should remark that the construction of Rudin-Shapiro sequences for compact Abelian groups is more complicated for groups which do not have a torus as a factor group; see Gaudry [3].

(ii) In the notation of [4], $G$ has the property of Corollary 4.5 if and only if there exists a finite subset $\gamma_1, \cdots, \gamma_k$ of $\Gamma$ such that $[\gamma_1, \cdots \gamma_k]$ is infinite. This follows from (28.10) and (28.6) of [4] combined with the fact that a compact group $G$ is a Lie group if and only if its dual $\Gamma$ is finitely generated.

*Proof of* 4.5. Suppose that $G_0$ is a closed normal subgroup of $G$ and that $\Gamma_0$ is the dual (hypergroup) of $G/G_0$. Let $A_0 = A(\Gamma, G_0)$ denote the annihilator of $G_0$ in $\Gamma$; then there exists an isomorphism $\varphi$ between hypergroups $A_0$ and $\Gamma_0$ in such a manner that for each $\gamma \in A_0$ we can choose $D_{\varphi(\gamma)}$ so that

$$D_{\varphi(\gamma)} \circ \pi = D_\gamma ,$$

where $\pi$ denotes the natural projection from $G$ onto $G/G_0$. For the sequel we suppose that the $D_{\varphi(\gamma)}$ are chosen in this manner. Thus, for example, if $f$ is an integrable function on $G/G_0$ and

$$f \sim \sum_{\gamma_0 \in \Gamma_0} d(\gamma_0)\, tr[\hat{f}(D_{\gamma_0})D_{\gamma_0}] ,$$

then

$$(4.11) \qquad f \circ \pi \sim \sum_{\gamma \in A_0} d(\gamma) tr[\hat{f}(D_{\varphi(\gamma)}) D_\gamma] .$$

For each $\mu \in \mathfrak{E}(\Gamma_0)$, we define $\mu' \in \mathfrak{E}(\Gamma)$ by $\mu' = \mu \circ \varphi$ on $A_0$, and zero otherwise. Corollary 4.5 is an immediate consequence of Theorem 4.4 and the fact that $\mu \in \mathfrak{M}(L^p(G/G_0), L^q(G/G_0))$ if and only if $\mu' \in \mathfrak{M}(L^p(G), L^q(G))$. The proof of this final equivalence is routine. (For example, see Lemma 4.6 of [3]; use can also be made of equations of the form (4.11) above and (A.3) (A.5) and (A.6) in the appendix of [1]).

*Added in proof.* The authors have been able to show that Theorem 4.4 (and hence also 4.2) are valid for an unrestricted compact group. The proof will appear elsewhere.

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UNIVERSITY OF GENOA,
GENOA, ITALY

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