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CHARACTERIZATIONS OF AMENABLE AND STRONGLY AMENABLE C^* -ALGEBRAS

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In this paper it is proved that a C^* -algebra A is strongly amenable iff A satisfies a certain fixed point property when acting on a compact convex set, or iff a certain Hahn-Banach type extension theorem is true for all Banach A -modules. It is proved that a C^* -algebra A is amenable iff A satisfies a weaker Hahn-Banach type extension theorem.

A topological group G is said to be amenable if there is a left invariant mean on the space of bounded continuous complex functions on G . A number of papers have been published which give equivalent definitions of amenability (for example, see the papers [4, 7, 11] or the book [3]). It has recently been proven that a locally compact group G is amenable iff for all two-sided $L^1(G)$ -modules X and bounded derivations D of $L^1(G)$ into X^* , we have that D is the inner derivation induced by an element of X^* [5, Theorem 2.5]. This result motivates the definition of amenable and strongly amenable C^* -algebras [5, sections 5 and 7]. In §2 of this paper we give some conditions on a C^* -algebra that are equivalent to amenability or strong amenability and are analogous to some of the known equivalent definitions of amenable groups. In §3 we show that the generalized Stone-Weierstrass theorem for separable C^* -algebras is true when the C^* -subalgebra in question is strongly amenable.

1. Preliminaries. Let A be a C^* -algebra. Then a complex Banach space X is called a Banach A -module if it is a two-sided A -module and there exists a positive real number M such that for all $a \in A$ and $x \in X$ we have

$$\|ax\| \leq M\|a\|\|x\|$$

and

$$\|xa\| \leq M\|x\|\|a\|,$$

If X is a Banach A -module, then the dual space X^* becomes a Banach A -module if we define for $a \in A$, $f \in X^*$, and $x \in X$,

$$\begin{aligned}(af)(x) &= f(xa) \\ (fa)(x) &= f(ax).\end{aligned}$$

A derivation from A into X^* is a bounded linear map D from A into

X^* such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. If $f \in X^*$, the function $\delta(f)$ from A into X^* given by

$$\delta(f)(a) = af - fa$$

is called the inner derivation induced by f .

DEFINITION 1. [5, § 5]. A C^* -algebra A is said to be *amenable* if every derivation from A into X^* is inner for all Banach A -modules X .

DEFINITION 2. [5, § 7]. A C^* -algebra A is said to be *strongly amenable* if, whenever X is a Banach A -module and D is a derivation of A into X^* , there is a $f \in \text{co} \{D(u)u^*: u \in U(A_e)\}$ with $D = -\delta(f)$, where A_e is the C^* -algebra obtained by adjoining the identity e to A , X is made into a unital A_e -module by defining $xe = ex = x$ for all $x \in X$, D is extended to A_e by defining $D(e) = 0$, $U(A_e)$ is the unitary group of A_e , and $\text{co } S$ denotes the w^* -closed convex hull of a set S contained in X^* .

A C^* -algebra A is strongly amenable iff A_e is strongly amenable, and a C^* -algebra A with identity is strongly amenable iff the definition is satisfied for all unital A -modules X with A_e replaced throughout by A [5, § 7]. The class of strongly amenable C^* -algebras includes all C^* -algebras which are *GCR*, uniformly hyperfinite, or the C^* -group algebra of a locally compact amenable group [5, § 7]. It is not known if there exist amenable C^* -algebras which are not strongly amenable.

For A a C^* -algebra, let $A \hat{\otimes} A$ be the completion of the algebraic tensor product $A \otimes A$ in the greatest cross-norm. Then we can identify $(A \hat{\otimes} A)^*$ with the space of bounded bilinear functionals on $A \times A$ [13, p. 438]. We see that $A \hat{\otimes} A$ becomes a Banach A -module if we define for $a, b, c \in A$,

$$\begin{aligned} a(b \otimes c) &= ab \otimes c \\ (b \otimes c)a &= b \otimes ca. \end{aligned}$$

Hence $(A \hat{\otimes} A)^*$ becomes a Banach A -module under the dual action: If $f \in (A \hat{\otimes} A)^*$ and $a, b, c \in A$,

$$\begin{aligned} (af)(b \otimes c) &= f(b \otimes ca) \\ (fa)(b \otimes c) &= f(ab \otimes c). \end{aligned}$$

We can also make $A \hat{\otimes} A$ and $(A \hat{\otimes} A)^*$ into Banach A -modules by defining for $f \in (A \hat{\otimes} A)^*$ and $a, b, c \in A$:

$$\begin{aligned} a \circ (b \otimes c) &= b \otimes ac \\ (b \otimes c) \circ a &= ba \otimes c \end{aligned}$$

$$\begin{aligned}(a \circ f)(b \otimes c) &= f(ba \otimes c) \\ (f \circ a)(b \otimes c) &= f(b \otimes ac) .\end{aligned}$$

Note that the two operations on $A \hat{\otimes} A$ do not interact; that is, if $a, b, c, d \in A$,

$$\begin{aligned}a \circ (b(c \otimes d)) &= b(a \circ (c \otimes d)) \\ ((c \otimes d)b) \circ a &= ((c \otimes d) \circ a)b \\ a \circ ((c \otimes d)b) &= (a \circ (c \otimes d))b\end{aligned}$$

and so forth.

2. Amenable and strongly amenable C^* -algebras.

PROPOSITION 1. *Let A be a C^* -algebra with unit e . Then the following seven statements are equivalent:*

- (a) *A is strongly amenable.*
- (b) *For all unital Banach A -modules X and $f \in X^*$, there exists $g \in \text{co}\{ufu^*: u \in U(A)\}$ such that $ag = ga$ for all $a \in A$.*
- (c) *For any $f \in (A \hat{\otimes} A)^*$ there exists $g \in \text{co}\{ufu^*: u \in U(A)\}$ such that $ag = ga$ for all $a \in A$.*
- (d) *There is a linear map T of $(A \hat{\otimes} A)^*$ into $C = \{g \in (A \hat{\otimes} A)^*: ag = ga \text{ all } a \in A\}$ such that $T(a \circ f) = a \circ T(f)$, $T(f \circ a) = T(f) \circ a$, and $T(f) \in \text{co}\{ufu^*: u \in U(A)\}$ for all $a \in A, f \in (A \hat{\otimes} A)^*$.*
- (e) *Let X be a Banach A -module, S a w^* -closed convex subset of X^* such that $usu^* \in S$ for all $s \in S, u \in U(A)$. Then there exists an element $s \in S$ such that $usu^* = s$ for all $u \in U(A)$.*
- (f) *Let Y be a Banach A -module and X a subspace of Y such that $uxu^* \in X$ for all $x \in X, u \in U(A)$. Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X, u \in U(A)$. Then for any $g \in Y^*$ which extends f , there is an $h \in \text{co}\{ugu^*: u \in U(A)\}$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y$ and $u \in U(A)$.*
- (g) *Let Y be a Banach A -module and X a two-sided A -submodule of Y . Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X, u \in U(A)$. Then for any $g \in Y^*$ which extends f , there is an $h \in \text{co}\{ugu^*: u \in U(A)\}$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y$ and $u \in U(A)$.*

Before proving the proposition, we make some remarks. The implications (a) implies (b) and (b) implies (d) were proven in [1]. The map T in (d) takes the place of the invariant mean that is present in amenable groups. The condition in (e) is a fixed point property; it is known that a locally compact group is amenable iff it has a certain fixed point property [7]. The condition (f) and (g) might be called the strong invariant extension property for subspaces,

and the strong invariant extension property for submodules respectively. A locally compact group is amenable iff it has a certain Hahn-Banach type extension property similar to (f) and (g) [11].

Proof of Proposition 1. (a) implies (b): Let $f \in X^*$ and $\delta(f)$ be the inner derivation induced by f . Then there is a $a \in \text{co} \{ \delta(f)(u)u^*: u \in U(A) \}$ such that $\delta(f) = -\delta(g)$. But $\delta(f)(u)u^* = ufu^* - f$, hence $f + g \in \text{co} \{ ufu^*: u \in U(A) \}$. Also $\delta(f)(a) = -\delta(g)(a)$ for all $a \in A$, thus $(f + g)a = a(f + g)$ for all $a \in A$.

(b) implies (c): Clear.

(c) implies (d): The proof is an adaption to the present situation of a proof of J. Schwartz [10, Lemma 5]. Let \mathcal{A} be the set of all linear mappings T of $(A \hat{\otimes} A)^*$ into $(A \hat{\otimes} A)^*$ such that $T(f) \in \text{co} \{ ufu^*: u \in U(A) \}$ and $T(a \circ f) = a \circ T(f)$, $T(f \circ a) = T(f) \circ a$ for all $f \in (A \hat{\otimes} A)^*$, $a \in A$. The set \mathcal{A} is nonempty since the identity map is in \mathcal{A} . We order \mathcal{A} by defining $T_1 \geq T_2$ if for all $f \in (A \hat{\otimes} A)^*$,

$$\text{co} \{ uT_1(f)u^*: u \in U(A) \} \subseteq \text{co} \{ uT_2(f)u^*: u \in U(A) \}.$$

Then \geq defines a quasi-order on \mathcal{A} . Suppose $\{T_\alpha: \alpha \in \mathcal{A}\}$ is a totally ordered subset of \mathcal{A} . We have $\|T_\alpha(f)\| \leq \|f\|$, thus for all $d \in A \hat{\otimes} A$, $|T_\alpha(f)(d)| \leq \|f\| \|d\|$ and $\{T_\alpha(f)(d): \alpha \in \mathcal{A}\}$ is a bounded function on \mathcal{A} . Let LIM be a Banach limit on the directed set \mathcal{A} (see [10, p. 21] for information on Banach limits). Then set $T(f)(d) = \text{LIM } T_\alpha(f)(d)$ for all $f \in (A \hat{\otimes} A)^*$ and $d \in A \hat{\otimes} A$. Then T is a bounded linear map from $(A \hat{\otimes} A)^*$ into $(A \hat{\otimes} A)^*$. An easy calculation shows that $T(a \circ f) = a \circ T(f)$ and $T(f \circ a) = T(f) \circ a$. We show that $T(f) \in \text{co} \{ ufu^*: u \in U(A) \}$ and $T \geq T_\alpha$ for all $\alpha \in \mathcal{A}$. If $\beta \geq \alpha$ and $f \in (A \hat{\otimes} A)^*$ then

$$T_\beta(f) \in \text{co} \{ uT_\alpha(f)u^*: u \in U(A) \} = K.$$

Suppose, for contradiction, that $T(f) \notin K$. Then by the strong separation theorem, there exists $d \in A \hat{\otimes} A$, λ real and $\varepsilon > 0$ such that for all $g \in K$,

$$\text{Re } T(f)(d) \leq \lambda < \lambda + \varepsilon \leq \text{Re } g(d).$$

Hence $\text{Re } T(f)(d) \leq \lambda < \lambda + \varepsilon \leq \text{Re } T_\beta(f)(d)$ for all $\beta \geq \alpha$. But applying the Banach limit to this equation we obtain $\text{Re } T(f)(d) < \text{Re } T(f)(d)$. Hence $T(f) \in K$. Thus

$$\text{co} \{ uT(f)u^*: u \in U(A) \} \subseteq \text{co} \{ uT_\alpha(f)u^*: u \in U(A) \}$$

for all $\alpha \in \mathcal{A}$. Hence $T \in \mathcal{A}$ and $T \geq T_\alpha$. Hence \mathcal{A} is inductive, so by Zorn's lemma \mathcal{A} has a maximal element T . We show that $T(f) \in C$ for all $f \in (A \hat{\otimes} A)^*$. If $g \in (A \hat{\otimes} A)^*$ is such that $T(g) \notin C$, then $\text{co} \{ uT(g)u^*: u \in U(A) \}$ contains more than one element. Since we

assumed (c), $C \cap \text{co} \{uT(g)u^*: u \in U(A)\}$ is nonempty. Let $\sum \lambda uT(g)u^*$ be a net indexed by a directed set \mathcal{A} which converge w^* to an element h of C (we suppress all indices in the sum). Define for $f \in (A \hat{\otimes} A)^*$ and $d \in (A \hat{\otimes} A)$,

$$T'(f)(d) = \text{LIM} \sum \lambda uT(f)u^*(d) .$$

Then T' is a bounded linear map from $(A \hat{\otimes} A)^*$ to $(A \hat{\otimes} A)^*$ and another application of the strong separation theorem shows that $T'(f) \in \text{co} \{uT(f)u^*: u \in U(A)\}$. If we show that $T'(a \circ f) = a \circ T'(f)$ and $T'(f \circ a) = T'(f) \circ a$, we will know that $T' \in \mathcal{A}$ and $T' \geq T$. But

$$\begin{aligned} T'(a \circ f)(b \otimes c) &= \text{LIM} \sum \lambda uT(a \circ f)u^*(b \otimes c) \\ &= \text{LIM} \sum \lambda (a \circ T(f))(u^*b \otimes cu) \\ &= \text{LIM} \sum \lambda T(f)(u^*ba \otimes cu) \\ &= T'(f)(ba \otimes c) \\ &= (a \circ T'(f))(b \otimes c) . \end{aligned}$$

Hence $T'(a \circ f) = a \circ T'(f)$ and likewise $T'(f \circ a) = T'(f) \circ a$. But the net $\sum \lambda uT(g)u^*(d)$ has the actual limit $h(d)$. Thus $T'(g) = h$ and $\text{co} \{uT'(g)u^*: u \in U(A)\} = \{h\}$, and $\{h\}$ is properly contained in $\text{co} \{uT(g)u^*: u \in U(A)\}$, hence it is not true that $T \geq T'$. But this contradicts the maximality of T , and we have that $T(f) \in C$ for all f . This completes the proof.

(d) *implies* (e): Let $x \in X$ and fix $s \in S$. Define a bounded bilinear function $F(x, s)$ on $A \times A$ by $F(x, s)(a, b) = s(axb)$ for all $a, b \in A$. We consider $F(x, s)$ as an element of $(A \hat{\otimes} A)^*$. Let $G(s)(x) = T(F(x, s))(e \otimes e)$. Then clearly $G(s) \in X^*$. Now if $u \in U(A)$, then

$$\begin{aligned} F(u^*xu, s)(a \otimes b) &= s(au^*xub) \\ &= F(x, s)(u \circ (a \otimes b) \circ u^*) \\ &= (u^* \circ F(x, s) \circ u)(a \otimes b) . \end{aligned}$$

Thus $F(u^*xu, s) = u^* \circ F(x, s) \circ u$. Hence for all $x \in X$ and $u \in U(A)$, by using the properties of the map T ,

$$\begin{aligned} (uG(s)u^*)(x) &= G(s)(u^*xu) \\ &= T(F(u^*xu, s))(e \otimes e) \\ &= T(u^* \circ F(x, s) \circ u)(e \otimes e) \\ &= (uT(F(x, s))u^*)(e \otimes e) \\ &= T(F(x, s))(e \otimes e) \\ &= G(s)(x) . \end{aligned}$$

Thus $uG(s)u^* = G(s)$ for all $u \in U(A)$. We will be done when we show that $G(s) \in S$. If $G(s) \notin S$, then there exists $x \in X$, a real number

λ and $\varepsilon > 0$ such that for all $t \in S$ we have

$$\operatorname{Re} G(s)(x) \leq \lambda < \lambda + \varepsilon \leq \operatorname{Re} t(x).$$

Now $T(F(x, s)) \in \operatorname{co} \{uF(x, s)u^*: u \in U(A)\}$, and $(uF(x, s)u^*)(e \otimes e) = (usu^*)(x)$. Since $usu^* \in S$ for all $u \in U(A)$, this implies that $\operatorname{Re} G(s)(x) \geq \lambda + \varepsilon$. This contradiction proves that $G(s) \in S$.

(e) *implies* (f): Let X, Y and $f \in X^*$ be as in (f). Let $g \in Y^*$ be any extension of f and let $S = \operatorname{co} \{ugu^*: u \in U(A)\}$. Then S is w^* -closed convex subset of Y^* and if $s \in S$ then $usu^* \in S$ for all $u \in U(A)$. Hence by (e) there is an element $h \in S$ such that $uhu^* = h$ for all $u \in U(A)$. Since $uXu^* \subseteq X$ for all $u \in U(A)$, $f(uxu^*) = f(x)$ for all $x \in X$ and g extends f , it is easily seen that h extends f .

(f) *implies* (g): Clear.

(g) *implies* (c): Given $g \in (A \hat{\otimes} A)^*$, let $Y = A \hat{\otimes} A$, $X = \{0\}$, $f = 0$ and apply (g). Thus there is an $h \in \operatorname{co} \{ugu^*; u \in U(A)\}$ such that $h(uyu^*) = h(y)$ for all $y \in A \hat{\otimes} A$ and $u \in A$; that is, $ah = ha$ for all a in A .

(d) *implies* (a): Let $D: A \rightarrow X^*$ be a derivation. Let $x \in X$ and define a bounded bilinear functional $f(x)$ on $A \times A$ by $f(x)(b, c) = D(b)(xc)$. Then define an element $G \in X^*$ by $G(x) = T(f(x))(e \otimes e)$. We will show that $D = \delta(G)$ and $-G \in \operatorname{co} \{D(u)u^*: u \in U(A)\}$. For $x \in X$ and $a \in A$, define a bounded bilinear functional $g(x, a)$ on $A \times A$ by $g(x, a)(b, c) = D(a)(xcb)$. Then a computation shows that $a \circ f(x) = f(ax) + g(x, a)$ and $f(xa) = f(x) \circ a$. If $a \in A$, $x \in X$, then

$$\begin{aligned} (\delta(G)(a))(x) &= (aG - Ga)(x) \\ &= G(xa - ax) \\ &= T(f(xa - ax))(e \otimes e) \\ &= T(f(x) \circ a - a \circ f(x) + g(x, a))(e \otimes e) \\ &= T(f(x))(e \otimes a) - T(f(x))(a \otimes e) + T(g(x, a))(e \otimes e) \\ &= T(g(x, a))(e \otimes e). \end{aligned}$$

The last equality is true because T maps into C . Now for $u \in U(A)$, $(ug(x, a)u^*)(e \otimes e) = g(x, a)(u^* \otimes u) = D(a)(x)$, and $T(g(x, a))$ is in $\operatorname{co} \{ug(x, a)u^*: u \in U(A)\}$, hence $(\delta(G)(a))(x) = D(a)(x)$ for all $x \in X$ and $a \in A$, thus $D = \delta(G)$. An application of the strong separation theorem shows that $-G \in \operatorname{co} \{D(u)u^*: u \in U(A)\}$. Thus (d) implies (a) and the proof of Proposition 1 is complete.

REMARKS. (1) The equivalence of (a) and (c) shows that, to

check strong amenability of A , it is only necessary to consider the A -module $A \hat{\otimes} A$; this gives another proof of the Proposition 7.15 of [5].

(2) In the notation of [6], a C^* -algebra A is amenable iff the first cohomology group $H_c^1(A, X^*)$ is zero for all Banach A -modules X . The reduction of dimension argument of [5, §1. a] then shows that all the cohomology groups $H_c^n(A, X^*)$ are zero. If A is strongly amenable, then the proof of (d) implies (a) above can be changed to show directly that $H_c^n(A, X^*)$ is zero for all n and all Banach A -modules X ; this proof is similar to the proof of Theorem 3.3 in [6], with the map T taking the place of the invariant mean which is present in that theorem.

(3) In [1] the author used the existence of the function T to generalize the well-known Dixmier-Mackey theorem on amenable groups by proving that every continuous representation of a strongly amenable C^* -algebra on a Hilbert space is similar to a $*$ -representation. However, this fact can be proved in a more elementary fashion as follows: Let A be a strongly amenable C^* -algebra and let π be a continuous representation of A as bounded operators on a Hilbert space H . It suffices to assume A has an identity e and $\pi(e) = I$. Make $B(H)$ into a Banach A -module by the operations $aT = \pi(a)T$, $Ta = T\pi(a^*)^*$ for $a \in A$, $T \in B(H)$. Then $B(H)$ is the dual Banach A -module of the trace class operators. Define a bounded linear map D of A into $B(H)$ by $D(a) = \pi(a) - \pi(a^*)^*$. Then

$$\begin{aligned} aD(b) + D(a)b &= \pi(a)(\pi(b) - \pi(b^*)^*) + (\pi(a) - \pi(a^*)^*)\pi(b)^* \\ &= \pi(ab) - \pi((ab)^*)^* \\ &= D(ab). \end{aligned}$$

Hence D is a derivation. Thus, since A is strongly amenable, there is an operator T in $\text{co}\{D(u)u^*: u \in U(A)\}$ such that $D = -\delta(T)$. Then for $a \in A$, $\pi(a) - \pi(a^*)^* = -aT + Ta = T\pi(a^*)^* - \pi(a)T$, thus $\pi(a)(I + T) = (I + T)\pi(a^*)^*$. We let $R = T + I$. Now $D(u)u^* = \pi(u)\pi(u)^* - I$, thus $R \in \text{co}\{\pi(u)\pi(u)^*: u \in U(A)\}$. For $x \in H$, $(\pi(u)\pi(u)^*x, x) = \|\pi(u)^*x\|^2$, and $\|x\|^2 = \|\pi(u^*)^*\pi(u)^*x\|^2 \leq \|\pi\|^2 \|\pi(u)^*x\|^2$. Hence $(1/\|\pi\|^2) \|x\|^2 \leq \|\pi(u)^*x\|^2$. Thus R is positive and invertible. Let S be the positive square root of R . Then $\pi(a)S^2 = S^2\pi(a^*)^*$, and $S^{-1}\pi(a)S = S\pi(a^*)^*S^{-1}$. If we define $\pi_1(a) = S^{-1}\pi(a)S$, then π_1 is clearly a representation of A , and $\pi_1(a^*)^* = (S^{-1}\pi(a^*)^*S)^* = S\pi(a^*)^*S^{-1} = S^{-1}\pi(a)S = \pi_1(a)$. Hence π_1 is a $*$ -representation.

We now give some equivalent conditions for a C^* -algebra to be amenable.

PROPOSITION 2. *Let A be a C^* -algebra with unit e . Then the*

following three statements are equivalent:

(a) A is amenable.

(b) There is a bounded linear map T of $(A \hat{\otimes} A)^*$ into $C = \{f \in (A \hat{\otimes} A)^* : af = fa \text{ all } a \in A\}$ such that T restricted to C is the identity on C and $T(a \circ f) = a \circ T(f)$, $T(f \circ a) = T(f) \circ a$ for all $a \in A$, $f \in (A \hat{\otimes} A)^*$.

(c) Let Y be a Banach A -module and X a two sided A -submodule of Y . Let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X$, $u \in U(A)$. Then there is a $h \in Y^*$ such that h extends f and $h(uyu^*) = h(y)$ for all $y \in Y$, $u \in U(A)$.

Proof. (a) implies (b): Let $Y = (A \hat{\otimes} A)^* \hat{\otimes} (A \hat{\otimes} A)$ and let Z , W and X be as in the proof of (g) implies (d) of Proposition 1. Let $F \in Y^*$ be defined by $F(f \otimes t) = f(t)$ and let D_1 be the inner derivation induced by F . Then for $a \in A$, $f \in (A \hat{\otimes} A)^*$, and $t \in (A \hat{\otimes} A)$, $D_1(a)(f \otimes t) = (af - fa)(t)$. Hence $D_1(a)$ is zero on W . A calculation using the fact that the two A -module operations on $(A \hat{\otimes} A)$ do not interact (see the comment at the end of §1) shows that $D_1(a)$ is zero on Z . Hence there is a map D from A into $(Y/X)^*$ given by $D(a)(\bar{y}) = D_1(a)(y)$, where \bar{y} is the coset in Y/X of an element $y \in Y$. It is easily seen that D is a derivation, hence since A is amenable there is an element $G_1 \in (Y/X)^*$ such that $D = \delta(G_1)$. Let $G \in Y^*$ be defined by $G(y) = G_1(\bar{y})$ for all $y \in Y$. Define a bounded linear map T_1 from $(A \hat{\otimes} A)^*$ into $(A \hat{\otimes} A)^*$ by $T_1(f)(t) = G(f \otimes t)$ for all $f \in (A \hat{\otimes} A)^*$, $t \in (A \hat{\otimes} A)$. Now $D(a) = aG_1 - G_1a$ for all $a \in A$, thus

$$\begin{aligned} D(a)(\overline{f \otimes t}) &= D_1(a)(f \otimes t) \\ &= (af - fa)(t) \\ &= (aG_1 - G_1a)(\overline{f \otimes t}) \\ &= \overline{G(f \otimes (ta - at))} \\ &= T_1(f)(ta - at) \\ &= (aT_1(f) - T_1(f)a)(t). \end{aligned}$$

Hence $(af - fa)(t) = (aT_1(f) - T_1(f)a)(t)$ for all $t \in (A \hat{\otimes} A)$, and we thus have $f - T_1(f) \in C$. Let T be the bounded linear map from $(A \hat{\otimes} A)^*$ into C given by $T(f) = f - T_1(f)$. If $f \in C$, then $T_1(f)(t) = G_1(\overline{f \otimes t}) = 0$, thus $T(f) = f$ if $f \in C$. Similarly, since G is zero on Z , we have that $T(a \circ f) = a \circ T(f)$ and $T(f \circ a) = T(f) \circ a$. This completes the proof of (a) implies (b).

(b) implies (c): Let Y be a Banach A -module, X a submodule of Y , and let $f \in X^*$ be such that $f(uxu^*) = f(x)$ for all $x \in X$, $u \in U(A)$. Let $f_1 \in Y^*$ be any extension of f and for each $y \in Y$ define an element $F(y)$ of $(A \hat{\otimes} A)^*$ by $F(y)(a \otimes b) = f_1(ayb)$. Then let $h \in Y^*$ be defined

by $h(y) = T(F(y))(e \otimes e)$. A calculation shows that for all $u \in U(A)$, $F(u^*yu) = u^* \circ F(y) \circ u$, so that $h(u^*yu) = h(y)$. Also, if $x \in X$, then it is easily seen that $F(x) \in C$, hence $h(x) = T(F(x))(e \otimes e) = F(x)(e \otimes e) = f_1(x) = f(x)$. Thus h has the desired properties.

(c) *implies* (b): The proof is essentially the same as the proof of (g) implies (d) in Proposition 1; we omit the details.

(b) *implies* (a): Again, the proof is essentially the same as the proof of (d) implies (a) in Proposition 1.

While we can not settle the question of whether every amenable C^* -algebra is strongly amenable, we think that the relationship between conditions (c) of Proposition 2 and conditions (f) and (g) of Proposition 1 may be useful in settling the question.

3. A Stone-Weierstrass type theorem. For A a C^* -algebra, let $ES(A)$ be the set of pure states of A . Let B be a C^* -subalgebra of A which separates $ES(A) \cup \{0\}$. The generalized Stone-Weierstrass question for C^* -algebras [9, section 4.7] asks when is A equal to B ? Using a method introduced by Sakai [8], we can show that $A = B$ if A is separable and B is strongly amenable.

PROPOSITION 3. *Let A be a separable C^* -algebra. If B is a strongly amenable C^* -subalgebra of A which separates $ES(A) \cup \{0\}$, then $A = B$.*

Proof. By [8, Lemma 1] we can assume that A has an identity which is also in B . Then as in [8, proof of Proposition 2] if $B \neq A$, there is a $*$ -representation π of A on a separable Hilbert space such that $(\pi(B))'' \neq (\pi(A))''$. Then by [12, Theorem 12.2] there is a Hilbert space H and a von Neumann algebra $D \subseteq B(H)$ such that D is $*$ -anti-isomorphic to D' and such that $(\pi(B))''$ is $*$ -isomorphic to D by a $*$ -isomorphism S . Now $*$ -anti-isomorphisms are clearly order isomorphisms and hence are ultraweakly continuous [2, A27]. Thus the image of $\pi(B)$ under the $*$ -anti-isomorphism is weakly dense in D' . It was proven in [5, Section 7] that the weak closure of any $*$ -representation of a strongly amenable C^* -algebra has Schwartz's Property P [10, Definition 1]; essentially the same proof shows that the weak closure of any $*$ -anti-representation of a strongly amenable C^* -algebra has Property P . Hence, the von Neumann algebra D' has Property P . Thus by [10, Lemma 5] there is a linear norm-decreasing map P from $B(H)$ onto D which is the identity on D . Now consider S as a $*$ -representation of $(\pi(B))''$ on H , then by [2, 2.10.2], there is

a Hilbert space K containing H and a $*$ -representation T of $(\pi(A))''$ on K such that $S(x) = T(x)|_H$ for all $x \in (\pi(B))''$. Let p be the projection of K onto H , and define a linear norm-decreasing map R from $B(K)$ onto $B(H)$ by $Ry = py|_H$ for all $y \in B(K)$. Then $S^{-1} \circ P \circ R \circ T$ is a linear norm-decreasing map from $(\pi(A))''$ onto $(\pi(B))''$ which is the identity on $(\pi(B))''$. Then by [8, Theorem 1], we have that $(S^{-1} \circ P \circ R \circ T)x = x$ for all $x \in (\pi(A))''$. Hence $(\pi(A))'' = (\pi(B))''$. This contradiction shows that $A = B$.

We remark that Sakai [9, 4.7.8] has proved Proposition 3 in the case when B is the uniform closure of an increasing directed set of Type I C^* -subalgebras. The author does not know of an example of a strongly amenable C^* -algebra which is not the uniform closure of an increasing directed set of Type I C^* -subalgebras.

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Max K. Agoston, <i>An obstruction to finding a fixed point free map on a manifold</i>	543
Nadim A. Assad and William A. Kirk, <i>Fixed point theorems for set-valued mappings of contractive type</i>	553
John Winston Bunce, <i>Characterizations of amenable and strongly amenable C^*-algebras</i>	563
Erik Maurice Ellentuck and Alfred Berry Manaster, <i>The decidability of a class of AE sentence in the isols</i>	573
U. Haussmann, <i>The inversion theorem and Plancherel's theorem in a Banach space</i>	585
Peter Lawrence Falb and U. Haussmann, <i>Bochner's theorem in infinite dimensions</i>	601
Peter Fletcher and William Lindgren, <i>Quasi-uniformities with a transitive base</i>	619
Dennis Garbanati and Robert Charles Thompson, <i>Classes of unimodular abelian group matrices</i>	633
Kenneth Hardy and R. Grant Woods, <i>On c-realcompact spaces and locally bounded normal functions</i>	647
Manfred Knebusch, Alex I. Rosenberg and Roger P. Ware, <i>Grothendieck and Witt rings of hermitian forms over Dedekind rings</i>	657
George M. Lewis, <i>Cut loci of points at infinity</i>	675
Jerome Irving Malitz and William Nelson Reinhardt, <i>A complete countable $L_{\omega_1}^Q$ theory with maximal models of many cardinalities</i>	691
Wilfred Dennis Pepe and William P. Ziemer, <i>Slices, multiplicity, and Lebesgue area</i>	701
Keith Pierce, <i>Amalgamating abelian ordered groups</i>	711
Stephen James Pride, <i>Residual properties of free groups</i>	725
Roy Martin Rakestraw, <i>The convex cone of n-monotone functions</i>	735
T. Schwartzbauer, <i>Entropy and approximation of measure preserving transformations</i>	753
Peter F. Stebe, <i>Invariant functions of an iterative process for maximization of a polynomial</i>	765
Kondagunta Sundaresan and Wojbor Woyczynski, <i>L-orthogonally scattered measures</i>	785
Kyle David Wallace, <i>C_λ-groups and λ-basic subgroups</i>	799
Barnet Mordecai Weinstock, <i>Approximation by holomorphic functions on certain product sets in C^n</i>	811
Donald Steven Passman, <i>Corrections to: "Isomorphic groups and group rings"</i>	823
Don David Porter, <i>Correction to: "Symplectic bordism, Stiefel-Whitney numbers, and a Novikov resolution"</i>	825
John Ben Butler, Jr., <i>Correction to: "Almost smooth perturbations of self-adjoint operators"</i>	825
Constantine G. Lascarides, <i>Correction to: "A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer"</i>	826
George A. Elliott, <i>Correction to: "An extension of some results of Takesaki in the reduction theory of von Neumann algebras"</i>	826
James Daniel Halpern, <i>Correction to: "On a question of Tarski and a maximal theorem of Kurepa"</i>	827