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**$C_\lambda$ -GROUPS AND  $\lambda$ -BASIC SUBGROUPS**

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## $C_\lambda$ -GROUPS AND $\lambda$ -BASIC SUBGROUPS

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The groups considered in this paper will be abelian primary groups. For  $\lambda$  a fixed but arbitrary countable limit ordinal, C. K. Megibben studied that class  $C_\lambda$  consisting of all  $p$ -groups  $G$  such that  $G/p^\alpha G$  is a direct sum of countable groups for all  $\alpha < \lambda$ .

Fundamental to the development of  $C_\lambda$ -theory was the introduction of the concept of a  $\lambda$ -basic subgroup, which generalized the familiar concept of a basic subgroup, and the following existence theorem: A primary group  $G$  contains a  $\lambda$ -basic subgroup if and only if  $G$  is a  $C_\lambda$ -group. This paper extends, in a natural fashion, the concepts of " $C_\lambda$ -group" and " $\lambda$ -basic subgroup" to an arbitrary limit ordinal  $\lambda$ , and considers the analogous question of existence. This is used to examine the structure of  $p^\lambda$ -pure subgroups of  $C_\lambda$ -groups for limit ordinals  $\lambda$  such that  $\lambda \neq \beta + \omega$  for any ordinal  $\beta$ . For an ordinal  $\lambda$  of this type, if  $H$  is a  $p^\lambda$ -pure subgroup of the  $C_\lambda$ -group  $G$  then both  $H$  and  $G/H$  are  $C_\lambda$ -groups.

The classical theory of torsion abelian groups corresponds to Megibben's  $C_\omega$ -theory, in that the class of all  $p$ -groups coincides with  $C_\omega$ .

1. Preliminaries. In this section we assemble the basic concepts which are crucial in the following development. For pertinent results related to these concepts, we refer the reader to [2].

A subgroup  $H$  of the  $p$ -group  $G$  is said to be a  $p^\alpha$ -pure subgroup if  $H \twoheadrightarrow G \twoheadrightarrow G/H$  represents an element of  $p^\alpha \text{Ext}(G/H, H)$ . This notion is due to Nunke and shall assume the same role in our theory as that played by ordinary purity (i.e.  $p^\alpha$ -purity for  $p$ -groups) in the classical theory.

The subgroup  $H$  is said to be a  $p^\alpha$ -high subgroup of  $G$  if  $H$  is maximal among the subgroups of  $G$  that intersect  $p^\alpha G$  trivially. From [5] or [7], if  $H$  is a  $p^\alpha$ -pure subgroup of  $G$  then  $H \cap p^\beta G = p^\beta H$  for all  $\beta \leq \alpha$  and  $p^\beta(G/H)[p] = (p^\beta G)[p] + H/H$  for all  $\beta < \alpha$ . Moreover, if  $G/H$  is divisible, where  $H$  is a  $p^\alpha$ -pure subgroup of  $G$  and  $\alpha$  is a limit ordinal, then  $H + p^\beta G/p^\beta G = G/p^\beta G$  for all  $\beta < \alpha$ . If  $H$  is a  $p^\alpha$ -high subgroup of  $G$ , then  $H$  is a  $p^{\alpha+1}$ -pure subgroup of  $G$  and  $H + p^\alpha G/p^\alpha G$  is  $p^\alpha$ -pure in  $G/p^\alpha G$  (see [3]).

A subgroup  $H$  of the  $p$ -group  $G$  is *neat* if  $pG \cap H = pH$ . From [3], if  $H$  is a neat subgroup of the  $p$ -group  $G$  and

$$G[p] = H[p] + p^\beta G[p]$$

for each  $\beta < \alpha$ , then  $H$  is  $p^\alpha$ -pure in  $G$ . Moreover, if  $A$  is a neat subgroup of  $p^\alpha G$  and if  $B \cong A$  is maximal in  $G$  with respect to  $B \cap p^\alpha G = A$ , then  $B$  is  $p^{\alpha+1}$ -pure in  $G$ .

A subgroup  $H$  of the  $p$ -group  $G$  is *nice* in  $G$  if each coset  $g+H$  contains an element  $g+h$  that has maximal height in  $G$ . If  $g$  has maximal height in the coset  $g+H$ , we say  $g$  is *proper* with respect to  $H$ .

Totally projective groups as introduced by Nunke provide a generalization of the concept of a direct sum of countable reduced groups. A  $p$ -group  $G$  is  $p^\alpha$ -projective if  $p^\alpha \text{Ext}(G, C) = 0$  for all groups  $C$ . A reduced  $p$ -group  $G$  is *totally projective* if  $Gp^\alpha G$  is  $p^\alpha$ -projective for every ordinal  $\alpha$ . The following characterization of totally projective groups, given and utilized by Hill [4] to show that the Ulm invariants suffice to classify totally projective  $p$ -groups, is used extensively.

**THEOREM A.** *A reduced  $p$ -group  $G$  is totally projective if and only if  $G$  has a collection  $\mathcal{E}_G$  of nice subgroups satisfying the following conditions:*

(0)  $0$  is a member of  $\mathcal{E}_G$ .

(1)  $\mathcal{E}_G$  is closed with respect to group-theoretic union.

(2) If  $A \in \mathcal{E}_G$  and  $H$  is a subgroup of  $G$  such that  $(H+A)/A$  is countable, there exists  $B \in \mathcal{E}_G$  such that  $B \cong H+A$  and  $B/A$  is countable.

*In the sequel, we shall refer to these conditions as the third axiom of countability and to condition (2) as the countable extension property.*

An ordinal  $\lambda$  is said to be *confinal* with  $\omega$  if  $\lambda$  is the limit of a countable ascending sequence of ordinals. From [7], if  $\alpha$  is confinal with  $\omega$  then every  $p^\alpha$ -pure subgroup of a  $p^\alpha$ -projective group is  $p^\alpha$ -projective.

To extend the concepts of  $C_\lambda$ -group and  $\lambda$ -basic subgroup to an arbitrary limit ordinal, we introduce the following definitions. For a fixed but arbitrary limit ordinal  $\lambda$ ,  $C_\lambda$  shall designate that class of all  $p$ -groups  $G$  such that  $G/p^\alpha G$  is totally projective for all  $\alpha < \lambda$ . Groups in the class  $C_\lambda$  will be referred to as  $C_\lambda$ -groups.  $B$  is said to

be a  $\lambda$ -basic subgroup of  $G$  if

- (1)  $B$  is totally projective of length at most  $\lambda$ ,
- (2)  $B$  is a  $p^\lambda$ -pure subgroup of  $G$ , and
- (3)  $G/B$  is divisible.

For  $B$  a  $\lambda$ -basic subgroup of  $G$  and  $\alpha < \lambda$ , a routine argument yields that the  $\alpha$ -th Ulm invariant of  $B$  coincides with the  $\alpha$ -th Ulm invariant of  $G$ . Hence, by Hill's version of Ulm's Theorem, we obtain the following analogue to a well-known property of ordinary basic subgroups

**PROPOSITION 1.1.** *If  $B$  and  $\bar{B}$  are  $\lambda$ -basic subgroups of  $G$  then  $B \cong \bar{B}$ .*

We shall require for technical convenience the notion of a  $\lambda$ -high confinal tower. Let  $\lambda$  be an ordinal confinal with  $\omega$ , and  $G$  an abelian  $p$ -group. A  $\lambda$ -high confinal tower of  $G$  is an ascending sequence  $\{G_n\}$  of subgroups of  $G$  such that:

- (1) For each positive integer  $n$ ,  $G_n$  is a  $p^{\alpha(n)}$ -high subgroup of  $G$ ;
- (2)  $\lambda = \sup \{\alpha(n)\}$ ,  $\alpha(n) < \alpha(n + 1)$ ;
- (3) If  $\lambda = \beta + \omega$  for some limit ordinal  $\beta$ , then  $\alpha(n) = \beta + m$  for some positive integer  $m$ ;
- (4) If  $\lambda \neq \beta + \omega$  for any ordinal  $\beta$ , then  $\alpha(n) = \beta(n) + \omega$  for some limit ordinal  $\beta(n)$ .

**2. The existence theorem.** In this section we determine, for an arbitrary but fixed limit ordinal  $\lambda$ , that class of all abelian  $p$ -groups  $G$  such that  $G$  contains a  $\lambda$ -basic subgroup (see Theorem 2.7).

**LEMMA 2.1.** *Suppose  $G/p^\beta G$  is totally projective and  $B$  is a basic subgroup of  $p^\beta G$ . If  $H$  is a subgroup of  $G$  such that*

$$G/B = H/B \oplus p^\beta G/B$$

*then  $H$  is totally projective.*

*Proof.* If  $H$  is a subgroup of  $G$  such that  $G/B = H/B \oplus p^\beta G/B$ , then  $G = H + p^\beta G$  and  $H$  is maximal in  $G$  with respect to  $H \cap p^\beta G = B$ . Thus  $H$  is  $p^{\beta+1}$ -pure in  $G$ . Consequently  $p^\alpha H = p^\alpha G \cap H$  for all  $\alpha \leq \beta + 1$ , and in particular  $p^\beta H = p^\beta G \cap H = B$ . We now observe that  $H/p^\beta H$  is totally projective since

$$H/p^\beta H = H/p^\beta G \cap H \cong (H + p^\beta G)/p^\beta G = G/p^\beta G,$$

and  $p^\beta H = B$  is a direct sum of cyclic groups.

**LEMMA 2.2.** *Let  $\lambda$  be a limit ordinal confinal with  $\omega$  such that  $\lambda \neq \beta + \omega$  for any ordinal  $\beta$ . Suppose  $G = \mathbf{U}G_n$  with  $\{G_n\}$  a  $\lambda$ -high confinal tower of  $G$ . If  $A \subseteq G$  satisfies the conditions:*

(1)  *$A$  is the union of an ascending sequence of subgroups  $A_1 \subseteq A_2 \subseteq \dots$  such that  $A_n$  is nice in  $G_n$  for each  $n$ ,*

(2)  *$A \subseteq p^\alpha G + A_n$  for all  $\alpha < \alpha(n)$ ;*

*then  $A$  is nice in  $G$ .*

*Proof.* We show that each coset  $x + A$  contains an element  $x + a$  that is proper with respect to  $A$ .

Let  $x \in G - A$ , and choose  $n$  such that  $x \in G_n$ . Let

$$\beta = h_G(x) < \alpha(n).$$

For  $k \geq n$ , there exists  $a_k \in A_k$  such that  $h_G(x + a_k) = h_{G_k}(x + a_k) \geq h_{G_k}(x + a') = h_G(x + a')$  for any  $a' \in A_k$ . It suffices to show that the sequence  $h_G(x + a_n) \leq h_G(x + a_{n+1}) \leq \dots$  cannot be strictly increasing.

Suppose for some  $m \geq n$  that  $h_G(x + a_m) > \beta = h_G(x)$ . Then  $h_G(a_{m+i}) = h_G(x)$  for  $i = 1, 2, \dots$ . Let  $\gamma = h_G(x + a_m)$  and observe  $\gamma < \alpha(m)$  since  $x + a_m \in G_m$ . Moreover  $\gamma + 1 < \alpha(m)$  since  $\alpha(m)$  is a limit ordinal. We shall show that  $x + a_m$  is proper with respect to  $A$ . Suppose  $x + a_m$  is not proper with respect to  $A$ . Then for some  $k$ ,  $h_G(x + a_{m+k}) > h_G(x + a_m) = \gamma$  and  $x + a_{m+k} \in p^{\gamma+1}G$ . Since  $A \subseteq p^{\gamma+1}G + A_m$  we have  $a_{m+k} = g_k + a_{m,k}$  with  $g_k \in p^{\gamma+1}G$  and  $a_{m,k} \in A_m$ . Hence

$$x + a_{m,k} = x + g_k + a_{m,k} \in p^{\gamma+1}G$$

and  $x + a_{m,k} \in p^{\gamma+1}G$ . This however is absurd since

$$h_G(x + a_{m,k}) \leq h_G(x + a_m) = \gamma.$$

Consequently  $x + a_m$  is proper with respect to  $A$  and  $A$  is nice in  $G$ .

With  $\lambda$  and  $G$  as in Lemma 2.2, we shall now restrict our attention to the case where  $G_n$  is totally projective for each  $n$ . Let  $\mathcal{E}_n$  denote a collection of nice subgroups of  $G_n$  satisfying the third axiom of countability. Let  $\mathcal{C}$  be the collection of all subgroups  $A$  of  $G$  such that

(1)  $A = \mathbf{U} A_n$  with  $A_1 \subseteq A_2 \subseteq \dots$  and  $A_n \in \mathcal{E}_n$  for each  $n$ ,

(2)  $A \subseteq p^\alpha G + A_n$  for all  $\alpha < \alpha(n)$ .

The members of  $\mathcal{C}$  are nice by Lemma 2.2.

**LEMMA 2.3.**  *$\mathcal{C}$  has the countable extension property.*

*Proof.* For each  $n$ , we have  $\alpha(n) = \beta(n) + \omega$  with  $\beta(n)$  a limit ordinal. Thus  $\lambda = \sup\{\alpha(n)\} = \sup\{\beta(n)\}$ . We observe that to show

$B \subseteq p^\alpha G + B_n$  for each ordinal  $\alpha < \alpha(n)$ , it suffices to show

$$B \subseteq p^{\beta(n)+k}G + B_n \quad \text{for each } k < \omega .$$

Let  $A \in \mathcal{C}$  and  $H$  a subgroup of  $G$  such that  $H/A$  is countable. Let  $S = \{x_i\}_{i < \omega}$  be such that  $H = \langle A, S \rangle$  and let  $S_n = S \cap G_n$ . By induction, we shall construct, for each positive integer  $n$ , subgroups  $B_1^{(n)} \subseteq B_2^{(n)} \subseteq \dots \subseteq B_n^{(n)}$  such that

- (0)  $A_i \subseteq B_i^{(n)}$  for  $i \leq n$ ,
- (1)  $B_j^{(k)} \subseteq B_i^{(n)}$  for  $k \leq n$ ,
- (2)  $B_i^{(n)} \in \mathcal{C}_i$ ,
- (3)  $B_{i+1}^{(n)} \subseteq p^{\beta(i)+k}G + B_i^{(n)}$  for  $k < \omega$ ,
- (4)  $B_i^{(n)} \supseteq S_i$ ,
- (5)  $B_i^{(n)}/A_i$  is countable for  $i \leq n$ .

We now show that the existence of subgroups  $B_i^{(n)}$  satisfying the above conditions (0) – (5) will suffice to establish the lemma. For each  $i < \omega$ , let  $B_i = \bigcup_{n \geq i} B_i^{(n)}$  and observe that  $B_i \in \mathcal{C}_i$  and  $B_i \subseteq B_{i+1}$ . Moreover  $B_{i+1} = \bigcup_{n \geq i+1} B_{i+1}^{(n)} \subseteq \bigcup_{n \geq i+1} (p^{\beta(i)+k}G + B_i^{(n)}) = p^{\beta(i)+k}G + B_i$  for  $k < \omega$ , and by induction  $B_{i+m} \subseteq p^{\beta(i)+k}G + B_i$  for all  $m < \omega$ ,  $k < \omega$ . Let  $B = \bigcup_{i < \omega} B_i$ . Clearly  $B \supseteq H$  and  $B/A$  is countable. Moreover  $B \in \mathcal{C}$  since  $B \subseteq p^{\beta(i)+k}G + B_i$  for each  $i$  and  $k$ .

Suppose we have constructed  $B_i^{(n)}$ ,  $1 \leq i \leq n$ , satisfying (0) – (5) above. We shall now construct  $B_i^{(n+1)}$  for  $1 \leq i \leq n + 1$ .

For  $1 \leq i \leq n$ , let  $B_{i,0} = A_i$  and  $B_{i,1} = B_i^{(n)}$ . Set  $B_{n+1,0} = A_{n+1}$  and let  $B_{n+1,1}$  be a member of  $\mathcal{C}_{n+1}$  such that

$$B_{n+1,1} \supseteq \langle B_n^{(n)} + A_{n+1}, S_{n+1} \rangle$$

and  $B_{n+1,1}/A_{n+1}$  is countable. By induction, we shall construct a family of subgroups  $B_{i,j}$ , with  $1 \leq i \leq n + 1$  and  $j < \omega$ , satisfying the conditions

- (i)  $B_{i,j} \subseteq B_{i,k}$  for  $j \leq k$ ,
- (ii)  $B_{i,j} \in \mathcal{C}_i$ ,
- (iii)  $B_{i,j+1}/B_{i,j}$  is countable,
- (iv)  $B_{i+1,2j} \subseteq p^{\beta(i)+k}G + B_{i,2j}$  for all  $1 \leq i \leq n$  and  $j, k < \omega$ ;
- (v)  $B_{i,2j+1} \subseteq B_{i+1,2j+1}$  for all  $1 \leq i \leq n$  and  $j < \omega$ .

We define  $B_i^{(n+1)} = \bigcup_{j < \omega} B_{i,j}$  and observe that

$$\bigcup_{j < \omega} B_{i,2j} = B_i^{(n+1)} = \bigcup_{j < \omega} B_{i,2j+1} .$$

By (iv), we see that

$$B_{i+1}^{(n+1)} = \bigcup_{j < \omega} B_{i+1,2j} \subseteq \bigcup_{j < \omega} (p^{\beta(i)+k}G + B_{i,2j}) = p^{\beta(i)+k}G + B_i^{(n+1)}$$

for all  $k < \omega$ ,  $1 \leq i \leq n$ . By (v),  $|B_i^{(n+1)} = \bigcup_{j < \omega} B_{i,2j+1} \subseteq \bigcup_{j < \omega} B_{i+1,2j+1} = B_{i+1}^{(n+1)}$  for all  $1 \leq i \leq n$ . It is now easy to see that conditions

(0) – (4) are satisfied by the subgroups  $B_i^{(j)}$ ,  $1 \leq i \leq j \leq n + 1$ . Since  $B_{i,j+1}/B_{i,j}$  is countable for each

$$j < \omega, B_i^{(n+1)}/A_i = B_i^{(n+1)}/B_{i,0} = \bigcup_{j < \omega} B_{i,j+1}/B_{i,0}$$

is countable for all  $1 \leq i \leq n + 1$  and condition (5) is satisfied.

Suppose we have constructed  $B_{i,j}$  satisfying (i) – (v), for all  $1 \leq i \leq n + 1$  and all  $j \leq 2m + 1$ . We shall now construct  $B_{i,2m+2}$  for  $1 \leq i \leq n + 1$ . Define  $B_{n+1,2m+2} = B_{n+1,2m+1}$ . Assuming, for some positive integer  $l \leq n$ , that  $B_{i,2m+2}$  has been constructed for each  $l + 1 \leq i \leq n + 1$ , we let  $\{x_j\}_{j < \omega} \subseteq B_{l+1,2m+2}$  be such that  $B_{l+1,2m+2} = \langle B_{l+1,2m}, \{x_j\}_{j < \omega} \rangle$ . Since  $G \subseteq p^{\beta(l)+k}G + G_l$  we obtain decompositions  $x_j = g_{j,k} + x_{j,k}$ , with  $g_{j,k} \in p^{\beta(l)+k}G$  and  $x_{j,k} \in G_l$ , for each  $j, k < \omega$ . Let  $T_{l,2m+2} = \{x_{j,k}\}_{j,k < \omega} \subseteq G_l$ . Let  $B_{l,2m+2}$  be a member of  $\mathcal{E}_l$  such that  $B_{l,2m+2} \supseteq \langle B_{l,2m+1}, T_{l,2m+2} \rangle$  and  $B_{l,2m+2}/B_{l,2m+1}$  is countable. Observe, for each  $k < \omega$ ,  $B_{l+1,2m+2} \subseteq p^{\beta(l)+k}G + B_{l,2m+2}$  since

$$B_{l+1,2m} \subseteq p^{\beta(l)+k}G + B_{l,2m} \subseteq p^{\beta(l)+k}G + B_{l,2m+1} \subseteq p^{\beta(l)+k}G + B_{l,2m+2}$$

and

$$\{x_j\}_{j < \omega} \subseteq p^{\beta(l)+k}G + \langle T_{n,2m+2} \rangle \subseteq p^{\beta(l)+k}G + B_{l,2m+2} .$$

To conclude the proof, it suffices to construct  $B_{i,2m+3}$  for  $1 \leq i \leq n + 1$ , having been given a collection  $B_{i,j}$  satisfying (i) – (v), for all  $1 \leq i \leq n + 1$  and all  $j \leq 2m + 2$ . Define  $B_{1,2m+3} = B_{1,2m+2}$  and assume, for some positive integer  $l \leq n$ , that  $B_{i,2m+3}$  has been constructed for each  $1 \leq i \leq l$ . Since  $B_{l,2m+3}/B_{l,2m+1}$  is countable and  $B_{l,2m+1} \subseteq B_{l+1,2m+1} \subseteq B_{l+1,2m+2}$ ,  $(B_{l,2m+3} + B_{l+1,2m+2})/B_{l+1,2m+2}$  is countable. Thus there exists  $B_{l+1,2m+3} \in \mathcal{E}_{l+1}$  such that  $B_{l+1,2m+3} \supseteq B_{l+1,2m+3} + B_{l+1,2m+2}$  and  $B_{l+1,2m+3}/B_{l+1,2m+2}$  is countable. The collection of subgroups  $B_{i,j}$ , for  $1 \leq i \leq n + 1$  and  $0 \leq j \leq 2m + 3$ , clearly satisfies conditions (i) – (v).

**LEMMA 2.4.** *If  $\alpha$  is confinal with  $\omega$  and  $G/p^\alpha G$  is totally projective, then every  $p^\alpha$ -high subgroup of  $G$  is totally projective.*

*Proof.* Let  $\alpha$  be an ordinal confinal with  $\omega$ , and  $H$  a  $p^\alpha$ -high subgroup of  $G$ . Since  $H \cong (H + p^\alpha G)/p^\alpha G$  and  $(H + p^\alpha G)/p^\alpha G$  is  $p^\alpha$ -pure in the  $p^\alpha$ -projective group  $G/p^\alpha G$ ,  $H$  is  $p^\alpha$ -projective. Since  $\alpha$  is a limit ordinal,  $H/p^\beta H \cong G/p^\beta G$  is  $p^\beta$ -projective for all  $\beta < \alpha$ . Consequently  $H$  is totally projective.

**PROPOSITION 2.5.** *Let  $\lambda$  be a limit ordinal confinal with  $\omega$ , and*

$\{G_n\}$  a  $\lambda$ -high confinal tower of  $G$ . If  $G$  is a  $C_\lambda$ -group then  $\bigcup G_n$  is totally projective of length at most  $\lambda$ .

*Proof.* Clearly  $\bigcup G_n$  is an isotype subgroup of  $G$  and hence has length at most  $\lambda$ . The proof that  $\bigcup G_n$  is totally projective shall consist of two cases.

*Case 1.*  $\lambda = \beta + \omega$ .

Consider the subgroup  $(\bigcup G_n) \cap p^\beta G$  of  $p^\beta C$ , and observe that

$$G_n \cap p^{\beta+\omega} G = 0$$

for each  $n$ . Consequently  $p^\omega((\bigcup G_n) \cap p^\beta G) = \bigcup(G_n \cap p^{\beta+\omega} G) = 0$  and thus  $(\bigcup G_n) \cap p^\beta G$  is a  $p$ -group without elements of infinite height. Since

$$(\bigcup G_n) \cap p^\beta G = \bigcup (G_n \cap p^\beta G) = \bigcup p^\beta G_n$$

is the union of an ascending sequence of bounded subgroups, it follows, by the Kulikov criterion, that  $(\bigcup G_n) \cap p^\beta G$  is a direct sum of cyclic groups. It is easy to see that  $(\bigcup G_n) \cap p^\beta G$  is a pure subgroup of  $p^\beta G$  and that  $\bigcup G_n + p^\beta G = G$ . Consequently  $(\bigcup G_n) \cap p^\beta G$  is a basic subgroup of  $p^\beta G$ . Since  $G$  is a  $C_\lambda$ -group,  $G/p^\beta G$  is totally projective and, by Lemma 2.1, it follows that  $\bigcup G_n$  is totally projective.

*Case 2.*  $\lambda \neq \beta + \omega$  for any ordinal  $\beta$ .

By Lemma 2.4, it follows that in this case  $G_n$  is totally projective for each  $n$ . To show that  $\bigcup G_n$  contains a collection of nice subgroups satisfying the third axiom of countability, let  $\mathcal{E}$  denote the collection of nice subgroups of  $\bigcup G_n$  as defined preceding Lemma 2.3. Clearly  $0 \in \mathcal{E}$ . By Lemma 2.3,  $\mathcal{E}$  has the countable extension property. Thus it suffices to show that  $\mathcal{E}$  is closed with respect to group-theoretic union. Suppose  $\{A_\gamma\}_{\gamma \in I} \subseteq \mathcal{E}$  with  $A_\gamma = \bigcup_{n < \omega} A_{n,\gamma}$  where

- (1)  $A_{n,\gamma} \subseteq A_{k,\gamma}$  for  $n \leq k$ ,
- (2)  $A_{n,\gamma} \in \mathcal{E}_n$  for each  $n$ .
- (3) For each  $n$  and  $\alpha < \alpha(n)$ ,  $A_\gamma \subseteq p^\alpha G + A_{\gamma,n}$ .

Then  $\sum_{\gamma \in I} A_\gamma = \sum_{\gamma \in I} (\bigcup_{n < \omega} A_{n,\gamma}) = \bigcup_n (\sum_{\gamma \in I} A_{n,\gamma})$  with

$$\sum_{\gamma \in I} A_{n,\gamma} \subseteq \sum_{\gamma \in I} A_{k,\gamma} \quad \text{for } n \leq k,$$

and  $\sum_{\gamma \in I} A_{n,\gamma} \in \mathcal{E}_n$ . Moreover, for each  $n$  and  $\alpha < \alpha(n)$ , we have

$$\sum_{\gamma \in I} A_\gamma \subseteq \sum_{\gamma \in I} (p^\alpha G + A_{n,\gamma}) = p^\alpha G + (\sum_{\gamma \in I} A_{n,\gamma}).$$



Consequently  $\sum_{\gamma \in I} A_\gamma \in \mathcal{E}$ .

**LEMMA 2.6.** *Let  $\{G_n\}$  be a  $\lambda$ -high confinal tower of  $G$ . If  $H = \bigcup G_n$  then  $H$  is  $p^\lambda$ -pure in  $G$ .*

*Proof.* Let  $\alpha < \lambda$ , and recall that  $H$  is an isotype, and hence a neat, subgroup of  $G$ . There exists a positive integer  $n$  such that  $\alpha < \alpha(n)$ , and  $G[p] = G_n[p] + (p^\alpha G)[p] = H[p] + (p^\alpha G)[p]$ .

**THEOREM 2.7.** (a) *If  $G$  is a  $C_\lambda$ -group with  $\lambda$  confinal with  $\omega$  then  $G$  contains a  $\lambda$ -basic subgroup.*

(b) *If  $G$  is a reduced  $p$ -group which contains a proper  $\lambda$ -basic subgroup then  $G$  is a  $C_\lambda$ -group and  $\lambda$  is confinal with  $\omega$ .*

*Proof.* Part (a) follows from Proposition 2.5 and Lemma 2.6.

Conversely, suppose  $H$  is a proper  $\lambda$ -basic subgroup of the reduced  $p$ -group  $G$ . For  $\alpha < \lambda$ ,

$$G/p^\alpha G = (H + p^\alpha G)/p^\alpha G \cong H/(H \cap p^\alpha G) = H/p^\alpha H$$

is totally projective. Thus  $G$  is a  $C_\lambda$ -group. That  $\lambda$  must be confinal with  $\omega$  is immediate from (3.7) of [1] and (3.10) of [7].

**3.  $C_\lambda$ -Groups for  $\lambda \neq \beta + \omega$ .** The purpose of this section is to examine the structure of  $p^\lambda$ -pure subgroups of  $C_\lambda$ -groups. We shall restrict our attention to ordinals that cannot be expressed in the form  $\beta + \omega$  for any ordinal  $\beta$ . The techniques utilized are essentially those of Megibben in [6] and rely upon the existence of  $\lambda$ -basic subgroups as established in § 2.

The proofs given for Lemma 3 in [6] can, with the aid of § 2, be reproduced to yield the following lemmas.

**LEMMA 3.1.** *Let  $\lambda$  be an ordinal confinal with  $\omega$ . Suppose  $H$  is a  $p^\lambda$ -pure subgroup of  $G$  and that  $\{H_n\}$  is a  $\lambda$ -high confinal tower of  $H$ . Then there exists a  $\lambda$ -high confinal tower  $\{G_n\}$  of  $G$  such that, for each  $n$ ,  $H_n \subseteq G_n$  and  $H_n = H \cap G_n$ .*

**LEMMA 3.2.** *Let  $\lambda$  be an ordinal confinal with  $\omega$  such that  $\lambda \neq \beta + \omega$  for any  $\beta$ . Suppose  $G$  is totally projective and that  $G = \bigcup G_n$  where  $\{G_n\}$  is a  $\lambda$ -high confinal tower. If  $H$  is a  $p^\lambda$ -pure subgroup of  $G$  such that, for each  $n$ ,  $H \cap G_n$  is a  $p^{\alpha(n)}$ -high subgroup of  $H$ , then  $H$  is a direct summand of  $G$ .*

**THEOREM 3.3.** *Let  $\lambda$  be any limit ordinal such that  $\lambda \neq \beta + \omega$  for any  $\beta$ , and let  $G$  be a  $C_\lambda$ -group. If  $H$  is a  $p^\lambda$ -pure subgroup of  $G$  then  $H$  is a  $C_\lambda$ -group.*

*Proof.* It suffices to establish the proposition for ordinals  $\lambda$  such that  $\lambda$  is confinal with  $\omega$  and  $\lambda \neq \beta + \omega$  for any ordinal  $\beta$ . For such an ordinal  $\lambda$ , let  $\{H_n\}$  be a  $\lambda$ -high confinal tower of  $H$ . By Lemma 3.1, there exists a  $\lambda$ -high confinal tower  $\{G_n\}$  of  $G$  such that  $H_n = H \cap G_n$  for each  $n$ . Since  $G$  is a  $C_\lambda$ -group,  $\bigcup G_n$  is totally projective, by Proposition 2.5. By Lemma 3.2, it follows that  $\bigcup H_n$  is a  $\lambda$ -basic subgroup of  $H$  and consequently, by Theorem 2.7,  $H$  is a  $C_\lambda$ -group.

**LEMMA 3.4.** *Let  $\lambda$  be confinal with  $\omega$ ,  $\lambda \neq \beta + \omega$  for any  $\beta$ . Let  $A$  be a totally projective group of length at most  $\lambda$  and suppose  $A$  is a  $p^\lambda$ -pure subgroup of the  $C_\lambda$ -group  $G$ . Then there exists a subgroup  $C$  of  $G$  such that  $A \oplus C$  is a  $\lambda$ -basic subgroup of  $G$ .*

*Proof.* Since  $A$  is a totally projective group of length at most  $\lambda$ , it follows from Proposition 1.1 that  $A$  is the union of a  $\lambda$ -high confinal tower  $\{A_n\}$  of itself. By Lemma 3.1, there exists a  $\lambda$ -high confinal tower  $\{G_n\}$  of  $G$  such that  $A_n = A \cap G_n$  for each  $n$ . Let  $B = \bigcup G_n$ . By the proof of Theorem 2.7,  $B$  is a  $\lambda$ -basic subgroup of  $G$ . But  $\{G_n\}$  is also a  $\lambda$ -high confinal tower of  $B$  and, by Lemma 3.2, we have the desired decomposition  $B = A \oplus C$ .

**THEOREM 3.5.** *Let  $\lambda$  be a limit ordinal such that  $\lambda \neq \beta + \omega$  for any ordinal  $\beta$ , and let  $G$  be a  $C_\lambda$ -group. If  $H$  is a  $p^\lambda$ -pure subgroup of  $G$  then  $G/H$  is a  $C_\lambda$ -group.*

*Proof.* It suffices to establish the result for an arbitrary but fixed ordinal  $\lambda$  satisfying the conditions that  $\lambda$  is confinal with  $\omega$  and  $\lambda \neq \beta + \omega$  for any ordinal  $\beta$ . Let  $\lambda$  be such an ordinal and  $H$  a  $p^\lambda$ -pure subgroup of the  $C_\lambda$ -group  $G$ . By Theorem 3.3,  $H$  is a  $C_\lambda$ -group and thus, by Theorem 2.7, contains a  $\lambda$ -basic subgroup. Let  $A$  be a  $\lambda$ -basic subgroup of  $H$  and choose  $C$ , by Lemma 3.4, such that  $A \oplus C$  is a  $\lambda$ -basic subgroup of  $G$ . If  $x \in (H \cap C)[p]$ , we can write, for each  $\alpha < \lambda$ ,  $x = a_\alpha + z_\alpha$  where  $a_\alpha \in A[p]$  and  $z_\alpha \in p^\alpha H$ . Thus  $-a_\alpha + x \in p^\alpha(A \oplus C) = p^\alpha A \oplus p^\alpha C$  and  $x \in \bigcap p^\alpha C = p^\lambda C = 0$ . We then have a direct decomposition  $H \oplus C$ . If  $pg \in H \oplus C$ , then

$$pg = a + ph + c$$

where  $a \in A, h \in H$  and  $c \in C$ . Since  $pG \cap (A \oplus C) = p(A \oplus C)$ , we

conclude that  $pG \cap (H \oplus C) = p(H \oplus C)$  and  $H \oplus C$  is neat in  $G$ . Moreover,  $G[p] \cong (A \oplus C)[p] + p^\alpha G \cong (H \oplus C)[p] + p^\alpha G$  for all  $\alpha < \lambda$  and therefore  $H \oplus C$  is a  $p^\lambda$ -pure subgroup of  $G$ . Consequently,  $(H \oplus C)/H$  is  $p^\lambda$ -pure in  $G/H$ . Also  $(H \oplus C)/H \cong C$  is totally projective of length at most  $\lambda$ , and

$$(G/H)/(H \oplus C/H) \cong (G/A \oplus C)/(H \oplus C/A \oplus C)$$

is divisible. We have constructed a  $\lambda$ -basic subgroup of  $G/H$  and we conclude that  $G/H$  is indeed a  $C_\lambda$ -group.

As easy consequences of Theorem 3.5, we have the following analogues of familiar properties of pure subgroups.

**COROLLARY 3.6.** *Suppose  $\lambda$  is a limit ordinal such that*

$$\lambda \neq \beta + \omega$$

*for any ordinal  $\beta$ . A subgroup  $H$  of a  $C_\lambda$ -group  $G$  is a  $p^\lambda$ -pure subgroup if and only if  $(H + p^\alpha G)/p^\alpha G$  is a direct summand of  $G/p^\alpha G$  for all  $\alpha < \lambda$ .*

**COROLLARY 3.7.** *Suppose  $\lambda$  is a limit ordinal such that*

$$\lambda \neq \beta + \omega$$

*for any ordinal  $\beta$ . If  $H$  is a  $p^\lambda$ -pure subgroup of the  $C_\lambda$ -group  $G$  and if  $p^\alpha H = 0$  for some  $\alpha < \lambda$ , then  $H$  is a direct summand of  $G$ .*

**4. Remark.** As noted above, we have not dealt with the problems of  $p^\lambda$ -pure subgroups of  $C_\lambda$ -groups where  $\lambda$  is a limit ordinal which may be expressed in the form  $\lambda = \beta + \omega$ . It would not be surprising, however, if the results of §3 fail to hold for certain of such ordinals.

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