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THE EVALUATION MAP AND EHP SEQUENCES

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Let $L(\Sigma B, X)$ be the space of maps from ΣB (the reduced suspension of B) to X with the compact-open topology, let $\angle: \Sigma B \rightarrow X$ and $L(\Sigma B, X; \angle)$ the path component of $L(\Sigma B, X)$ containing \angle . For nice spaces the evaluation map $\omega: L(\Sigma B, X, \angle) \rightarrow X$ defined by $\omega(f) = f(*)$ is a fibration and gives rise to a long exact sequence in homotopy. The purpose of this paper is to show that the boundary map in that long exact sequence can be given by a generalized Whitehead product and that the sequence generalizes the EHP sequence of G. W. Whitehead.

1. Preliminary definitions. All spaces are assumed to be CW complexes with base point at a vertex. Maps are base point preserving. The cartesian product $A \times B$ is assumed to be based at (a_0, b_0) , the unit interval, I , is based at 0, and quotient spaces are based at the image of the base point under the natural quotient map. Where the space is clear $*$ will denote the base point as well as the constant map with image at the base point.

We use the following notations. $L(A, B)$ will denote the space of maps from A to B with the compact-open topology and $L(A, B; \angle)$ the path component of $L(A, B)$ containing $\angle: A \rightarrow B$. $L_0(A, B)$ and $L_0(A, B; \angle)$ will denote the space of base point preserving maps in $L(A, B)$ and $L(A, B; \angle)$ respectively. Let $A \vee B$ and $A \# B$ denote the one point union and smash product respectively.

Since spaces are assumed to be CW complexes the smash product can be taken as $A \times B$ with $A \vee B$ identified with (a_0, b_0) . $q: A \times B \rightarrow A \# B$ will denote the quotient map. Note that $S^{p+q} = \Sigma^p S^q = S^p \# S^q$, $\Sigma^p A = S^p \# A$, and $\Sigma(A \vee B) = \Sigma A \vee \Sigma B$.

Let $p_1, p_2: A \times B \rightarrow A \vee B$ be defined by $p_1(a, b) = a \vee b_0$ and $p_2(a, b) = a_0 \vee b$. Define $k: \Sigma(A \times B) \rightarrow \Sigma A \vee \Sigma B$ by $k = \Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2$. Since $k| \Sigma(A \vee B)$ homotopically trivial, by the homotopy extension property there is a map $k': \Sigma(A \times B) \rightarrow \Sigma A \vee \Sigma B$, homotopic to k , such that $k'| \Sigma(A \vee B) = *$. k' then induces a map $\tilde{k}: \Sigma(A \# B) \rightarrow \Sigma A \vee \Sigma B$. Arkowitz [1] has shown that $[\tilde{k}]$ is uniquely determined by the requirement $k \cong \tilde{k} \circ \Sigma q$. The following definition is due to Arkowitz [1].

DEFINITION 1.1. For $\alpha = [f] \in [\Sigma A, X]$ and $\beta = [g] \in [\Sigma B, X]$, the generalized Whitehead product $[\alpha, \beta]$ is defined by $[\alpha, \beta] = [(f \vee g) \circ \tilde{k}] \in [\Sigma(A \# B)X]$.

Hardie shows (Theorem 2.3 in [2]) that the map $\Sigma p_1 + \Sigma p_2 + \Sigma q: \Sigma(A \times B) \rightarrow \Sigma A \vee \Sigma B \vee \Sigma(A \# B)$ is a homotopy equivalence for A and B CW complexes with a single vertex. Then there is a map $\phi: \Sigma(A \# B) \rightarrow \Sigma(A \times B)$ such that $\Sigma q \circ \phi \cong 1_{\Sigma(A \# B)}$.

DEFINITION 1.2. If $f: A \times B \rightarrow X$, where A and B have a single vertex, the element obtained from f by the *generalized Hopf construction* is defined to be the map $\Sigma f \circ \phi: \Sigma(A \# B) \rightarrow \Sigma X$.

Hardie shows in [2] that if A and B are spheres, Definition 1.2 reduces to the classical definition of the Hopf construction.

Let $\phi_r: S^r \rightarrow S^r \vee S^r$ be the map which identifies the equator of S^r . G. W. Whitehead (Theorem 1.17 in [6]) shows for $n < p + q + \min(p, q) - 3$ that $\pi_n(S^p \vee S^q) = \pi_n(S^p) \oplus \pi_n(S^q) \oplus \pi_n(S^{p+q-1})$. Let $Q: \pi_n(S^p \vee S^q) \rightarrow \pi_n(S^{p+q-1})$ be the natural projection onto the direct summand $\pi_n(S^{p+q-1})$.

DEFINITION 1.3. For $n < 3r - 3$ the *generalized Hopf invariant* $\tilde{H}: \pi_n(S^r) \rightarrow \pi_n(S^{2r-1})$ is defined by $\tilde{H} = Q \circ \phi_{r*}$.

DEFINITION 1.4. For $\lambda = [\angle] \in [\Sigma B, X]$ the λ -Whitehead homomorphism $P_\lambda: [\Sigma A, X] \rightarrow [\Sigma(A \# B), X]$ is defined by $P_\lambda(\alpha) = [\alpha, \lambda]$.

DEFINITION 1.5. If $F: A \rightarrow L(B, X)$ the map $G: A \times B \rightarrow X$ given by $G(a, b) = F(a)(b)$ is said to be an associated map for F .

2. The λ -component *EHP* sequence. The purpose of this section is to show that the map P_λ of Definition 1.4 is embedded in a long exact sequence resulting from the fibration $\omega: L(\Sigma B, X; \angle) \rightarrow X$. Each $\lambda \in [\Sigma B, X]$ determines a path component of $L(\Sigma B, X)$ and ω restricted to each path component determine a fibration and a long exact sequence. In §3 the relationship between these sequences and the James suspension sequence is explored and it is shown that G. W. Whitehead's *EHP* sequence [7] is a special case of an ι_n -component *EHP* sequence where $\iota_n = [1_{S^n}]$ in $\pi_n(S^n)$.

LEMMA 2.1. For $\angle \in L_0(\Sigma B, X)$, $L_0(\Sigma B, X; *)$ is homotopy equivalent to $L_0(\Sigma B, X; \angle)$.

Proof. Let $\hat{\angle}: L_0(\Sigma B, X; *) \rightarrow L_0(\Sigma B, X; \angle)$ be defined by $\hat{\angle}(g) = g + \angle$ and $\hat{\angle}^{-1}: L_0(\Sigma B, X; \angle) \rightarrow L_0(\Sigma B, X; *)$ by $\hat{\angle}^{-1}(g) = g - \angle$. Then it is clear that $\hat{\angle}^{-1}$ is a two sided homotopy inverse of $\hat{\angle}$.

In remaining parts of this section the map $\hat{\angle}$ will be taken to be given by

$$\hat{\epsilon}(g)(b, t) = \begin{cases} g\left(b, \frac{5}{4}t\right) & 0 \leq t \leq \frac{4}{5} \\ \epsilon(b, 5t - 4) & \frac{4}{5} \leq t \leq 1 \end{cases}.$$

LEMMA 2.2. $[\Sigma(A \# B), X]$ is isomorphic to $[A, L_0(\Sigma B, X; *)]$.

This fact is well known. For the remainder of this section the isomorphism will be denoted by $\theta: [\Sigma(A \# B), X] \rightarrow [A, L_0(\Sigma B, X; *)]$ defined as follows. If $f: \Sigma(A \# B) \rightarrow X$, $\theta(f)(a)$ is the map taking (b, t) to $f((a, b), t)$ in X .

DEFINITION 2.3. $A @ B$ is defined as $A \times B$ with $A \times \{b_0\}$ identified with (a_0, b_0) .

Let $m: A \times \Sigma B \rightarrow (A \# \Sigma B) \vee (A @ \Sigma B)$ be defined by

$$(m(a, (b, t))) = \begin{cases} \left(a, \left(b, \frac{5}{4}t\right)\right) \vee * & 0 \leq t \leq \frac{4}{5} \\ * \vee (a, (b, 5t - 4)) & \frac{4}{5} \leq t \leq 1 \end{cases}.$$

Now let $G: A \# \Sigma B \rightarrow X$ be a map associated with $[g] \in [A, L_0(\Sigma B, X; *)]$, $\epsilon \in L_0(\Sigma B, X)$, and $p_2: A @ \Sigma B \rightarrow \Sigma B$ the natural projection.

The following lemma can be easily verified.

LEMMA 2.4. $(G \vee (\epsilon \circ p_2)) \circ m: A \times \Sigma B \rightarrow X$ is an associated map for $\hat{\epsilon}_*([g]) \in [A, L_0(\Sigma B, X; \epsilon)]$.

Let $h_1: A \times SB \rightarrow \Sigma(A \times B)$ be defined by $h_1(a, (b, t)) = ((a, b), t)$, where SA is the unreduced suspension.

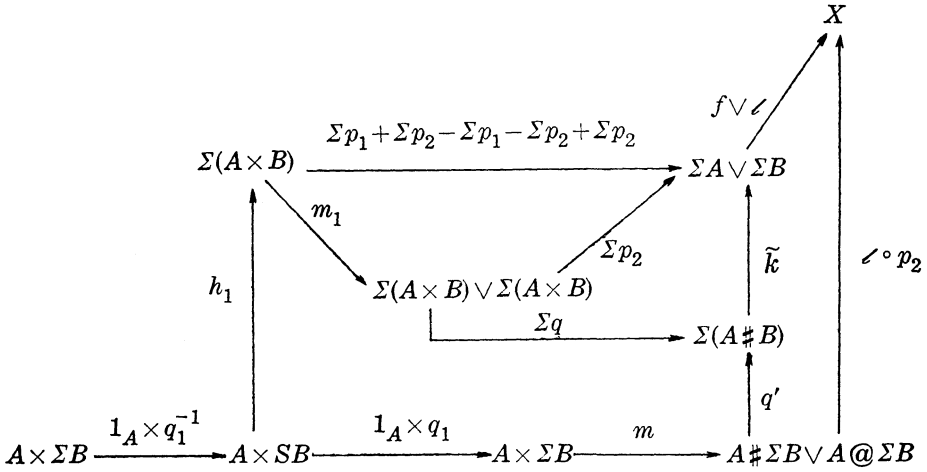
By the homotopy extension property the quotient map $q_1: SB \rightarrow \Sigma B$ is a homotopy equivalence. Its homotopy inverse will be denoted $q_1^{-1}: \Sigma B \rightarrow SB$.

LEMMA 2.5. Let $\alpha = [f] \in [\Sigma A, X]$ and $\lambda = [\epsilon] \in [\Sigma B, X]$, then $(f \vee \epsilon) \circ (\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2 + \Sigma p_2) \circ h_1 \circ (1_A \times q_1^{-1}): A \times \Sigma B \rightarrow X$ is a map associated with $\hat{\epsilon}_* \circ \theta([\alpha, \lambda]) \in [A, L_0(\Sigma B, X; \epsilon)]$.

Proof. Let $m_1: \Sigma(A \times B) \rightarrow \Sigma(A \times B) \vee \Sigma(A \times B)$ be given by

$$m_1((a, b), t) = \begin{cases} \left((a, b), \frac{5}{4}t\right) \vee * & 0 \leq t \leq \frac{4}{5} \\ * \vee ((a, b), 5t - 4) & \frac{4}{5} \leq t \leq 1 \end{cases}.$$

Consider the following diagram:



$q': A \# SB \rightarrow \Sigma(A \# B)$ is the homomorphism defined by $q'((a, (b, t))) = ((a, b), t)$ and k is as in Definition 1.1. It is easiest to check the homotopy commutativity of this diagram by looking first at the lower four fifths of the t coordinate in SB and then at the upper fifth.

Part 1. $\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2 + \Sigma p_2 \cong ((\tilde{k} \circ \Sigma q) \vee \Sigma p_2) \circ m_1$.

The lower four fifths of $\Sigma(A \times B)$ is mapped in one case by $\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2$ and in the other by $\tilde{k} \circ \Sigma q$. But these are homotopic by the definition of \tilde{k} . The upper fifth is mapped by Σp_2 in either case.

Part 2. $(f \vee \ell) \circ (\tilde{k} \circ \Sigma q \vee \Sigma p_2) \circ m_1 \circ h_1 = (((f \vee \ell) \circ \tilde{k} \circ q') \vee \ell \circ p_2) \circ m \circ (1_A \times q_1)$.

On the lower four fifths the maps "meet" at $\Sigma(A \# B)$. In either case the point $(a, (b, t)) \in A \times SB$ is mapped to $((a, b), (5/4)t) \in \Sigma(A \# B)$. On the upper fifth both maps are given by taking $(a, (b, t))$ to $\ell((b, 5t - 4))$ in X .

By Lemma 2.4 and the definition of $[\alpha, \lambda]$, $((f \vee \ell) \circ \tilde{k} \circ q') \vee (\ell \circ p_2) \circ m$ is an associated map for $\ell_* \theta([\alpha, \lambda])$. This is the lower route in the above diagram. Since q_1 and q_1^{-1} are homotopy inverses

$$\begin{aligned}
 & (((f \vee \ell) \circ \tilde{k} \circ q') \vee \ell \circ p_2) \circ m \\
 & \cong (((f \vee \ell) \circ \tilde{k} \circ q') \vee \ell \circ p_2) \circ m \circ (1_A \times q_1) \circ (1_A \times q_1^{-1}) \\
 & \cong (f \vee \ell) \circ (\tilde{k} \circ \Sigma q \vee \Sigma p_2) \circ m_1 \circ h_1 \circ (1_A \times q_1^{-1}) \\
 & \cong (f \vee \ell) \circ (\Sigma p_1 + \Sigma p_2 - \Sigma p_1 - \Sigma p_2 + \Sigma p_2) \circ h_1 \circ (1_A \times q_1^{-1}).
 \end{aligned}$$

The last two homotopies follow from Part 2 and Part 1 respectively. But the last map is the one claimed to be an associated map for

$\hat{\iota}_* \theta([\alpha, \lambda])$ and the lemma is proven.

For $\lambda = [\not\prec] \in [\Sigma B, X]$ the evaluation map $\omega: L(\Sigma B, X; \not\prec) \rightarrow X$ is a fibration with fiber $L_0(\Sigma B, X; \not\prec)$. Then there is a long exact sequence of homotopy groups

$$\begin{aligned} \dots \longrightarrow [\Sigma^{r+1}A, X] &\xrightarrow{\partial} [\Sigma^rA, L_0(\Sigma B, X; \not\prec)] \xrightarrow{i_*} \\ &[\Sigma^rA, L(\Sigma B, X; \not\prec)] \xrightarrow{\omega_*} [\Sigma^rA, X] \xrightarrow{\partial} \dots \longrightarrow [\Sigma A, X] \xrightarrow{\partial} \\ &[A, L_0(\Sigma B, X; \not\prec)] \xrightarrow{i_*} [A, L(\Sigma B, X; \not\prec)] \xrightarrow{\omega_*} [A, X], \end{aligned}$$

where exactness at the last two stages is as pointed sets. Recall that Lemma 2.2 shows there is an isomorphism $\theta: [\Sigma(A \# B), X] \rightarrow [A, L(\Sigma B, X; *)]$.

THEOREM 2.6. For $\alpha \in [\Sigma^rA, X]$, $\partial(\alpha) = \hat{\iota}_* \circ \theta \circ P_i(\alpha)$.

Proof. Let α be represented by a map $f: \Sigma^rA \rightarrow X$ and let $q_2: C(\Sigma^{r-1}A) \rightarrow \Sigma^rA$ be the natural quotient map from the cone to the suspension. Define $F: C(\Sigma^{r-1}A) \times SB \rightarrow \Sigma(\Sigma^{r-1}A) \vee \Sigma B$ by

$$F((a, r), (b, t)) = \begin{cases} q_2(a, r + 3t) \vee * & 0 \leq t \leq \frac{1}{3} \quad r \leq -3t + 1 \\ * & 0 \leq t \leq \frac{1}{3} \quad r \geq -3t + 1 \\ * \vee q_1(b, 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ * & \frac{2}{3} \leq t \leq 1 \quad r \geq 3t - 2 \\ q_2(a, 3 + r - 3t) \vee * & \frac{2}{3} \leq t \leq 1 \quad r \leq 3t - 2 \end{cases}$$

where $(a, r) \in C(\Sigma^{r-1}A)$, r being the level on the cone and $(b, t) \in SB$, t being the level on the suspension. At $t = 1/3$ or $2/3$ and on the lines $r = -3t + 1$ and $r = 3t - 2$, the image of F is at $*$ and F is well defined and continuous at these points. Since F is independent of a at $r = 1$ and independent of b at $t = 1$ and $t = 0$, F is well defined. Let $\Sigma^{r-1}A \times SB \rightarrow C(\Sigma^{r-1}A) \times SB$ be induced by including $\Sigma^{r-1}A$ at the 0 level of $C(\Sigma^{r-1}A)$. Consider the following diagram:

$$\begin{array}{ccc} C(\Sigma^{r-1}A) \vee SB & & \\ \downarrow & \searrow q_2 \vee q_1 & \\ C(\Sigma^{r-1}A) \times SB & \xrightarrow{F} & \Sigma(\Sigma^{r-1}A) \vee \Sigma B \\ \uparrow & & \downarrow \Sigma p_1 + \Sigma p_2 - \Sigma p_1 \\ \Sigma^{r-1}A \times SB & \xrightarrow{h_1} & \dot{\Sigma}(\Sigma^{r-1}A \times B). \end{array}$$

The map $(\Sigma p_1 + \Sigma p_2 - \Sigma p_1) \circ h_1$ is given by

$$(a, (b, t)) \longrightarrow \begin{cases} (a, 3t) \vee * & 0 \leq t \leq \frac{1}{3} \\ * \vee q_1(b, 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ (a, 3 - 3t) \vee * & \frac{2}{3} \leq t \leq 1. \end{cases}$$

But this is the same as $F((a, 0), (b, t))$, that is $F| \Sigma^{r-1}A \times SB$. Therefore the lower square commutes. In the upper triangle, when $t = 0$ (the base point of SB), $F((a, r), (b, 0)) = q_2(a, r)$ by definition. At the base point of $C(\Sigma^{r-1}A)$ consider

$$F((a_0, 0), (b, t)) = \begin{cases} q_2(a_0, 3t) \vee * = * & 0 \leq t \leq \frac{1}{3} \\ * \vee q_1(b, 3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ q_2(a_0, 3 - 3t) \vee * = * & \frac{2}{3} \leq t \leq 1. \end{cases}$$

But this is clearly homotopic to $* \vee q_1$, thus the upper triangle commutes up to homotopy. Now consider the map $\tilde{F}: C(\Sigma^{r-1}A) \times \Sigma B \rightarrow X$ given by $\tilde{F} = (f \vee \ell) \circ F \circ (1_{C(\Sigma^{r-1}A)} \times q_1^{-1})$. \tilde{F} is then an associated map for an element of $[(C(\Sigma^{r-1}A), \Sigma^{r-1}A), (L(\Sigma B, X; \ell), L_0(\Sigma B, X; \ell))]$. Since $\tilde{F}| C(\Sigma^{r-1}A) \times *$ is given by $f \circ q_2$, \tilde{F} is associated to the class $[f] \in [\Sigma^r A, X]$ under the bijection (see p. 104 in [5]) $\omega_*: [(C(\Sigma^{r-1}A), \Sigma^{r-1}A), (L(\Sigma B, X; \ell), L_0(\Sigma B, X; \ell))] \rightarrow [\Sigma^r A, X]$. Then by definition of the boundary homomorphism, $\partial([f]) = \partial(\alpha)$ has associated map $\tilde{F}| \Sigma^{r-1}A \times \Sigma B$. But by commutativity of the above diagram $\tilde{F}| \Sigma^{r-1}A \times \Sigma B \cong (f \vee \ell) \circ (\Sigma p_1 + \Sigma p_2 - \Sigma p_1) \circ h_1 \circ (1_{\Sigma^{r-1}A} \times q_1^{-1})$ and by Lemma 2.5 this is an associated map for $\hat{\ell}_* \circ \theta \circ ([\alpha, \lambda]) = \hat{\ell}_* \circ \theta \circ P_\lambda(\alpha)$.

The existence of the λ -component EHP sequence now can be shown. Let $i'_*: [\Sigma(\Sigma^{r-1}A \# B), X] \rightarrow [\Sigma^{r-1}A, L(\Sigma B, X; \ell)]$ be given by $i'_* = i_* \circ \hat{\ell}_* \circ \theta$.

THEOREM 2.7. *There is a long exact sequence*

$$\begin{aligned} \dots \longrightarrow [\Sigma^r A, L(\Sigma B, X; \ell)] &\xrightarrow{\omega_*} [\Sigma^r A, X] \xrightarrow{P_\lambda} \\ [\Sigma(\Sigma^{r-1}A \# B); X] &\xrightarrow{i'_*} [\Sigma^{r-1}A, L(\Sigma B, X; \ell)] \longrightarrow \dots \end{aligned}$$

Proof. Since $\hat{\ell}_*$ and θ are isomorphisms, the exactness of this sequence is immediate from the exactness of the homotopy exact sequence of the fibration $\omega_*: L(\Sigma B, X; \ell) \rightarrow X$ and Theorem 2.6.

3. **The Whitehead and James sequences.** The purpose of this section is to compare the λ -component EHP sequence with the classical EHP sequence of George W. Whitehead [7] and the suspension sequence of I. M. James [4]. The spaces A and B will be assumed to be CW complexes with a single vertex. For $\alpha \in [A, L(\Sigma B, X; \angle)]$ the element $H(\alpha) \in [\Sigma(A \# \Sigma B), \Sigma X]$ is defined by the element obtained from a map associated with α by the Hopf construction of Definition 1.2. The homomorphism $E: [A \# \Sigma B, X] \rightarrow [\Sigma(A \# \Sigma B), \Sigma X]$ is defined by $E([f]) = [\Sigma f]$.

LEMMA 3.1. *The following diagram commutes:*

$$\begin{array}{ccc}
 [A \# \Sigma B, X] & \xrightarrow{E} & [\Sigma(A \# \Sigma B), \Sigma X] \\
 \cong \downarrow \theta & & \downarrow H \\
 [A, L_0(\Sigma B, X; *)] & & \\
 \cong \downarrow \hat{\angle}_* & & \downarrow \\
 [A, L_0(\Sigma B, X; \angle)] & \xrightarrow{i_*} & [A, L(\Sigma B, X; \angle)] .
 \end{array}$$

Proof. Let $f: A \# \Sigma B \rightarrow X$ represent an element of $[A \# \Sigma B, X]$. Then $\hat{\angle}_* \circ \theta([f])$ has an associated map $F: A \times \Sigma B \rightarrow X$ given by

$$F(a, (b, t)) = \begin{cases} f(a, (b, 2t)) & 0 \leq t \leq \frac{1}{2} \\ \angle(b, 2t - 1) & \frac{1}{2} \leq t \leq 1 . \end{cases}$$

Let $\phi: \Sigma(A \# \Sigma B) \rightarrow \Sigma(A \times \Sigma B)$ and $q: A \times \Sigma B \rightarrow A \# \Sigma B$ be as in the comments preceding Definition 1.2. Consider the following diagram:

$$\begin{array}{ccc}
 & & \Sigma X \\
 & \nearrow \Sigma F & \uparrow \Sigma f \vee \Sigma l \\
 \Sigma(A \times \Sigma B) & \xrightarrow{\Sigma q + \Sigma p_2} & \Sigma(A \# \Sigma B) \vee \Sigma \Sigma B \\
 \uparrow \phi + * & \nearrow i_1 & \\
 \Sigma(A \# \Sigma B) & &
 \end{array}$$

where i_1 is the inclusion of $\Sigma(A \# \Sigma B)$ in $\Sigma(A \# \Sigma B) \vee \Sigma \Sigma B$. The homotopy commutativity of this diagram will establish the result since $\Sigma F \circ (\phi + *) \cong \Sigma F \circ \phi$ which by definition is the element obtained from

$\hat{\mathcal{L}}_* \circ \theta([f])$ by the Hopf construction and $(\Sigma f \vee \Sigma \mathcal{L}) \circ i_1 = \Sigma f$, a representative of $E([f])$.

In the lower triangle of the diagram $(\Sigma q + \Sigma p_2) \circ (\phi + *) \cong \Sigma q \circ \phi$ which is homotopic to i_1 by the definition of ϕ .

In the upper triangle

$$\Sigma F(a, (b, t), r) = \begin{cases} (f(a, (b, 2t)), r) & 0 \leq t \leq \frac{1}{2} \\ (\mathcal{L}(b, 2t - 1), r) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and

$$(\Sigma f \vee \Sigma \mathcal{L}) \circ (\Sigma q + \Sigma p_2)(a, (b, t), r) = \begin{cases} (f(a, (b, t)), 2r) & 0 \leq r \leq \frac{1}{2} \\ (\mathcal{L}(b, t), 2r - 1) & \frac{1}{2} \leq r \leq 1. \end{cases}$$

The usual homotopy to interchange the roles of t and r for homotopy will work in this case since f is defined on $A \# \Sigma B$ and \mathcal{L} is independent of a . Thus the upper triangle is homotopy commutative and the lemma is established.

DEFINITION 3.2. The *classical EHP sequence* is given by:

$$\begin{aligned} \pi_{3n-2}(S^n) &\xrightarrow{E} \pi_{3n-1}(S^{n+1}) \longrightarrow \dots \longrightarrow \pi_{n+p}(S^n) \longrightarrow \\ \pi_{n+p+1}(S^{n+1}) &\xrightarrow{H'} \pi_p(S^n) \xrightarrow{P} \pi_{p+n-1}(S^n) \xrightarrow{E} \dots \end{aligned}$$

where E is the suspension homomorphism, $H' = -E^{-n-1} \circ \tilde{H}$ where \tilde{H} is the Hopf invariant of Definition 1.8, $P = P_{\iota_n}$, where $\iota_n = [1_{S^n}] \in \pi_n(S^n)$.

This sequence was shown exact in [7]; the form used in Definition 3.2 is that of P. J. Hilton and J. H. C. Whitehead in [3]. The classical *EHP* sequence can now be compared with the ι_n -component *EHP* sequence for the fibration $\omega: L(S^n, S^n; 1_{S^n}) \rightarrow S^n$.

THEOREM 3.3. For $q \leq 3n - 2$ the following exact ladder is commutative and H is an isomorphism:

$$\begin{array}{ccccccc} \longrightarrow & \pi_q(S^n) & \xrightarrow{E} & \pi_{q+1}(S^{n+1}) & \xrightarrow{H'} & \pi_{q-n}(S^n) & \xrightarrow{P} & \pi_{q-1}(S^n) & \longrightarrow \\ & \uparrow 1 & & \uparrow H & & \uparrow 1 & & \uparrow 1 & \\ \longrightarrow & \pi_q(S^n) & \xrightarrow{i'_*} & \pi_{q-n}L(S^n, S^n; 1_{S^n}) & \xrightarrow{\omega*} & \pi_{q-n}(S^n) & \xrightarrow{P_{\iota_n}} & \pi_{q-1}(S^n) & \longrightarrow . \end{array}$$

Proof. The left square commutes by Lemma 3.1 since, by defini-

tion, $i'_* = i_* \circ \hat{\epsilon}_* \circ \theta$. The right square commutes by the definition of P . For the range $q \leq 3n - 2$, G. W. Whitehead shows (Corollary 6-4 in [7]) that every element $\alpha \in \pi_{q+1}(S^{n+1})$ is obtainable from a map $F: S^{q-n} \times S^n \rightarrow S^n$ of type $(H'(\alpha), \iota_n)$ by a Hopf construction. Thus if F is considered as an associated map for an element $\beta \in \pi_{q-n}(L(S^n, S^n; 1_{S^n}))$, β has type $(\omega_*(\beta), \iota_n)$ and $H(\beta)$ is obtainable by a Hopf construction from a map of type $(\omega_*(\beta), \iota_n)$ as well as a map of type $(H'(\alpha), \iota_n)$. But then by 5.1 in [6], $\omega_*(\beta) * \iota_n = H'(\alpha) * \iota_n$, where $*$ is the join operation. Since ι_n is the homotopy class of 1_{S^n} , $E^{n+1}\omega_*(\beta) = \omega_*(\beta) * \iota_n = H'(\alpha) * \iota_n = E^{n+1}H'(\alpha)$. Now $q \leq 3n - 2$ so $q - n \leq 2n - 2$ and by the Freudenthal suspension theorem E^{n+1} is an isomorphism, thus $\omega^*(\beta) = H'(\alpha) = H'(H(\beta))$. This establishes the commutivity of the ladder. That H is an isomorphism follows from the five lemma.

Since the bottom line is the ι_n -component EHP sequence, the classical EHP sequence can be considered as the ι_n -component EHP sequence for spheres in the range $q \leq 3n - 2$.

Some definitions will be required before describing the suspension sequence of James. Let D^n denote the solid n -ball. Then $\partial D^n = S^{n-1} = D_+^{n-1} \cup D_-^{n-1}$ where D_+^{n-1} and D_-^{n-1} are the northern and southern hemispheres of S^{n-1} respectively. Note that $D_+^{n-1} \cap D_-^{n-1} = S^{n-2}$.

DEFINITION 3.4. For A and B subspaces of X such that $A \cap B \neq \emptyset$ let $\pi_n(X; A, B)$ be the set of homotopy classes maps of $f: (D^n, D_+^{n-1}, D_-^{n-1}) \rightarrow (X, A, B)$.

There are natural boundary operators $\partial_1: \pi_n(X; A, B) \rightarrow \pi_{n-1}(A, A \cap B)$ and $\partial_2: \pi_{n-1}(A, A \cap B) \rightarrow \pi_{n-2}(A \cap B)$ defined by restriction to (D_+^{n-1}, S_-^{n-2}) and S^{n-2} respectively.

DEFINITION 3.5. The repeated boundary operator $\Delta: \pi_n(X; A, B) \rightarrow \pi_{n-2}(A \cap B)$ is defined by $\Delta = \partial_2 \circ \partial_1$.

The following result of James will be useful.

THEOREM 3.6. There is a pairing $\{\beta, \gamma\} \in \pi_{p+q+1}(\Sigma X; C_+X, C_-X)$ for $\beta \in \pi_p(X)$ and $\gamma \in \pi_q(X)$ such that

- (i) $\Delta\{\beta, \gamma\} = [\beta, \gamma] \in \pi_{p+q+1}(X)$, the usual Whitehead product and
- (ii) If $i_*: \pi_{p+q+1}(\Sigma X) \rightarrow \pi_{p+q+1}(\Sigma X; C_+X, C_-X)$ is the natural inclusion, an element $\alpha \in \pi_{p+q+1}(\Sigma X)$ is obtainable by a Hopf construction of type (β, γ) iff $i_*(\alpha) = \{\beta, \gamma\}$.

Proof. See §4 and Theorem 2.17 in [4].

DEFINITION 3.7. The James suspension sequence is

$$\longrightarrow \pi_{p+q}(X) \xrightarrow{E} \pi_{p+q+1}(\Sigma X) \xrightarrow{i_*} \pi_{p+q+1}(\Sigma X; C_+X, C_-X) \xrightarrow{\Delta} \pi_{p+q-1}(X)$$

where E is the suspension homomorphism and Δ is the repeated boundary operator.

THEOREM 3.8. *The following exact ladder is commutative:*

$$\begin{array}{ccccccc} \longrightarrow & \pi_{p+q}(X) & \xrightarrow{i'_*} & \pi_p(L(S^q, X; \angle)) & \xrightarrow{\omega_*} & \pi_p(X) & \xrightarrow{P_\lambda} & \pi_{p+q-1}(X) & \longrightarrow \\ & \downarrow 1 & & \downarrow H & & \downarrow \tilde{P} & & \downarrow 1 & \\ \longrightarrow & \pi_{p+q}(X) & \xrightarrow{E} & \pi_{p+q+1}(\Sigma X) & \xrightarrow{i_*} & \pi_{p+q+1}(\Sigma X; C_+X, C_-X) & \xrightarrow{\Delta} & \pi_{p+q-1}(X) & \longrightarrow \end{array}$$

where H is as in Lemma 3.1, $\tilde{P}(\alpha) = \{\alpha, \lambda\}$, and $\lambda = [\angle]$ for any $[\angle] \in \pi_q(X)$.

Proof. The left square commutes by Lemma 3.1 since $i'_* = i_* \circ \angle_* \circ \theta$ by definition. If $\alpha \in \pi_p(L(S^q, X; \angle))$ then by definition of H , $H(\alpha)$ is obtainable by a Hopf construction of type $(\omega_*(\alpha), \lambda)$ and by Theorem 3.6, (ii), $i_*H(\alpha) = \{\omega_*(\alpha), \lambda\} = \tilde{P} \circ \omega_*(\alpha)$. Thus the middle square commutes. The right square commutes by Theorem 3.6, (i).

Theorem 3.8 clearly indicates the extent to which the map $i'_* = i_* \circ \angle_* \circ \theta$ of the λ -component EHP sequence approximates the suspension homomorphism. Indeed, $E = H \circ i'_*$. While the James sequence contains the suspension homomorphism in a straight forward form, the λ -component EHP sequence contains the generalized Whitehead product in a more direct form.

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