Pacific Journal of Mathematics

A CLASS OF OPERATORS ON EXCESSIVE FUNCTIONS

MICHAEL J. SHARPE

Vol. 44, No. 1

May 1973

A CLASS OF OPERATORS ON EXCESSIVE FUNCTIONS

MICHAEL J. SHARPE

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a special standard Markov process with state space (E, \mathscr{E}) and transition semigroup (P_t) . We emphasize here that the \mathscr{F}_t are the usual completions of the natural σ -fields for the process. In this paper, we associate with certain multiplicative functionals of X operators on the class of excessive functions which are related to the operators P_M but which are a bit unusual in probabilistic potential theory in that they are not generally determined by kernels on $E \times \mathscr{E}$. An application is given to a problem treated by P.-A. Meyer concerning natural potentials dominated by an excessive function.

2. The operator associated with a natural multiplicative functional.¹ By a multiplicative functional of X, we mean a progressively measurable process M which satisfies, in addition to the standard conditions ([1], III, (1.1)) the following condition:

(2.1) almost surely, $M_{\xi} = 0, t \to M_t$ is decreasing on $[0, \infty)$ and if $S = \inf \{t > 0: M_t = 0\}$, then $t \to M_t$ is right continuous on [0, S), and $M_t M_S \circ \theta_t = M_{t+S \circ \theta_t}$ a.s. for all $t \ge 0$.

A simple example which illustrates some possibilities is obtained by considering X to be uniform motion to the right on the real line and $M_t = f(X_t)/f(X_0)$ on $\{f(X_0) > 0\}, M_t = 0$ for all t on $\{f(X_0) = 0\}, f(X_0) = 0\}$ where f is a decreasing positive function on the line, f(0+) = 0, f is right continuous on $(-\infty, 0)$ and $f(0) \leq f(0-)$.

If M is a multiplicative functional, then S is a terminal time and so $M_t \mathbf{1}_{[0,S)}(t)$ is a multiplicative functional which is right continuous. For a given M, the modified functional will be denoted \widetilde{M} . Let us denote by E_M the set $\{x \in E: P^x \{S > 0\} = 1\} = E_{\widetilde{M}}$ and call M exact if \widetilde{M} is exact. Note that M and \widetilde{M} generate the same resolvent, but not necessarily the same semigroup.

It should be emphasized that one will not have the freedom to replace M by an equivalent multiplicative functional, for the operator to be associated with M will not respect equivalence.

Let M be a given MF; for almost all ω , let $(-dM_t(\omega))$ denote the measure on $(0, \zeta(\omega))$ generated by the increasing function $t \to 1 - M_{t \wedge S}(\omega)$. Care should be taken when computing with $(-dM_t)$, since $(-dM_t)$ is generally not the restriction of $(-d\tilde{M}_t)$ to (0, S].

¹ The reader is referred to the books of Blumenthal and Getoor [1] and Meyer [2] for unexplained terminology.

DEFINITION 2.2. A multiplicative functional M is called natural if, almost surely, the trajectories $t \to M_t$ and $t \to X_t$ have no common discontinuity on [0, S), and $X_s = X_{s-}$ on $\{M_s < M_{s-}, S < \zeta\}$.

We now associate with a natural MF M an operator \bar{P}_{M}^{α} on the class \mathscr{S}^{α} of α -excessive functions for X.

DEFINITION 2.3. If M is a natural MF and $f \in S^{\alpha}$, let

$$ar{P}^{lpha}_{_M}f(x) \,=\, E^x igg\{ \int_{_{(0,\,\zeta)}} e^{-lpha t} f(X_t)_-(-dM_t) \,+\, e^{-lpha S} f(X_S) M_S igg\} \,, \quad x \in E_{_M} \ =\, f(x) \,, \quad x
otin E_{_M} \,,$$

By $f(X_t)_-$ is meant the left limit of the trajectory $s \to f(X_s)$ at t if t > 0, and $f(X_0)$ if t = 0. Recall that if M is a right continuous MF, $\alpha \ge 0$ and \mathscr{C}_+^* , one defines $P_M^{\alpha} f$ by

(2.4)
$$P_{M}^{\alpha}f(x) = E^{x}\int_{(0,\zeta)}e^{-\alpha t}f(X_{t})(-dM_{t}), \quad x \in E_{M}$$
$$= f(x) \qquad , \quad x \notin E_{M}.$$

One obtains $P_{M}^{\alpha}U^{\alpha}f + V^{\alpha}f = U^{\alpha}f$, where (V^{α}) is the resolvent for the subprocess (X, M) and it follows that if M is exact, $P_{M}^{\alpha}g \in \mathscr{S}^{\alpha}$ for all $g \in \mathscr{S}^{\alpha}$. If $f \in \mathscr{S}^{\alpha}$ is regular, in particular if $f = U^{\alpha}g$ for some $g \in \mathscr{C}_{+}^{*}$, then for M natural, $\bar{P}_{M}^{\alpha}f = P_{M}^{\alpha}f$. In general though, the trajectory $t \to f(X_{t})$ can jump at the same time as does the trajectory $t \to M_{t}$ and $\bar{P}_{M}^{\alpha}f$ will differ from $P_{M}^{\alpha}f$. Because of the assumption that X is special standard, it follows from [1], IV, (4.21) that $f(T_{T})_{-} \geq f(X_{T})$ for any accessible stopping time T, and therefore

(2.5)
$$\overline{P}_{\widetilde{M}}^{\alpha}f(x) \ge P_{\widetilde{M}}^{\alpha}f(x) \text{ for all } x \text{ if } f \in \mathscr{S}^{\alpha}.$$

We shall show that $\bar{P}_{\scriptscriptstyle M}^{\,\alpha}f \leq f$ and $\bar{P}_{\scriptscriptstyle M}^{\,\alpha}f \in \mathscr{S}^{\,\alpha}$ if $f \in \mathscr{S}^{\,\alpha}$. The fact that the action of $\bar{P}_{\scriptscriptstyle M}^{\,\alpha}$ on α -potentials is the same as that of $P_{\scriptscriptstyle M}^{\,\alpha}$, but that $\bar{P}_{\scriptscriptstyle M}^{\,\alpha}$ may differ from $P_{\scriptscriptstyle M}^{\,\alpha}f$ shows that generally, $\bar{P}_{\scriptscriptstyle M}^{\,\alpha}$ is not determined by a kernel on $E \times \mathscr{C}$.

The first lemma shows that although it may not be determined by a kernel, \bar{P}^{α}_{M} does respect certain increasing limits. Obviously $\bar{P}^{\alpha}_{M}f \leq \bar{P}^{\alpha}_{M}g$ if $f, g \in \mathscr{S}^{\alpha}$ and $f \leq g$.

LEMMA 2.6. If $f \in \mathscr{S}^{\alpha}$, $\overline{P}_{M}^{\alpha}(f \wedge n)$ increases to $\overline{P}_{M}^{\alpha}f$ as $n \to \infty$.

Proof. It suffices to prove that $(f \wedge n)(X_t)_{-}$ increases to $f(X_t)_{-}$ for all $t \in (0, \zeta)$, almost surely. If the trajectory $s \to f(X_s)$ is right continuous and has left limits on $(0, \zeta)$, then for each $t < \zeta$, if $f(X_t)_{-} > \beta$, then there exists $\varepsilon > 0$ such that $f(X_s) > \beta$ on $[t - \varepsilon, t)$. Therefore, if $n > \beta$, $(f \wedge n)(X_s) > \beta$ on $[t - \varepsilon, t)$ and hence $(f \wedge n)(X_t)_{-} \ge \beta$.

We remark at this point that $\alpha \to \overline{P}_{M}^{\alpha}f(x)$ is right continuous for every fixed choice of M, f and x.

THEOREM 2.7. If M is an exact natural MF, $0 \leq \alpha < \infty$ and $f \in \mathscr{S}^{\alpha}$, then $\bar{P}_{\mathbb{M}}^{\alpha} f \leq f$ and $\bar{P}_{\mathbb{M}}^{\alpha} f \in \mathscr{S}^{\alpha}$.

Proof. Because of (2.6) it may be assumed that f is bounded. We may also assume $\alpha > 0$, since the case $\alpha = 0$ will follow by a trivial limit argument. Let

$$egin{aligned} N_t &= M_t, \, t < S \ &= M_s, \, t \geqq S \, ext{ on } \{S < \zeta\} \ &= M_{t-}, \, t \geqq \zeta \, ext{ on } \{S = \zeta\} \;. \end{aligned}$$

One then has $-dN_t = -dM_t$ almost surely, and for $x \in E_M$, $\overline{P}_M^{\alpha} f(x) = E^x \left\{ \int_0^\infty e^{-\alpha t} f(X_t)_{-}(-dN_t) + e^{-\alpha s} f(X_s) M_s \right\}$. Define a family $\{T_s; 0 < s < 1\}$ of (\mathscr{F}_t) stopping times by

$$T_s = \inf \{u > 0: 1 - N_u > s\}$$
.

It is clear that $s \to T_s$ is almost surely increasing and right continuous, $T_s = \infty$ a.s. on $\{T_s > S\}$, $\{T_s = 0 \text{ for some } s\} = \{M_{0+} = 0\}$ and $\{T_s \leq S\} = \{T_s < \zeta\}$ almost surely. By the change of variable formula,

$$\int_{(0,\zeta)} e^{-\alpha t} f(X_t)_{-}(-dM_t) = \int_0^1 e^{-\alpha T_s} f(X_{T_s})_{-} \mathbf{1}_{\{T_s < \zeta\}} ds .$$

Let $Z_t = e^{-\alpha(t \wedge S)} f(X_{t \wedge S})$. Since $\alpha > 0$,

$$\int_{0}^{1} Z_{T_{s}} ds = \int_{0}^{1} Z_{T_{s}} \mathbf{1}_{\{T_{s} \leq S\}} ds + \int_{0}^{1} Z_{T_{s}} \mathbf{1}_{\{T_{s} = \infty\}} ds$$

 $= \int_{0}^{1} e^{-lpha T_{s}} f(X_{T_{s}}) \mathbf{1}_{\{T_{s} \leq S\}} ds + \int_{0}^{1} e^{-lpha S} f(X_{s}) \mathbf{1}_{\{T_{s} = \infty\}} ds$
 $= \int_{(0,\zeta)} e^{-lpha t} f(X_{t}) (-dM_{t}) + e^{-lpha S} f(X_{s}) M_{S} .$

Upon checking separately the case $x \notin E_M$, one finds

(2.8)
$$\overline{P}_{M}^{\alpha}f(x) = E^{x}\int_{0}^{1}Z_{T_{s}}ds, x \in E.$$

We now need a fact which will be of use at a subsequent point in the proof.

(2.9) For any initial measure μ , the set of $s \in (0, 1)$ for which T_s is a.s. P^{μ} equal to an accessible stopping time has full Lebesgue measure.

To demonstrate (2.9), we let

$$egin{aligned} I(\omega) &= \{\infty\} \cup [0,\,\zeta(\omega)) - \{t \in (0,\,\zeta(\omega)) \colon N_{t+arepsilon}(\omega) < N_t(\omega) \ & ext{for all } arepsilon > 0 ext{ and } N_{t-arepsilon}(\omega) = N_t(\omega) ext{ for some } arepsilon > 0 \} \end{aligned}$$

Obviously $[0, \zeta) - I$ is countable and $\int_{[0,\zeta)-I} (-dM_t) = 0$ a.s., and consequently $\int_0^1 \frac{1}{T_s \notin I} ds = 0$ a.s., by the change of variable formula. If we prove that T_s is accessible on $\{T_s \in I\}$, we shall have proven (2.9), for by Fubini,

$$0 = E^{\mu} \int_{0}^{1} 1_{\{T_s \notin I\}} ds = \int_{0}^{1} P^{\mu} \{T_s \notin I\} ds$$
.

On $\{T_s = 0\} \cup \{T_s = \infty\}$, T_s is trivially accessible. It is easy to check that $\{T_s \in I, 0 < T_s < \zeta\} = \{0 < T_s = T_{s-} < \zeta\}$, and on $\{T_s \in I, 0 < T_s < \zeta\} \cap \{X_{T_s} = X_{T_s-}\}$, T_s is accessible by the famous theorem of Meyer, whilst on $\{T_s \in I, 0 < T_s < \zeta\} \cap \{X_{T_s} \neq X_{T_s-}\}$, $N_{T_s} = N_{T_s-}$ since M is natural, and it follows that a.s., $T_{s-s} < T_s$ for all $\varepsilon \in (0, s)$. The accessibility of T_s on $\{T_s \in I\}$ is now evident.

To obtain $\overline{P}_{M}^{\alpha}f \leq f$, we invoke (2.8) to see that $\overline{P}_{M}^{\alpha}f(x) = \int_{0}^{1} E^{x}Z_{T_{s}}ds$, and conclude by observing that $(Z_{t}, \mathscr{F}_{t}, P^{x})$ is a bounded non-negative right-continuous supermartingale and that for almost all $s \in (0, 1)$, T_{s} is a.s. P^{x} accessible to find $E^{x}Z_{T_{s}} \leq E^{x}Z_{0} = f(x)$ for almost all s.

We prove next that $\overline{P}_{M}^{\alpha}f$ is α -super-mean-valued. It is enough to give a proof in case $\alpha > 0$. From (2.8) we see that

$$P_t^{\alpha} \overline{P}_{M}^{\alpha} f(x) = E^x e^{-\alpha t} E^{X_t} \int_0^1 Z_{T_s} ds = \int_0^1 E^x e^{-\alpha t} Z_{T_s} \circ \theta_t ds .$$

Our first step is to show

$$(2.10) P_t^{\alpha} \overline{P}_M^{\alpha} f(x) \leq \int_0^1 E^x (Z_{t+T_s \circ \theta_t})_{-} ds , \qquad x \in E.$$

On $\{S \ge t + T_s \circ \theta_t\}$, either S > t or S = t and $T_s \circ \theta_t = 0$. It is a matter of checking cases to see that

$$e^{-lpha t}Z_{{}^{T_s-}}\circ heta_t=(Z_{{}^{t+T_s}\circ heta_t})_-$$
 on $\{S>t\}$,

and a.s. on $\{S = t, T_s \circ \theta_t = 0\}$,

$$e^{-\alpha t} Z_{T_{s^{-}}} \circ \theta_{t} = e^{-\alpha t} f(X_{t}) = e^{-\alpha t} f(X_{t-}) \leq e^{-\alpha t} f(X_{t})_{-} = (Z_{t+T_{s}} \circ \theta_{t})_{-}.$$

 $\underset{t}{\overset{t}{\underset{t}{\mapsto}}} \text{Hence } e^{-\alpha t} Z_{T_s -} \circ \theta_t \leq (Z_{t+T_s \circ \theta_t})_{-} \text{ a.s. on } \{S \geq t + T_s \circ \theta_t\}. \text{ On } \{S < t + T_s \circ \theta_t\}, (Z_{t+T_s \circ \theta_t})_{-} = e^{-\alpha S} f(X_S), \text{ while }$

$$\begin{split} e^{-\alpha t} Z_{T_s-} \circ \theta_t &\leq e^{-\alpha (t+T_s \circ \theta_t)} f(X_{t+T_s \circ \theta_t})_{-} \text{ on } \{S < t + T_s \circ \theta_t, \ T_s \circ \theta_t \leq S \circ \theta_t\} \text{ ,} \\ &= e^{-\alpha (t+S \circ \theta_t)} f(X_{t+S \circ \theta_t}) \text{ on } \{S < t + T_s \circ \theta_t, \ T_s \circ \theta_t > S \circ \theta_t\} \text{ .} \end{split}$$

One sees readily from (2.9) that for fixed $x, t + T_s \circ \theta_t$ is a.s. P^x equal to an accessible stopping time for almost all s and so for almost all choices of s, there exists an increasing sequence $\{R_n\}$ of stopping times with limit $t + T_s \circ \theta_t$ such that $P^x \{R_n < t + T_s \circ \theta_t\} = 1$ for every n. Then $L_n = R_n \wedge (t + S \circ \theta_t)$ increases to $t + T_s \circ \theta_t$ strictly from below (a.s. P^x) on $\{S < t + T_s \circ \theta_t, T_s \circ \theta_t \leq S \circ \theta_t\}$ and R_n is eventually equal to $t + S \circ \theta_t$ on $\{S < t + T_s \circ \theta_t, T_s \circ \theta_t > S \circ \theta_t\}$. One then has

$$egin{aligned} E^x e^{-lpha t} Z_{T_{s^-}} \circ heta_t &= E^x \{ e^{-lpha t} Z_{T_{s^-}} \circ heta_t [\mathbf{1}_{\{S \geq t+T_s \circ heta_t\}} + \mathbf{1}_{\{S < t+T_s \circ heta_t\}}] \} \ &\leq E^x \{ (Z_{t+T_s \circ heta_t}) - \mathbf{1}_{\{S \geq t+T_s \circ heta_t\}} + \lim_n e^{-lpha L_n} f(X_{L_n}) \mathbf{1}_{\{S < t+T_s \circ heta_t\}} \} \;. \end{aligned}$$

But $t + S \circ \theta_t \geq S$ a.s. and so $L_n \geq S$ eventually, a.s., on $\{S < t + T_s \circ \theta_t\}$ and it follows from the fact that $\{e^{-\alpha t}f(X_t), \mathscr{F}_t, P^s\}$ is a bounded nonnegative right-continuous supermartingale that $E^x e^{-\alpha t} Z_{T_s} \circ \theta_t \leq E^x (Z_{t+T_s \circ \theta_t})_-$ for almost all $s \in (0, 1)$. This proves (2.10).

Now observe that a.s., $T_s \leq t + T_s \circ \theta_t$ on $\{T_s \leq S\}$ and $t + T_s \circ \theta_t > S$ on $\{T_s > S\}$. For, on $\{T_s \leq S\} \cap \{M_t > 0\}$,

$$egin{aligned} t + \ T_s \circ heta_t &= \inf \left\{ u + t {: \ u > 0, \ N_u \circ heta_t < 1 - s}
ight\} \ &\geq \inf \left\{ u + t {: \ u > 0, \ M_u \circ heta_t < 1 - s}
ight\} \ &= \inf \left\{ v > t {: \ M_v < (1 - s) M_t}
ight\} \ &\geq \inf \left\{ v > 0 {: \ M_v < 1 - s}
ight\} = T_s \;, \end{aligned}$$

and on $\{T_s \leq S\} \cap \{M_t = 0\}, t \geq S$ so $T_s \leq S \leq t \leq t + T_s \circ \theta_t$. On $\{T_s > S\} \cap \{M_t > 0\}$, the same calculation as above gives $t + T_s \circ \theta_t \geq inf \{v > 0: M_v < 1 - s\}$ a.s., and so $t + T_s \circ \theta_t \leq S$ would imply $T_s \leq S$. On $\{T_s > S\} \cap \{M_t = 0\}, M_s > 0$ so t > S and $t + T_s \circ \theta_t > S$ almost surely.

For almost all $s \in (0, 1)$, T_s and $t + T_s \circ \theta_t$ are (a.s. P^x) accessible stopping times and it follows simply from the order relation observed above and the fact that $(Z_t, \mathscr{F}_t, P^x)$ is bounded nonnegative supermartingale that $E^x(Z_{t+T_s} \circ \theta_t)_- \leq E^x Z_{T_s}$ for almost all $s \in (0, 1)$, whence $P_t^{\alpha} \bar{P}_M^{\alpha} f(x) \leq \bar{P}_M^{\alpha} f(x)$.

It remains to show $P_t^{\alpha} \overline{P}_{\mathcal{M}}^{\alpha} f(x) \to \overline{P}_{\mathcal{M}}^{\alpha} f(x)$ as $t \to 0$. If $x \in E_{\mathcal{M}}$, then $X_t \in E_{\mathcal{M}}$ a.s. on $\{t < S\}$, and so

$$\begin{split} P_{t}^{\alpha}\bar{P}_{M}^{\alpha}f(x) &= E^{x}e^{-\alpha t}\bar{P}_{M}^{\alpha}f(X_{t})\\ &\geq E^{x}e^{-\alpha t}\mathbf{1}_{\{t$$

By Fatou's lemma, if $x \in E_M$

$$\begin{split} &\lim \inf_{(t \to 0)} P_t^{\alpha} \bar{P}_{M}^{\alpha} f(x) \\ & \geq E^{x} \lim_{(t \to 0)} \mathbb{1}_{\{t < S\}} M_t^{-1} \Big\{ \int_{(t,\zeta)} e^{-\alpha u} f(X_u)_{-} (-dM_u) + f(X_S) M_S e^{-\alpha S} \Big\} \\ & = E^{x} \Big\{ \int_{(0,\zeta)} e^{-\alpha u} f(X_u)_{-} (-dM_u) + e^{-\alpha S} M_S f(X_S) \Big\} = \bar{P}_{M}^{\alpha} f(x) \; . \end{split}$$

Consequently $P_t^{\alpha} \bar{P}_{\tilde{M}}^{\alpha} f(x) \to \bar{P}_{\tilde{M}}^{\alpha} f(x)$ if $x \in E_M$. On the other hand, if $x \in E - E_M$, $P_t^{\alpha} \bar{P}_{\tilde{M}}^{\alpha} f(x) \ge P_t^{\alpha} P_{\tilde{M}}^{\alpha} f(x)$ which converges as $t \to 0$ to $P_{\tilde{M}}^{\alpha} f(x) = f(x) = \bar{P}_{\tilde{M}}^{\alpha} f(x)$, using exactness of \tilde{M} . Our proof is now complete.

3. Application to a problem treated by Meyer. Meyer [3] proved that if u is a natural potential of $X, f \in \mathscr{S}$ and $u \leq f$, and if in addition $u(X_t)_{-} \leq f(X_t)$ for all t such that $X_t = X_{t-}$, then $u = P_R f$ for some exact terminal time R on a possibly larger sample space. We give here a similar representation using an operator of the type discussed in the preceding section, one advantage being that one may remain on the original sample space, using only the fields (\mathscr{F}_t) , and another being that the last, somewhat unnatural, condition may be dropped.

THEOREM 3.1. Let $f \in \mathscr{S}$ be finite off a polar set and let u be a natural potential such that $u \leq f$. Then there exists a natural exact MF M of X such that $u = \overline{P}_M f$.

Proof. Let $u = u_B$, B a natural additive functional. Since u is finite, B is a.s. finite on $[0, \zeta)$, and by [1], IV, (4.29), if T is a stopping time which is accessible on Λ , then $B_T - B_{T-} = u(X_T) - u(X_T)$ a.s. on $\Lambda \cap \{T < \zeta\}$. For every $\varepsilon > 0$, let

$$A^arepsilon_t=\int_0^t(f(X_s)_-+arepsilon\,-\,u(X_s))^{-1}dB_s$$
 .

Clearly A^{ε} is a finite natural AF of X, and if T is an accessible stopping time, $A_T^{\varepsilon} - A_{T-}^{\varepsilon} = (f(X_T) - \varepsilon - u(X_T))^{-1}(u(X_T) - u(X_T))$ a.s. on $\{T < \zeta\}$ and so $A_T^{\varepsilon} - A_{T-}^{\varepsilon} < 1$ for any accessible T. There exists therefore a right continuous natural MF, M^{ε} , such that $S = \zeta$ and

$$(M^{arepsilon}_{t\,-})^{\scriptscriptstyle -1}(-dM^{arepsilon}_{t})=dA^{arepsilon}_{t}$$
 , $t<\zeta$.

Let $C_t = B_t^c$, the continuous part of B. Then for $t < \zeta$

$$egin{aligned} M^arepsilon_t &= \exp\left\{-\int_{\mathfrak{o}}^t [f(X_s)_- + arepsilon - u(X_s)]^{-1} dC_s
ight\} \ & imes \prod_{s\leq t} \left[1 - (f(X_s)_- + arepsilon - u(X_s))^{-1} arphi B_s
ight] \end{aligned}$$

and it is clear that a.s., M_t^{ε} decreases as ε decreases for all $t \ge 0$.

Let $M_t = \lim_{(\varepsilon \to 0)} M_t^{\varepsilon}$, $S = \inf \{t > 0 : M_t = 0\}$. We propose to show that M is a MF of the type considered in the second section. Obviously M is adapted, multiplicative, a.s. decreasing, $M_{\zeta} = 0$, $M_t M_s \circ \theta_t = M_{t+s \circ \theta_t}$, but it may well happen that $M_s > 0$. Upon taking the monotonic limit as $\varepsilon \to 0$ in the above representation, one sees that

(3.2)
$$M_{t} = \exp\left\{-\int_{0}^{t} [f(X_{s})_{-} - u(X_{s})]^{-1} dC_{s}\right\} \\ \times \prod \left[1 - (f(X_{s})_{-} - u(X_{s}))^{-1} \Delta B_{s}\right]$$

for all $t < \zeta$, and from (3.2) one finds

(3.3)
$$S = \inf \left\{ t > 0: \int_{0}^{t} [f(X_{s})_{-} - u(X_{s})]^{-1} dB_{s} = \infty \right\}.$$

REMARK. In the product term of (3.2), we take

$$[f(X_s)_- - u(X_s)]^{-1} \Delta B_s = 0$$
 if $\Delta B_s = 0$.

It is almost surely true that if $M_t > 0$, $M_s^{\varepsilon}/M_s \leq M_t^{\varepsilon}/M_t$ for all $s \leq t$ whence $M_s^{\varepsilon} \to M_s$ uniformly on [0, t] if $M_t > 0$. The right continuity of M on [0, S) follows immediately.

To see that M is natural, use (3.2) to observe that on [0, S), the only jumps of M must occur at jump times of B, and that on $\{M_s < M_{s-}, S < \zeta\}, \Delta B_s > 0$, implying that S is accessible on $\{M_{s-} > M_s\}$.

The exactness of M is a consequence of [1], III, (5.9) once it is established that if $P^{*}{S = 0} = 1$, then $E^{*}\widetilde{M}_{v-t} \circ \theta_{t} \to 0$ as $t \to 0$, for all v > 0. However, $\widetilde{M} \leq M$ and it is easy to see that $t \to M_{v-t} \circ \theta_{t}$ is an increasing function. Because of the monotonic convergence of M_{t}^{*} to M_{t} , it is legal to interchange limits to obtain

$$\lim_{(t o 0)} M_{v-t} \circ heta_t = \lim_{(t o 0)} \lim_{(arepsilon o 0)} M^{arepsilon}_{v-t} \circ heta_t$$

 $= \lim_{(arepsilon o 0)} \lim_{(t o 0)} M^{arepsilon}_{v-t} \circ heta_t = \lim_{(arepsilon o 0)} M^{arepsilon}_v = 0 ext{ a.s. } P^x ext{ ,}$

using the exactness of M^{ε} .

We remark at this point that $f(X_s) = u(X_s)$ a.s. on $\{S < \zeta\}$, for by (3.3), on $\{S < \zeta\}$, either $\Delta B_s > 0$ and $f(X_s)_- = u(X_s)$ or $\Delta B_s = 0$. In the first case, S is accessible on $\{\Delta B_s > 0\}$ and so $f(X_s) \leq f(X_s)_- = u(X_s) \leq f(X_s)$ whence $u(X_s) = f(X_s)$. In case $\Delta B_s = 0, t \to A_t = \int_{[0,t]} [f(X_s)_- - u(X_s)]^{-1} dB_s$ is left continuous at S. If $A_s = \infty$ and $T_n = \inf\{t > 0: A_t \geq n\}$ then T_n increases to S a.s. on $\{S < \zeta\}$ and $T_n < S$ a.s. on $\{0 < S < \zeta\}$. Thus S is accessible on $\{A_s = \infty, 0 < S < \zeta\}$ and a.s. on $\{A_s = \infty, 0 < S < \zeta\}$, $\liminf_{(t \to S_-)} [f(X_t)_- - u(X_t)] = 0$ which implies $f(X_s)_- = u(X_s)$. But $u(X_s)_- = u(X_s)$ since $\Delta B_s = 0$ and $f(X_s) \leq f(X_s)_-$ since S is accessible. This shows that $u(X_s) = f(X_s)$ a.s. on $\{0 < S < \zeta, A_s = \infty\}$. On $\{S < \zeta, A_s < \infty\}$, one sees from (3.3) that $\liminf_{(t \to S^+)} [f(X_t)_- - u(X_t)] = 0$, whence $f(X_s) = u(X_s)$, proving finally that $u(X_s) = f(X_s)$ a.s. on $\{S < \zeta\}$.

From (3.2), we find that a.s. on [0, S)

$$(-dM_t)(f(X_t)_- - u(X_t)) = M_t_- dB_t$$

and a.s. on $\{S < \zeta\}$

$$(M_{s-} - M_s)f(X_s)_- = (M_{s-} - M_s)u(X_s) + M_{s-}\Delta B_s$$
 .

Thus

$$\begin{split} \int_{(0,\zeta)} f(X_t)_{-}(-dM_t) &= \int_{(0,S)} f(X_t)_{-}(-dM_t) \\ &+ [f(X_S)_{-}(M_{S-} - M_S) + f(X_S)M_S]\mathbf{1}_{(S<\zeta)} \\ &= \int_{(0,S)} u(X_t)(-dM_t) + \int_{(0,S)} M_{t-}dB_t \\ &+ [(M_{S-} - M_S)u(X_S) + M_{S-}\Delta B_S + u(X_S)M_S]\mathbf{1}_{(S<\zeta)} \\ &= \int_{0}^{\infty} u(X_t)(-d\widetilde{M}_t) + \int_{0}^{\infty} \widetilde{M}_{t-}dB_t \ . \end{split}$$

Since $u(X_T)1_{\{T<\infty\}} = E^{*}\{(B_{\infty} - B_T)1_{\{T<\infty\}} | \mathscr{F}_T\}$ for all stopping times T, Meyer's integration lemma ([2], VII, T. 15) applies to give

$$E^x \int_0^\infty u(X_t)(-d\widetilde{M}_t) = E^x \int_0^\infty (B_\infty - B_t)(-d\widetilde{M}_t)$$
.

Thus, for $x \in E_M$,

$$ar{P}_{\scriptscriptstyle M}f(x) \,=\, E^x\!\!\int_{_0}^{^\infty}\!\!(B_{\scriptscriptstyle\infty}\,-\,B_t)(-d\widetilde{M}_t)\,+\,\int_{_0}^{^\infty}\!\!\widetilde{M}_{t-}dB_t
onumber \ =\, E^x(B_{\scriptscriptstyle\infty})\,+\,E^x\!\!\int_{_0}^{^\infty}\!\!\widetilde{M}_tdB_t\,-\,\int_{_0}^{^\infty}\!\!B_t(-d\widetilde{M}_t)
onumber \ =\, u(x)$$

upon integrating by parts.

If $x \notin E_M$, $\overline{P}_M f(x) = f(x) = E^x f(X_S) = E^x u(X_S) = u(x)$, and the theorem is completely proven.

4. REMARKS. It is natural to ask for a specification of the class $\{\bar{P}_{M}^{a}f: M \text{ a natural exact } MF\}$, for a given $f \in \mathscr{S}$. The following example shows that although it contains f and all natural potentials, it need not include all excessive functions dominated by f. Let X be uniform motion to the right on the real line, let $f \equiv 1$ and $u \equiv 1/2$. Obviously $\bar{P}_{M}f(x) = P_{\widetilde{M}}1(x)$ for all x, and because we can write down (up to equivalence) the form of \widetilde{M} , it is a simple matter to check that $P_{\widetilde{M}}1 = 1/2$ has no solution for \widetilde{M} .

A particular example of an operator \overline{P}_{M} which may be of interest is obtained by taking, for a fixed Borel subset B of E,

$$M_t = \mathbf{1}_{[0, T_B \land \zeta)}(t) + \mathbf{1}_{\{t = T_B < \zeta, X_t = \neq X_t\}}$$
 .

Then $S = \inf \{t > 0: M_t = 0\} = T_B \land \zeta$, and using the fact that S is totally inaccessible on $\{X_s \neq X_{s-}, S < \zeta\}, P^x\{t = T_B < \zeta, X_{t-} \neq X_t\} = 0$ for all $t \ge 0$ and $x \in E$. It follows readily that M is a MF satisfying (2.1). Define, for $f \in \mathcal{S}$,

$$ar{P}_{\scriptscriptstyle B}f(x) = ar{P}_{\scriptscriptstyle M}f(x) = E^x\{f(X_{{\scriptscriptstyle T}_{\scriptscriptstyle B}})_-;\, T_{\scriptscriptstyle B} < \zeta,\, X_{{\scriptscriptstyle T}_{\scriptscriptstyle B}} = X_{{\scriptscriptstyle T}_{\scriptscriptstyle B}-}\} \ + \, E^x\{f(X_{{\scriptscriptstyle T}_{\scriptscriptstyle B}});\, T_{\scriptscriptstyle B} < \zeta,\, X_{{\scriptscriptstyle T}_{\scriptscriptstyle R}}
e X_{{\scriptscriptstyle T}_{\scriptscriptstyle R}-}\} \;.$$

Because of Theorem (2.7), $\overline{P}_{\scriptscriptstyle B}f \in \mathscr{S}$ if $f \in \mathscr{S}$.

One simple use of the operator \overline{P}_{B} is afforded by the following example. Let *B* be a finely closed Borel subset of *E* and let *f* be a uniformly integrable excessive function. Assume that *X* is a Hunt process. Let f^{B} be the lower envelope of the family of excessive functions which dominate *f* on a (variable) neighborhood of *B*. In [1], VI, (2.12)-(2.15), it is shown, under different hypotheses, that $f^{B} = P_{B}f$ off a certain exceptional set provided *f* is "admissible". However, under the hypotheses given above without assuming *f* to be admissible, it is a simple matter, using [1], I, (11.3) together with certain facts from [1], VI, (2.12)-(2.15), to obtain $\overline{P}_{B}f \leq f^{B}$ everywhere, and $\overline{P}_{B}f(x) = f^{B}(x)$ except possibly on $B - B^{r}$. It does not seem to be easy to remove the restrictions imposed above to obtain a general representation of f^{B} .

References

1. R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York (1968).

2. P.-A. Meyer, Probability and Potentials, Ginn (Blaisdell), Boston (1966).

3. _____, Quelques résultats sur les processus de Markov, Invent. Math., 1 (1966), 101-115.

Received September 16, 1971. Research supported by NSF Grant GP-8770.

UNIVERSITY OF CALIFORNIA, SAN DIEGO

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON Stanford University Stanford, California 94305

C. R. HOBBY University of Washington Seattle, Washington 98105 J. DUGUNDJI Department of Mathematics University of Southern California Los Angeles, California 90007

RICHARD ARENS University of California Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

UNIVERSITY OF BRITISH COLUMBIA

UNIVERSITY OF CALIFORNIA

OREGON STATE UNIVERSITY

UNIVERSITY OF NEVADA NEW MEXICO STATE UNIVERSITY

UNIVERSITY OF OREGON

OSAKA UNIVERSITY

MONTANA STATE UNIVERSITY

CALIFORNIA INSTITUTE OF TECHNOLOGY

B. H. NEUMANN F. WOLF

 SUPPORTING
 INSTITUTIONS

 COLUMBIA
 UNIVERSITY OF SOUTHERN CALIFORNIA

 F TECHNOLOGY
 STANFORD UNIVERSITY

 VIA
 UNIVERSITY OF TOKYO

 SITY
 UNIVERSITY OF UTAH

 WASHINGTON STATE UNIVERSITY

 VERSITY
 UNIVERSITY OF WASHINGTON

K. YOSHIDA

* * * AMERICAN MATHEMATICAL SOCIETY NAVAL WEAPONS CENTER

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics Vol. 44, No. 1 May, 1973

Jimmy T. Arnold, <i>Power series rings over Prüfer domains</i>	1
Maynard G. Arsove, On the behavior of Pincherle basis functions	13
Jan William Auer, <i>Fiber integration in smooth bundles</i>	33
George Bachman, Edward Beckenstein and Lawrence Narici, Function algebras	
over valued fields	45
Gerald A. Beer, <i>The index of convexity and the visibility function</i>	59
James Robert Boone, A note on mesocompact and sequentially mesocompact	
spaces	69
Selwyn Ross Caradus, Semiclosed operators	75
John H. E. Cohn, <i>Two primary factor inequalities</i>	81
Mani Gagrat and Somashekhar Amrith Naimpally, Proximity approach to	
semi-metric and developable spaces	93
John Grant, Automorphisms definable by formulas	107
Walter Kurt Hayman, <i>Differential inequalities and local valency</i>	117
Wolfgang H. Heil, <i>Testing 3-manifolds for projective planes</i>	139
Melvin Hochster and Louis Jackson Ratliff, Jr., Five theorems on Macaulay	
rings	147
Thomas Benton Hoover, <i>Operator algebras with reducing invariant subspaces</i>	173
James Edgar Keesling, <i>Topological groups whose underlying spaces are separable</i>	
Fréchet manifolds	181
Frank Leroy Knowles, <i>Idempotents in the boundary of a Lie group</i>	191
George Edward Lang, <i>The evaluation map and EHP sequences</i>	201
Everette Lee May, Jr, <i>Localizing the spectrum</i>	211
Frank Belsley Miles, <i>Existence of special K-sets in certain locally compact abelian</i>	
groups	219
Susan Montgomery, A generalization of a theorem of Jacobson. II	233
T. S. Motzkin and J. L. Walsh, <i>Equilibrium of inverse-distance forces in</i>	
three-dimensions	241
Arunava Mukherjea and Nicolas A. Tserpes, <i>Invariant measures and the converse</i>	
of Haar's theorem on semitopological semigroups	251
James Waring Noonan, <i>On close-to-convex functions of order</i> β	263
Donald Steven Passman, <i>The Jacobian of a growth transformation</i>	281
Dean Blackburn Priest, A mean Stieltjes type integral	291
Joe Bill Rhodes, <i>Decomposition of semilattices with applications to topological</i>	
lattices	299
Claus M. Ringel, Socle conditions for QF – 1 rings	309
Richard Rochberg, <i>Linear maps of the disk algebra</i>	337
Roy W. Ryden, <i>Groups of arithmetic functions under Dirichlet convolution</i>	355
Michael J. Sharpe, A class of operators on excessive functions	361
Erling Stormer, Automorphisms and equivalence in von Neumann algebras	371
Philip C. Tonne, Matrix representations for linear transformations on series	
analytic in the unit disc	385