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**AUTOMORPHISMS AND EQUIVALENCE IN VON NEUMANN
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Let \mathfrak{R} be a von Neumann algebra acting on a Hilbert space \mathfrak{H} . Let G be a group and let $t \rightarrow U_t$ be a unitary representation of G on \mathfrak{H} such that $U_t^* \mathfrak{R} U_t = \mathfrak{R}$ for all $t \in G$. Two projections E and F in \mathfrak{R} are called G -equivalent, written $E \sim_G F$, if there is for each $t \in G$ an operator $T_t \in \mathfrak{R}$ such that $E = \sum_{t \in G} T_t T_t^*$, $F = \sum_{t \in G} U_t^* T_t^* T_t U_t$. The main results in this paper state that this relation is indeed an equivalence relation (Thm. 1), that "semi-finiteness" is equivalent to the existence of a faithful normal semi-finite G -invariant trace on \mathfrak{R}^+ (Thm. 2), and that "finiteness" together with countable decomposability of \mathfrak{R} is equivalent to the existence of a faithful normal finite G -invariant trace on \mathfrak{R} (Thm. 3).

There are two approaches which can be used to prove these theorems. The most natural one would be to develop a comparison theory for projections in \mathfrak{R} and then to construct the traces. This can be done by means of modifications and extensions of the theory developed by Kadison and Pedersen [4]. The other approach, which we shall follow, is to consider the cross product $\mathfrak{R} \times G$, and then show that the canonical imbedding of \mathfrak{R} into the von Neumann algebra $\mathfrak{R} \times G$ is close to being an isomorphism of \mathfrak{R} with the structure of G -equivalence into $\mathfrak{R} \times G$ with the usual equivalence relation between projections.

Our main theorems form a link between von Neumann algebras and ergodic theory. If G is the one element group the equivalence relation \sim_G reduces to the usual one defined by Murray and von Neumann [8] for projections in a von Neumann algebra. We thus obtain extensions of the theorems on existence of traces in finite and semi-finite von Neumann algebras. If the von Neumann algebra \mathfrak{R} is abelian we show (Thm. 5), using theorems on the existence of invariant measures, that the equivalence relation \sim_G is the same as the one defined by Hopf [3] in ergodic theory. He showed that, with some extra assumptions, "finiteness" of the partial ordering is equivalent to the existence of an invariant normal state. Later on the "semi-finite" case was taken care of by Kawada [6] in a well ignored paper, and then independently by Halmos [2]. Thus our theorems are also generalizations of well known results on invariant measures.

We refer the reader to the book of Dixmier [1] for the theory of von Neumann algebras. The author is indebted to the referee for

several valuable comments.

2. **Statements of results.** In the present section we state the main results and definitions. The proofs will be given in §3.

THEOREM 1. *Let \mathfrak{R} be a von Neumann algebra acting on a Hilbert space \mathfrak{H} . Let G be a group and $t \rightarrow U_t$ a unitary representation of G on \mathfrak{H} such that $U_t^* \mathfrak{R} U_t = \mathfrak{R}$ for all $t \in G$. If E and F are projections in \mathfrak{R} we write $E \sim_G F$ if for each $t \in G$ there is an operator $T_t \in \mathfrak{R}$ such that*

$$E = \sum_{t \in G} T_t T_t^*, \quad F = \sum_{t \in G} U_t^* T_t^* T_t U_t.$$

Then \sim_G is an equivalence relation on the projections in \mathfrak{R} .

REMARK 1. If G is the one element group then the equivalence relation \sim_G is the same as the usual equivalence relation \sim for projections in a von Neumann algebra.

REMARK 2. If G is the additive group of \mathfrak{R} and the representation $t \rightarrow U_t$ is the trivial representation, so $U_t = I$ for $t \in G$, then the equivalence relation \sim_G is the one defined by Kadison and Pedersen [4, Def. A].

REMARK 3. If \mathfrak{R} is abelian and countably decomposable the equivalence relation \sim_G coincides with the one defined by Hopf [3] in ergodic theory. For this see Theorem 5 and Remark 6.

REMARK 4. If E and F are equivalent projections in \mathfrak{R} , i.e. there is a partial isometry $V \in \mathfrak{R}$ such that $E = VV^*$, $F = V^*V$, then $E \sim_G F$. This is clear from the definition of \sim_G , putting $T_e = V$, $T_t = 0$ for $t \neq e$.

DEFINITION 1. With notation as in Theorem 1 we say two projections E and F in \mathfrak{R} are G -equivalent if $E \sim_G F$. We write $E <_G F$ if $E \sim_G F_0 \leq F$. A projection F is said to be \sim_G -finite if $E \leq F$ and $E \sim_G F$ implies $E = F$. \mathfrak{R} is said to be \sim_G -finite if the identity operator I is \sim_G -finite. \mathfrak{R} is said to be \sim_G -semi-finite if every non-zero projection in \mathfrak{R} majorizes a nonzero \sim_G -finite projection.

THEOREM 2. *With notation as in Theorem 1 there exists a faithful normal semi-finite G -invariant trace on \mathfrak{R}^+ if and only if \mathfrak{R} is \sim_G -semi-finite.*

THEOREM 3. *With notation as in Theorem 1 there exists a faith-*

ful normal finite G -invariant trace on \mathfrak{R} if and only if \mathfrak{R} is \sim_G -finite and countably decomposable.

3. *Proofs.* We first introduce some notation and follow [1, Ch. I, §9] closely. Following the notation in Theorem 1 \mathfrak{R} acts on a Hilbert space \mathfrak{H} , G is a group, considered as a discrete group, and $t \rightarrow U_t$ is a unitary representation of G on \mathfrak{H} such that $U_t^* \mathfrak{R} U_t = \mathfrak{R}$ for all $t \in G$. For $t \in G$ let \mathfrak{H}_t be a Hilbert space of the same dimension as \mathfrak{H} and J_t an isometry of \mathfrak{H} onto \mathfrak{H}_t . Let $\tilde{\mathfrak{H}} = \sum_{t \in G} \oplus \mathfrak{H}_t$. We write an operator $R \in \mathfrak{B}(\tilde{\mathfrak{H}})$ —the bounded operators on $\tilde{\mathfrak{H}}$ —as a matrix $(R_{s,t})_{s,t \in G}$, where $R_{s,t} = J_s^* R J_t \in \mathfrak{B}(\mathfrak{H})$. For each $T \in \mathfrak{R}$ let $\Phi(T)$ denote the element in $\mathfrak{B}(\tilde{\mathfrak{H}})$ with matrix $(R_{s,t})$, where $R_{s,t} = 0$ if $s \neq t$, and $R_{s,s} = T$ for all $s \in G$. Then Φ is a *-isomorphism of \mathfrak{R} onto a von Neumann subalgebra $\tilde{\mathfrak{R}}$ of $\mathfrak{B}(\tilde{\mathfrak{H}})$. For $y \in G$ let \tilde{U}_y be the operator in $\mathfrak{B}(\tilde{\mathfrak{H}})$ with matrix $(R_{s,t})$, where $R_{s,t} = 0$ if $st^{-1} \neq y$, $R_{yt,t} = U_y$ for all $t \in G$. Then (see [1, Ch. I, §9]) $y \rightarrow \tilde{U}_y$ is a unitary representation of G on $\tilde{\mathfrak{R}}$ such that

$$\tilde{U}_y^* \Phi(T) \tilde{U}_y = \Phi(U_y^* T U_y), \quad y \in G, T \in \mathfrak{R}.$$

If \mathfrak{B} denotes the von Neumann algebra generated by $\tilde{\mathfrak{R}}$ and the $\tilde{U}_y, y \in G$, then each operator in \mathfrak{B} is represented by a matrix $(R_{s,t})$ where $R_{s,t} = T_{st^{-1}} U_{st^{-1}}$, $T_{st^{-1}} \in \mathfrak{R}$.

We denote by \mathfrak{R}^G the von Neumann subalgebra of \mathfrak{R} consisting of the G -invariant operators in \mathfrak{R} . \mathfrak{C} shall denote the center of \mathfrak{R} , and \mathfrak{D} shall denote $\mathfrak{C} \cap \mathfrak{R}^G$. Whenever we write $P \sim Q$ for two projections in \mathfrak{B} we shall mean they are equivalent as operators in \mathfrak{B} , i.e. there is a partial isometry $V \in \mathfrak{B}$ such that $VV^* = P$, $V^*V = Q$, and we shall not consider P and Q as equivalent in a von Neumann subalgebra of \mathfrak{B} . The next lemma includes Theorem 1 and shows more, namely that \sim_G -equivalence is the same as equivalence in \mathfrak{B} .

LEMMA 1. *Let E and F be projections in \mathfrak{R} . Then $E \sim_G F$ if and only if $\Phi(E) \sim \Phi(F)$. Hence \sim_G is an equivalence relation on the projections \mathfrak{R} .*

Proof. Suppose $E \sim_G F$. Then for each $t \in G$ there is $T_t \in \mathfrak{R}$ such that

$$E = \sum_{t \in G} T_t T_t^*, \quad F = \sum_{t \in G} U_t^* T_t^* T_t U_t.$$

Then we have

$$\begin{aligned} \Phi(E) &= \sum \Phi(T_t T_t^*) = \sum \Phi(T_t) \Phi(T_t)^* \\ &= \sum (\Phi(T_t) \tilde{U}_t) (\Phi(T_t) \tilde{U}_t)^*, \end{aligned}$$

and

$$\begin{aligned}\Phi(F) &= \sum \Phi(U_t^* T_t^* T_t U_t) = \sum \tilde{U}_t^* \Phi(T_t^* T_t) \tilde{U}_t \\ &= \sum (\Phi(T_t) \tilde{U}_t)^* (\Phi(T_t) \tilde{U}_t) .\end{aligned}$$

Thus by a result of Kadison and Pedersen [4, Thm. 4.1] $\Phi(E) \sim \Phi(F)$.

Conversely assume $\Phi(E) \sim \Phi(F)$. Then there is a partial isometry $V \in \mathfrak{B}$ such that $VV^* = \Phi(E)$, $V^*V = \Phi(F)$. Say $V = (T_{st^{-1}}U_{st^{-1}})$. Then an easy calculation shows

$$E = \sum_{t \in G} T_t T_t^* , \quad F = \sum_{t \in G} U_t^* T_t^* T_t U_t ,$$

hence $E \sim_G F$. The proof is complete.

LEMMA 2. *Let $S = (T_{st^{-1}}U_{st^{-1}})$ belong to the center of \mathfrak{B} . Then for each $s \in G$ we have*

(i) $TT_s = T_s U_s T U_s^*$ for all $T \in \mathfrak{R}$,

(ii) $T_{sy} = U_y^* T_y U_y$ for all $y \in G$.

In particular $T_e \in \mathfrak{D}$. Furthermore, if $R \in \mathfrak{D}$ then $\Phi(R)$ belongs to the center of \mathfrak{B} .

Proof. Let $T \in \mathfrak{R}$. Then

$$(TT_{st^{-1}}U_{st^{-1}}) = \Phi(T)S = S\Phi(T) = (T_{st^{-1}}U_{st^{-1}}TU_{ts^{-1}}U_{st^{-1}}) ,$$

and (i) follows. Let $y \in G$. Then an easy computation shows

$$(T_{st^{-1}y^{-1}}U_{st^{-1}}) = S\tilde{U}_y = \tilde{U}_y S = (U_y T_{y^{-1}st^{-1}} U_y^* U_{st^{-1}}) .$$

Replacing y by y^{-1} and letting $t = e$, (ii) follows. By (i) $T_e T = TT_e$, so $T_e \in \mathfrak{C}$. By (ii) if $s = y^{-1}$ we find $T_e = U_y^* T_e U_y$, so $T_e \in \mathfrak{R}^G$, hence $T_e \in \mathfrak{D}$.

Finally let $R \in \mathfrak{D}$, and let $S' = (S_{st^{-1}}U_{st^{-1}}) \in \mathfrak{B}$. Then we have

$$\begin{aligned}\Phi(R)S' &= (RS_{st^{-1}}U_{st^{-1}}) = (S_{st^{-1}}RU_{st^{-1}}) \\ &= (S_{st^{-1}}U_{st^{-1}}R) = S'\Phi(R) ,\end{aligned}$$

hence $\Phi(R)$ belongs to the center of \mathfrak{B} . The proof is complete.

LEMMA 3. *Let E be a projection in \mathfrak{R} . Let D_E be the smallest operator in \mathfrak{D} majorizing E . Then D_E is a projection, and $\Phi(D_E)$ is the central carrier of $\Phi(E)$ in \mathfrak{B} .*

Proof. Since \mathfrak{D} is an abelian von Neumann algebra its positive operators form a complete lattice under infs and sups. Thus $D_E = \text{g.l.b.}\{A \in \mathfrak{D} : E \leq A \leq I\}$, and D_E is well defined. Since $E \leq D_E$ and both operators commute we have $E = E^2 \leq D_E^2$. But $D_E \leq I$, so $D_E^2 \leq$

D_E . Hence by minimality of D_E , $D_E = D_E^2$, so it is a projection. By Lemma 2 $\Phi(D_E)$ is a central projection in \mathfrak{B} , hence if $C_{\Phi(E)}$ denotes the central carrier of $\Phi(E)$ in \mathfrak{B} , then $\Phi(D_E) \geq C_{\Phi(E)}$. Now let $C_{\Phi(E)} = (T_{st^{-1}}U_{st^{-1}})$. By Lemma 2 $T_e \in \mathfrak{D}$, and since $C_{\Phi(E)} \geq \Phi(E)$, $T_e \geq E$. By definition of D_E , $T_e \geq D_E$. But $\Phi(D_E) \geq C_{\Phi(E)}$, so $D_E \geq T_e$, hence $T_e = D_E$. The operator $\Phi(D_E) - C_{\Phi(E)}$ is positive and has zeros on the main diagonal. Therefore it is 0, and $\Phi(D_E) = C_{\Phi(E)}$ as asserted.

LEMMA 4. *Let E be a projection in \mathfrak{R} . Let C_E be its central carrier in \mathfrak{R} , and let D_E be as in Lemma 3. Then $D_E = D_{C_E}$.*

Proof. Since $E \leq C_E$, $D_E \leq D_{C_E}$. But $D_E \in \mathfrak{C}$ and $D_E \geq E$, hence $D_E \geq C_E$. Therefore by definition of D_{C_E} , $D_E \geq D_{C_E}$, and they are equal.

LEMMA 5. *Let E be a countably decomposable projection in \mathfrak{R} . Then $\Phi(E)$ is countably decomposable in \mathfrak{B} .*

Proof. Let x be a vector in $E\mathfrak{H}$. Then x considered as a vector in $\sum_{t \in G} \oplus \mathfrak{H}_t$ belongs to \mathfrak{H}_e . Let F be the support of ω_x in $E\mathfrak{R}E$. Then F is countably decomposable, and ω_x is a faithful normal state of $F\mathfrak{R}F$. Let $\{F_\alpha\}_{\alpha \in J}$ be an orthogonal family of projections in \mathfrak{B} such that $\sum_{\alpha \in J} F_\alpha = \Phi(F)$. Let $F_\alpha = (T_{st^{-1}}^\alpha U_{st^{-1}})$. Then $F_\alpha \leq \Phi(F)$, so $T_e^\alpha \leq F$, hence $T_e^\alpha \in F\mathfrak{R}F$. Furthermore, since $x \in \mathfrak{H}_e$, $\omega_x(F_\alpha) = \omega_x(T_e^\alpha)$. Thus we have

$$1 = \omega_x(F) = \omega_x(\Phi(F)) = \sum \omega_x(F_\alpha) = \sum \omega_x(T_e^\alpha).$$

Therefore $\omega_x(T_e^\alpha) = 0$ except for a countable number of $\alpha \in J$. But then $T_e^\alpha = 0$ and hence $F_\alpha = 0$ except for a countable number of $\alpha \in J$. Thus $\Phi(F)$ is countably decomposable in \mathfrak{B} . Now E is a countable sum of orthogonal cyclic projections, hence $\Phi(E)$ is a countable sum of orthogonal countably decomposable projections. Hence $\Phi(E)$ is countably decomposable. The proof is complete.

DEFINITION 2. We say a projection E in \mathfrak{R} is \sim_G -abelian if $E\mathfrak{R}E = E\mathfrak{D}$.

Clearly a \sim_G -abelian projection is abelian.

LEMMA 6. *There is a projection $P \in \mathfrak{D}$ such that there exists a \sim_G -abelian projection $E \leq P$ with $D_E = P$, and $I - P$ has no nonzero \sim_G -abelian subprojection.*

Proof. Partially order the \sim_G -abelian projections in \mathfrak{R} by $E \ll F$ if $E \leq F$ and $D_{F-E} \leq I - D_E$. Then in particular $D_E F = E$. Let $\{E_\alpha\}$ be a totally ordered set of \sim_G -abelian projections, and let $E = \sup E_\alpha$,

so $E_\alpha \rightarrow E$ strongly. Then

$$D_{E_\alpha}E = D_{E_\alpha} \lim_{\beta > \alpha} E_\beta = \lim_{\beta > \alpha} D_{E_\alpha}E_\beta = E_\alpha,$$

hence if $A \in \mathfrak{R}$ then

$$EAED_{E_\alpha} = E_\alpha AE_\alpha = A_\alpha E_\alpha,$$

where $A_\alpha \in \mathfrak{D}D_{E_\alpha}$. Now it is well known that if Q_α is an increasing net of projections, and $Q_\alpha \rightarrow Q$ strongly, then $C_{Q_\alpha} \rightarrow C_Q$ strongly. Thus

$$\Phi(D_{E_\alpha}) = C_{\Phi(E_\alpha)} \rightarrow C_{\Phi(E)} = \Phi(D_E)$$

by Lemma 3, hence $D_{E_\alpha} \rightarrow D_E$ strongly. The same argument also shows

$$D_{E-E_\alpha} = \lim_{\beta > \alpha} D_{E_\beta - E_\alpha} \leq I - D_{E_\alpha}.$$

Thus $E = E(I - D_{E_\alpha}) + E_\alpha$, and since $A_\alpha = A_\alpha D_{E_\alpha}$ we have $EAED_{E_\alpha} = A_\alpha E \in E\mathfrak{D}$. Since $D_{E_\alpha} \rightarrow D_E$ it follows that $EAE = \lim_\alpha EAED_{E_\alpha} \in E\mathfrak{D}$. Therefore E is \sim_G -abelian. Now let E be a maximal \sim_G -abelian projection in \mathfrak{R} . Let $P = D_E$. Suppose F is a \sim_G -abelian subprojection of $I - P$. Then $E + F$ is \sim_G -abelian. Indeed, if $A \in \mathfrak{R}$ then there are $A_E \in D_E\mathfrak{D}$ and $A_F \in D_F\mathfrak{D}$ such that

$$\begin{aligned} (E + F)A(E + F) &= EAE + FAF = EA_E + FA_F \\ &= (E + F)(A_E + A_F) \in (E + F)\mathfrak{D}. \end{aligned}$$

Thus $E + F$ is \sim_G -abelian. Since $E \ll E + F$, the maximality of E implies $F = 0$. The proof is complete.

Thus in order to prove Theorems 2 and 3 we may consider two cases separately, namely the case when \mathfrak{R} has a \sim_G -abelian projection E with $D_E = I$, and the case when \mathfrak{R} has no nonzero \sim_G -abelian projection. We first treat the case with a \sim_G -abelian projection.

LEMMA 7. *Let E be a \sim_G -abelian projection in \mathfrak{R} . Then C_E is not G -equivalent to a proper central subprojection. Furthermore if Q is a central projection such that $Q \leq C_E$ then $Q = D_Q C_E$.*

Proof. Let Q be as in the statement of the lemma. Since E is \sim_G -abelian there is an operator $D \in \mathfrak{D}$ such that $QE = DE$, hence, since $E\mathfrak{C} \cong C_E\mathfrak{C}$, $Q = QC_E = DC_E$, and $D \geq Q$. By definition of D_Q , $D \geq D_Q$. But $D_Q \geq Q$, so $Q = QC_E \leq D_Q C_E \leq DC_E = Q$, so that $Q = D_Q C_E$. Now suppose P is a projection in \mathfrak{C} such that $P \leq C_E$ and $P \sim_G C_E$. Then in particular by Lemma 1 $\Phi(P) \sim \Phi(C_E)$, so they have the same central carrier in \mathfrak{B} , hence $D_P = D_{C_E} = D_E$ by Lemma 4. By the preceding, $P = D_P C_E = C_E$. The proof is complete.

LEMMA 8. Let E be a \sim_G -abelian projection in \mathfrak{R} . Let Q be a central projection orthogonal to C_E . Then if C_E and $C_E + Q$ are G -equivalent relative to \mathfrak{C} , i.e. the operators T_t defining the equivalence belong to \mathfrak{C} , then $Q = 0$.

Proof. Let $P = C_E$ and assume $P \sim_G P + Q$ relative to \mathfrak{C} . Then since \mathfrak{C} is abelian, for each $t \in G$ there is $A_t \in \mathfrak{C}^+$ such that $P = \sum_{t \in G} A_t$, $P + Q = \sum_{t \in G} U_t^* A_t U_t$. Since $E\mathfrak{C} = E\mathfrak{D}$ and $P\mathfrak{C} \cong E\mathfrak{C}$, we have $P\mathfrak{C} = P\mathfrak{D}$. Since $A_t \leq P$ there is $D_t \in \mathfrak{D}^+$ such that $A_t = PD_t$. Thus we have

$$\begin{aligned} \sum PD_t &= P = P(P + Q) = \sum PU_t^* A_t U_t \\ &= \sum PU_t^* PD_t U_t = \sum PD_t U_t^* PU_t. \end{aligned}$$

Now $PD_t U_t^* PU_t \leq PD_t$ for all t , hence we have $PD_t U_t^* PU_t = PD_t$ for all t . Let E_t denote the range projection of D_t . Then $E_t \in \mathfrak{D}$. Since $U_t^* PU_t PD_t = PD_t$, $U_t^* PU_t PE_t = PE_t$. Thus $U_t^* PU_t \geq PE_t$, and thus $U_t^* PE_t U_t = U_t^* PU_t E_t \geq PE_t$. Consequently $PE_t \geq U_t PE_t U_t^*$. By Lemma 7 $P = C_E$ is \sim_G -finite relative to \mathfrak{C} , hence so is PE_t . Therefore $PE_t = U_t PE_t U_t^*$, and $U_t^* PE_t U_t = PE_t$. Therefore we have

$$U_t^* A_t U_t = U_t^* PD_t U_t = U_t^* PE_t U_t D_t = PE_t D_t = PD_t = A_t,$$

and $P = P + Q$, so that $Q = 0$. The proof is complete.

LEMMA 9. Suppose E is a \sim_G -abelian projection in \mathfrak{R} with $D_E = I$. Then \mathfrak{R} is of type I, and there exists a faithful normal semi-finite G -invariant trace on \mathfrak{R}^+ .

Proof. Since E is abelian $C_E \mathfrak{R}$ is of type I. Since every $*$ -automorphism of \mathfrak{R} preserves the type I portion of \mathfrak{R} , and $D_E = I$, \mathfrak{R} is of type I.

E is a sum of orthogonal cyclic projections E_α . If we can show the lemma for each E_α then it holds for E . Therefore we may assume E is cyclic, say $E = [\mathfrak{R}'x]$. Then ω_x is faithful on $E\mathfrak{R}E$, hence faithful on $E\mathfrak{C}$. If $A \geq 0$ belongs to $C_E \mathfrak{C}$ and $\omega_x(A) = 0$, then $0 = \omega_x(EA)$, so $EA = 0$. Hence $A = AC_E = 0$. Thus ω_x is faithful on $C_E \mathfrak{C}$, so C_E is a countably decomposable projection in \mathfrak{C} .

We shall now apply the previous theory to $\mathfrak{A} = \mathfrak{C} \times G$ instead of $\mathfrak{B} = \mathfrak{R} \times G$. We use the same notation as before. By Lemma 7 C_E is \sim_G -finite. If $C_E = D_E = I$ then by Lemma 7 $\mathfrak{C} = \mathfrak{D}$, and it is trivial that there exists a faithful normal semi-finite G -invariant trace on \mathfrak{C}^+ . Assume $C_E \neq I$. Then there is $s \in G$ such that $U_s^* C_E U_s \neq C_E$. Since by Lemma 7 C_E is \sim_G -finite, and $U_s^* C_E U_s \sim_G C_E$, $U_s^* C_E U_s$ is not a subprojection of C_E . Thus $Q = U_s^* C_E U_s (I - C_E) \neq 0$. Since C_E is

countably decomposable, so is Q , and hence $C_E + Q$. By Lemma 5 $\Phi(C_E + Q)$ is countably decomposable in \mathfrak{A} . Since $I = D_E \leq D_{C_E} + Q$, the central carriers of $\Phi(C_E)$ and $\Phi(C_E + Q)$ are by Lemma 3 equal to I . If $\Phi(C_E)$ is properly infinite then by [1, Ch. III, §8, Cor. 5] $\Phi(C_E) \sim \Phi(C_E + Q)$, so by Lemma 1 $C_E \sim_G C_E + Q$, contradicting Lemma 8. Thus $\Phi(C_E)$ is not properly infinite, and there is a nonzero central projection P in \mathfrak{A} such that $P\Phi(C_E)$ is nonzero and finite. Since the central carrier of $\Phi(C_E)$ is I , $P\mathfrak{A}$ is semi-finite. Let φ be a normal semi-finite trace on \mathfrak{A}^+ with support P such that $\varphi(\Phi(C_E)) < \infty$. For $A \in \mathfrak{E}^+$ define $\tau(A) = \varphi(\Phi(A))$. Then τ is a normal G -invariant trace because $\tau(U_s^* A U_s) = \varphi(\tilde{U}_s^* \Phi(A) \tilde{U}_s) = \varphi(\Phi(A)) = \tau(A)$. Since $\tau(C_E) < \infty$ and $D_{C_E} = I$, τ is semi-finite, hence τ is a normal semi-finite G -invariant trace on \mathfrak{E}^+ . Let D be the support of τ . Then $0 \neq D \in \mathfrak{D}$. Now apply the preceding to $(I - D)\mathfrak{E}$ and $E(I - D)$, and use Zorn's lemma to obtain a family D_α of orthogonal projections in \mathfrak{D} with sum I , and a normal semi-finite G -invariant trace τ_α of \mathfrak{E}^+ with support D_α . Let $\tau = \sum \tau_\alpha$. Then τ is a faithful normal semi-finite G -invariant trace on \mathfrak{E}^+ .

Now since \mathfrak{R} is of type I there is a faithful normal center valued trace ψ on \mathfrak{R}^+ such that $U_s^* \psi(U_s A U_s^*) U_s = \psi(A)$ for each $s \in G$, $A \in \mathfrak{R}^+$, see [11, p. 3]. Then $\tau \circ \psi$ is a faithful normal semi-finite G -invariant trace on \mathfrak{R}^+ , see [1, Ch. III, §4, Prop. 2]. The proof is complete.

LEMMA 10. *Suppose \mathfrak{R} is \sim_G -semi-finite and there are no nonzero \sim_G -abelian projections in \mathfrak{R} . Then there is a faithful normal semi-finite G -invariant trace on \mathfrak{R}^+ .*

Proof. Let E be a nonzero countably decomposable \sim_G -finite projection in \mathfrak{R} . Since E is not \sim_G -abelian there is a projection $H \in E\mathfrak{R}E$ such that $H \neq ED_H$. Let $F = H + (I - D_H)E$. Then $F \leq E$, $F \neq E$, and $D_F = D_H + (I - D_H)D_E = D_E$. $\Phi(F)$ is not properly infinite in \mathfrak{B} . Indeed, if it were, then since $\Phi(E)$ is countably decomposable by Lemma 5, [1, Ch. III, §8, Cor. 5] would imply $\Phi(F) \sim \Phi(E)$, hence by Lemma 1, $F \sim_G E$, contradicting the \sim_G -finiteness of E . Therefore there is a nonzero central projection P in \mathfrak{B} such that $P\Phi(F)$ is finite and nonzero. Thus $P\Phi(D_E)\mathfrak{B} = P\Phi(D_F)\mathfrak{B}$ is semi-finite and nonzero. Let φ be a normal semi-finite trace on \mathfrak{B} with support $P\Phi(D_E)$ such that $\varphi(\Phi(F)) < \infty$. For $A \in \mathfrak{R}^+$ define $\tau(A) = \varphi(\Phi(A))$. As in the proof of Lemma 9 τ is a normal G -invariant trace on \mathfrak{R}^+ . Since $\tau(F) < \infty$ there is a nonzero central projection Q in \mathfrak{R} such that τ is faithful and semi-finite on $Q\mathfrak{R}$ [1, Ch. I, §6, Cor. 2]. Since τ is G -invariant $Q \in \mathfrak{D}$. Now a Zorn's Lemma argument completes the proof just as in Lemma 9.

Proof of Theorem 2. By Lemma 6 there is a projection $P \in \mathfrak{D}$ such that there exists a \sim_G -abelian projection $E \in P\mathfrak{R}$ with $D_E = P$, and $I - P$ has no nonzero \sim_G -abelian subprojection. By Lemma 9 there is a faithful normal semi-finite G -invariant trace τ_1 on $P\mathfrak{R}^+$. If \mathfrak{R} is \sim_G -semi-finite then by Lemma 10 there is a faithful normal semi-finite G -invariant trace τ_2 on $(I - P)\mathfrak{R}^+$. Thus $\tau = \tau_1 + \tau_2$ is a faithful normal semi-finite G -invariant trace on \mathfrak{R}^+ .

Conversely assume there exists a faithful normal semi-finite G -invariant trace τ on \mathfrak{R}^+ . Suppose E is a projection in \mathfrak{R} such that $\tau(E) < \infty$. Since $E \sim_G F$ implies $\tau(E) = \tau(F)$ it is clear that E is \sim_G -finite. Thus \mathfrak{R} is \sim_G -semi-finite. The proof is complete.

LEMMA 11. *Suppose \mathfrak{C} is countably decomposable and \mathfrak{R} is \sim_G -finite. Then there is a faithful normal finite G -invariant trace on \mathfrak{R} .*

Proof. Since \mathfrak{R} is \sim_G -finite \mathfrak{R} is in particular finite. By [1, Ch. III, §4, Thm. 3] there is a unique center valued trace ψ on \mathfrak{R} which is the identity on \mathfrak{C} . By uniqueness ψ is G -invariant, so if τ is a faithful normal finite G -invariant trace on \mathfrak{C} , then $\tau \circ \psi$ is one on \mathfrak{R} . Therefore we may assume $\mathfrak{R} = \mathfrak{C}$. Now there exists a projection $P \in \mathfrak{D}$ such that $P\mathfrak{C} = P\mathfrak{D}$, and G is freely acting on $(1 - P)\mathfrak{C}$, i.e. for each projection $E \neq 0$ in $(I - P)\mathfrak{C}$ there is a nonzero subprojection F of E and $s \in G$ such that $U_s^* F U_s \leq I - F$, see e.g. [5]. Since I is countably decomposable, so is P , and there is a faithful normal state on $P\mathfrak{C}$, hence a faithful normal finite G -invariant trace on $P\mathfrak{C}$. We may thus assume G is freely acting. Let F be a nonzero projection in \mathfrak{C} and s an element in G such that $U_s^* F U_s \leq I - F$. Let $E = I - F$. Then $D_E = I$, and $F <_G E$. As in the proof of Lemma 10 $\Phi(E)$ is not properly infinite, so we can choose a central projection $P \neq 0$ in \mathfrak{B} such that $P\Phi(E)$ is finite. Since $F <_G E$, $\Phi(F) < \Phi(E)$, by Lemma 1, hence $P\Phi(F) < P\Phi(E)$, so $P\Phi(F)$ is finite. Thus $P = P\Phi(E) + P\Phi(F)$ is finite in \mathfrak{B} , and $P\mathfrak{B}$ is finite. Since I is countably decomposable in $\mathfrak{C}(=\mathfrak{R})$ $\Phi(I)$ is countably decomposable in \mathfrak{B} by Lemma 5, hence so is P . Therefore by [1, Ch. I, §6, Prop. 9] there is a faithful normal finite trace φ on $P\mathfrak{B}$. Then τ defined by $\tau(A) = \varphi(\Phi(A))$ is a normal finite G -invariant trace on \mathfrak{C} with support $D \neq 0$ in \mathfrak{D} . A Zorn's Lemma argument now gives a family τ_α of normal finite G -invariant traces on \mathfrak{C} with orthogonal supports D_α in \mathfrak{D} . Since I is countably decomposable the family $\{\tau_\alpha\}$ is countable, and by multiplying each τ_α by a convenient positive scalar we may assume $\sum \tau_\alpha(D_\alpha) = 1$. Thus if $\tau = \sum \tau_\alpha$, then τ is a faithful normal finite G -invariant trace on \mathfrak{C} . The proof is complete.

Proof of Theorem 3. Suppose there is a faithful normal finite G -invariant trace τ on \mathfrak{R} . Then I is \sim_σ -finite, for if E is a projection in \mathfrak{R} which is G -equivalent to I then $\tau(E) = \tau(I)$, hence $\tau(I - E) = 0$, hence $I - E = 0$, since τ is faithful. Thus \mathfrak{R} is \sim_σ -finite. Again since τ is faithful, its support I is countably decomposable, i.e. \mathfrak{R} is countably decomposable. The converse follows from Lemma 11.

COROLLARY. *If \mathfrak{R} is \sim_σ -semi-finite then \mathfrak{B} is semi-finite. If \mathfrak{R} is \sim_σ -finite and there is an orthogonal family of countably decomposable projections in \mathfrak{D} with sum I , then \mathfrak{B} is finite.*

Proof. If \mathfrak{R} is \sim_σ -semi-finite, then by Theorem 2 there is a faithful normal semi-finite G -invariant trace on \mathfrak{R} . Thus there is a faithful normal semi-finite trace on \mathfrak{B} by [1, Ch. I, §9, Prop. 1], hence \mathfrak{B} is semi-finite. If P is a projection in \mathfrak{D} then by Lemma 2 $\Phi(P)$ is a central projection in \mathfrak{B} . Thus in order to show the last part of the corollary we may assume I is countably decomposable. Then by Theorem 3 there is a faithful normal finite G -invariant trace on \mathfrak{R} , hence by [1, Ch. I, §9, Prop. 1] there is a normal finite trace on \mathfrak{B} , so \mathfrak{B} is finite. The proof is complete.

REMARK 5. G. K. Pedersen has pointed out that the corollary can be sharpened. Indeed one can show that if E is a projection in \mathfrak{R} then E is \sim_σ -finite if and only if $\Phi(E)$ is finite in \mathfrak{B} . In particular \mathfrak{R} is \sim_σ -finite if and only if \mathfrak{B} is finite.

4. G -finite von Neumann algebras. Let notation be as in Theorem 1. Following [7] we say \mathfrak{R} is G -finite if there is a family \mathcal{F} of normal G -invariant states which separate \mathfrak{R}^+ , i.e. if $A \in \mathfrak{R}^+$, and $\omega(A) = 0$ for all $\omega \in \mathcal{F}$, then $A = 0$. For semi-finite von Neumann algebras it would be natural to compare this concept with those of \sim_σ -finite and \sim_σ -semi-finite. Since a \sim_σ -finite von Neumann algebra is necessarily finite we cannot expect a G -finite semi-finite von Neumann algebra to be \sim_σ -finite. We say G acts ergodically on \mathfrak{C} if $\mathfrak{D}(=\mathfrak{C} \cap \mathfrak{R}^G)$ is the scalars.

THEOREM 4. *Let \mathfrak{R} be a semi-finite von Neumann algebra acting on a Hilbert space \mathfrak{H} . Let G be a group and $t \rightarrow U_t$ a unitary representation of G on \mathfrak{H} such that $U_t^* \mathfrak{R} U_t = \mathfrak{R}$ for all $t \in G$. Assume either that G acts ergodically on the center of \mathfrak{R} or the center is elementwise fixed under G . Then \mathfrak{R} is G -finite if and only if there is a faithful normal semi-finite G -invariant trace τ on \mathfrak{R}^+ and an orthogonal family $\{E_\alpha\}$ of G -invariant projections in \mathfrak{R} with sum I and $\tau(E_\alpha) < \infty$ for each α .*

Proof. Assume \mathfrak{R} is G -finite. Suppose first that G acts ergodically on the center \mathfrak{C} of \mathfrak{R} , and suppose ω is a faithful normal G -invariant state on \mathfrak{R} . Then by [11] there is a faithful normal semi-finite G -invariant trace on \mathfrak{R}^+ , hence by Theorem 2 \mathfrak{R} is \sim_G -semi-finite. In general, by Zorn's Lemma there is a family $\{\omega_\alpha\}$ of normal G -invariant states with orthogonal supports E_α such that $\sum E_\alpha = I$. Then each E_α is G -invariant, and by the first part of the proof $E_\alpha \mathfrak{R} E_\alpha$ is \sim_G -semi-finite. In particular, E_α is the sup of an increasing net of \sim_G -finite projections. Let F be a projection in \mathfrak{R} . We show F has a nonzero \sim_G -finite subprojection. By the above considerations there are E_α and a \sim_G -finite subprojection F_α of E_α such that $C_{F_\alpha} F \neq 0$. Let $F_1 = C_{F_\alpha} F$. Then there is a nonzero subprojection F_0 of F_1 such that $F_0 \preceq F_\alpha$. Say $F_0 \sim G_\alpha \leq F_\alpha$. Since F_α is \sim_G -finite, so is G_α . Indeed, if $G_\alpha \sim_G H \leq G_\alpha$ then by Lemma 1 $\Phi(G_\alpha) \sim \Phi(H)$, hence $\Phi(F_\alpha) = \Phi(G_\alpha) + \Phi(F_\alpha - G_\alpha) \sim \Phi(H) + \Phi(F_\alpha - G_\alpha)$, so again by Lemma 1, $F_\alpha \sim_G H + F_\alpha - G_\alpha$, so that $H = G_\alpha$ by finiteness of F_α . Thus G_α is \sim_G -finite. Since G_α is in particular finite there is by [1, Ch. III, §2, Prop. 6] a unitary operator $U \in \mathfrak{R}$ such that $UF_0U^{-1} = G_\alpha$. But then F_0 is \sim_G -finite, for if $F_0 \sim_G F_2 \leq F_0$ then $UF_2U^{-1} \sim F_2 \sim_G G_\alpha$, so by transitivity $UF_2U^{-1} \sim_G G_\alpha$. Since $UF_2U^{-1} \leq G_\alpha$, they are equal by finiteness of G_α , so $F_2 = F_0$, and F_0 is \sim_G -finite. Therefore the projection F has a nonzero \sim_G -finite subprojection F_0 , and \mathfrak{R} is \sim_G -semi-finite.

Next assume $\mathfrak{C} = \mathfrak{D}$. Then every normal semi-finite trace on \mathfrak{R}^+ is G -invariant [10, Cor. 2.2], so there exists a faithful normal semi-finite G -invariant trace on \mathfrak{R}^+ , hence by Theorem 2, \mathfrak{R} is \sim_G -semi-finite.

Let by Theorem 2 τ be a faithful normal semi-finite G -invariant trace on \mathfrak{R}^+ . Let $\{\omega_\alpha\}$ be as before with orthogonal supports $\{E_\alpha\}$. Then there is a positive self-adjoint operator $H_\alpha \in L(\mathfrak{R}, \tau)$ affiliated with \mathfrak{R}^G such that $\omega_\alpha(T) = \tau(H_\alpha T)$ for $T \in \mathfrak{R}$, see e.g. [1, Ch. I, §6, no. 10]. Let E be a spectral projection of H_α with $\tau(E) < \infty$. Then E is G -invariant. A Zorn's Lemma argument now gives an orthogonal family of G -invariant projections in \mathfrak{R} with sum I and finite trace.

Conversely assume \mathfrak{R} has a faithful normal semi-finite G -invariant trace τ and an orthogonal family $\{E_\alpha\}$ of nonzero G -invariant projections with sum I such that $\tau(E_\alpha) < \infty$. Let $c_\alpha = \tau(E_\alpha)^{-1}$, and let $\omega_\alpha(T) = c_\alpha \tau(E_\alpha T)$. Then $\{\omega_\alpha\}$ is a separating family of normal G -invariant states on \mathfrak{R} , hence \mathfrak{R} is G -finite. The proof is complete.

The above theorem is probably true without the assumptions of the action of G on \mathfrak{C} . A direct proof of this would be quite interesting.

5. Abelian von Neumann algebras. Assume \mathfrak{R} is an abelian von Neumann algebra acting on a Hilbert space \mathfrak{H} . Let G be a group and suppose $t \rightarrow U_t$ is a unitary representation of G on \mathfrak{H} such that

$U_t^* \mathfrak{R} U_t = \mathfrak{R}$ for all $t \in G$. We say two projections E and F in \mathfrak{R} are *equivalent in the sense of Hopf* and write $E \sim_H F$ if there are an orthogonal family of projections $\{E_\alpha\}_{\alpha \in J}$ in \mathfrak{R} and $t_\alpha \in G$, for $\alpha \in J$, such that $E = \sum E_\alpha$, $F = \sum U_{t_\alpha}^* E_\alpha U_{t_\alpha}$. Since each $U_{t_\alpha}^* E_\alpha U_{t_\alpha}$ is a projection, and their sum is a projection, they are all mutually orthogonal. Since we can collect the E_α 's for which t_α coincide the definition of equivalence in the sense of Hopf is equivalent to the existence of an orthogonal family of projections $\{E_t\}_{t \in G}$ in \mathfrak{R} such that $E = \sum_{t \in G} E_t$, $F = \sum_{t \in G} U_t^* E_t U_t$. This equivalence was introduced by Hopf [3]. Just as for \sim_G we define \sim_H -finite, \sim_H -semi-finite, and $<_H$. Note that if $E \sim_H F$ as above, if we let $T_t = E_t$, then $E = \sum T_t T_t^*$, $F = \sum U_t^* T_t^* T_t U_t$, so $E \sim_G F$. If we assume \mathfrak{R} is countably decomposable, we shall now prove the converse via a proof which makes use of the known results on invariant measures if \mathfrak{R} is \sim_H -finite and \sim_H -semi-finite. A direct proof would be more desirable.

THEOREM 5. *Assume \mathfrak{R} is countably decomposable, and let notation be as above. Then two projections E and F in \mathfrak{R} are G -equivalent if and only if they are equivalent in the sense of Hopf.*

Outline of proof. It remains to be shown that if $E \sim_G F$ then $E \sim_H F$. Assume $E \sim_G F$. By Lemma 1 $\Phi(E) \sim \Phi(F)$, so they have the same central carrier C . By Lemma 3 $\Phi(D_E) = C = \Phi(D_F)$, so $D_E = D_F$. Suppose first E and F are such that EP and FP are \sim_H -infinite for all nonzero projections $P \in \mathfrak{D}$. In a von Neumann algebra two properly infinite countably decomposable projections with the same central carriers are equivalent [1, Ch. III, §8, Cor. 5]. Using the comparison theory for \mathfrak{R} with the Hopf ordering $<_H$, as developed in [6], see also [9], we can modify the proof of the quoted result for von Neumann algebras, to show $E \sim_H F$. If E is \sim_H -finite then since $D_E = D_F$, we may assume \mathfrak{R} is \sim_H -semi-finite, so by [6] there is a faithful normal semi-finite G -invariant trace τ on \mathfrak{R}^+ such that $\tau(E) < \infty$. From the comparison theorem on \mathfrak{R} [6, Lem. 16], or [9, Lem. 2.7], there exist two orthogonal projections P and Q in \mathfrak{D} with sum I such that $PE <_H PF$ and $QF <_H QE$. Since $PE \sim_G PF$ we have $\tau(PE) = \tau(PF)$. But if a proper subprojection F_0 of PF is such that $PE \sim_H F_0$ then $\tau(PE) = \tau(F_0) < \tau(PF) = \tau(PE)$, a contradiction. Thus $PE \sim_H PF$, and similar $QE \sim_H QF$. Thus $E \sim_H F$, and the proof is complete.

REMARK 6. Theorem 5 is undoubtedly true without the assumption that \mathfrak{R} is countably decomposable. If E is \sim_H -finite then it is still possible to find τ as above. If E is \sim_H -infinite the above

proof works as long as E is countably decomposable. Otherwise the theorem seems to be more difficult to prove, cf. proof of [4, Thm. 4.1].

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