Pacific Journal of Mathematics

MULTIPLICITY AND THE AREA OF AN $(n-1)$ CONTINUOUS MAPPING

RONALD FRANCIS GARIEPY

Vol. 44, No. 2 June 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 2, 1973

MULTIPLICITY AND THE AREA OF AN $(n-1)$ CONTINUOUS MAPPING

RONALD GARIEPY

For a class of mappings considered by Goffman and Ziemer [Annals of Math. 92 (1970)] it is shown that the area is given by the integral of a multiplicity function and a convergence theorem is proved.

1. **Introduction.** A theory of surface area for mappings beyond the class of continuous mappings was initiated in [2]. This theory includes certain essentially discontinuous mappings for which it seems natural that the area be given by the classical integral formula.

Let $Q = R^n \cap \{x\colon 0 < x_i < 1 \text{ for } 1 \leqq i \leqq n\}$. For each $i \in \{1, \cdots, n\}$ and $r \in I = \{t\colon 0 < t < 1\}$ let $P_i(r) = Q \cap \{x\colon x_i = r\}$. A mapping $f\colon Q \to R^m$, $n \leqq m$, is said to be $n-1$ continuous if, for each $i, f \mid P_i(r)$ is continuous for almost every (in the sense of 1-dimensional Lebesgue measure) $r \in I$. A sequence $\{f_j\}$ of mappings from Q into R^m is said to converge $n-1$ to f if, for each $i, f_j \mid P_i(r)$ converges uniformly to $f \mid P_i(r)$ for almost every $r \in I$.

The area of an $n-1$ continuous mapping $f\colon Q \to R^m$ is defined as

$$A(f) = \inf \varliminf_{j\to\infty} a(f_j)$$

where the infimum is taken over all sequences $\{f_j\}$ of quasilinear mappings converging $n-1$ to f and $a(f_j)$ denotes the elementary area of f_j. In [2] it was shown that $A(f)$ coincides with Lebesgue area if f is continuous.

For real $p \geqq 1$, let $W_p^1(Q)$ denote those functions in $L^p(Q)$ whose distribution first partial derivatives are functions in $L^p(Q)$. Suppose $f\colon Q \to R^m$ with $f = (f^1, \cdots, f^m)$ and $f^i \in W_{p_i}^1(Q)$, $p_i > n-1$ for $1 \leqq i \leqq m$ and $\sum_{j=1}^n 1/p_{i_j} \leqq 1$ whenever $1 \leqq i_1 < \cdots < i_n \leqq m$. It was shown in [3] that f is $n-1$ continuous and

$$A(f) = \int_Q |Jf(x)|\, dx\ .$$

In this paper we prove the following

THEOREM. *If $f\colon Q \to R^n$ with $f^i \in W_{p_i}^1(Q)$, $p_i > n-1$ and $\sum_{i=1}^n 1/p_i \leqq 1$, then there is a nonnegative integer valued lower semicontinuous function $N(f, y)$ on R^n such that*

$$A(f) = \int_{R^n} N(f, y)\, dy \tag{1}$$

and, if $\{f_j\}$ is any sequence of quasi-linear mappings converging $n-1$ to f with $A(f) = \lim_{j\to\infty} a(f_j)$, then

$$(2) \qquad \lim_{j\to\infty} \int_{R^n} | N(f, y) - N(f_j, y) | \, dy = 0$$

and

$$(3) \qquad \int_Q \phi(f(x)) Jf(x) dx = \lim_{j\to\infty} \int_Q \phi(f_j(x)) Jf_j(x) dx$$

whenever ϕ is a continuous real valued function on R^n with compact support.

2. *Proof of* (1) *and* (2). Suppose f satisfies the hypothesis of the theorem. By a full set of hyperplanes we will mean a subset P of $\{P_i(r): 1 \leqq i \leqq n \text{ and } 0 < r < 1\}$ such that, for each i, $P_i(r) \in P$ for almost every $r \in I$.

If $\pi \subset Q$ is an n-cube such that $f \mid \partial\pi$ is continuous and $y \in R^n - f(\partial\pi)$, let $0(f, \pi, y)$ denote the topological index of y with respect to the mapping $f \mid \partial\pi$ [4, p. 123]. If $y \in f(\partial\pi)$ let $0(f, \pi, y) = 0$.

Let P be a full set of hyperplanes such that $f \mid P_i(r)$ is continuous whenever $P_i(r) \in P$. In harmony with [1, page 173] let, for $y \in R^n$,

$$N(f, y) = \sup \sum | 0(f, \pi, y) |$$

where the supremum is taken over all finite collections G of non overlapping n-cubes $\pi \subset Q$ whose $n-1$ faces all lie in elements of P. From the properties of the topological index, it is easily seen that $N(f, y)$ is a lower semicontinuous function of y.

If $g: Q \to R^n$ is quasi-linear, then $N(g, y)$ is independent of the choice of P and

$$a(g) = \int_{R^n} N(g, y) dy \, .$$

By [3, 3.5] we know that f possesses a regular approximate differential almost everywhere in Q. Using the arguments of [1, page 424] one verifies that

$$\int_Q | Jf(x) | \, dx \leqq \int_{R^n} N(f, y) dy$$

whenever $N(f, y)$ is computed relative to a full set P of hyperplanes such that the restriction of f to each element of P is continuous.

Suppose $\{f_j\}$ is a sequence of quasi-linear mappings converging $n-1$ to f with $A(f) = \lim_{j\to\infty} a(f_j)$. Let P be a full set of hyperplanes on each of which the sequence converges uniformly to f and define $N(f, y)$ relative to P. For each $y \in R^n$ we have

$$N(f, y) \leqq \lim_{j\to\infty} N(f_j, y)$$

and hence

$$\int_{R^n} N(f, y)dy \leqq \lim_{j\to\infty} \int_{R^n} N(f_j, y)dy = A(f) .$$

If $\bar{P} \subset P$ is a full set of hyperplanes and $\bar{N}(f, y)$ is defined relative to $\bar{P}$, then, clearly $\bar{N}(f, y) \leqq N(f, y)$ for all $y \in R^n$. Since $A(f) = \int |Jf(x)| dx$, it follows that $N(f, y)$ is determined as an element of $L^1(R^n)$ independent of the choice of the sequence $\{f_j\}$. Thus (1) is proved and (2) follows because $N(f, y)$ is integer valued and

$$N(f, y) \leqq \lim_{j\to\infty} N(f_j, y)$$

for almost every $y \in R^n$ whenever $\{f_j\}$ is a sequence of quasilinear mappings converging $n-1$ to f with $A(f) = \lim_{j\to\infty} a(f_j)$.

Proof of (3). Suppose f and $\{f_j\}$ satisfy the conditions of the theorem and let P be a full set of hyperplanes on each of which $\{f_j\}$ converges uniformly to f.

For $y \in R^n$ let

$$N^{\pm}(f, y) = \sup \sum_{\pi \in G} \frac{1}{2}[| 0(f, \pi, y) | \pm 0(f, \pi, y)]$$

where the supremum is taken over all finite collections G of non overlapping n-cubes $\pi \subset Q$ whose $n-1$ faces all lie in elements of P. Clearly

$$N^{\pm}(f, y) \leqq N(f, y) \leqq N^+(f, y) + N^-(f, y) .$$

It is readily seen that

$$N^{\pm}(f, y) \leqq \lim_{j\to\infty} N^{\pm}(f_j, y)$$

and that the $N^{\pm}(f, y)$ are lower semicontinuous functions of y.

In case $g: Q \to R^n$ is quasi-linear, $N^{\pm}(g, y)$ are independent of the choice of P and

$$N(g, y) = N^+(g, y) + N^-(g, y)$$

for almost every $y \in R^n$.

For each positive integer j, let

$$E_j^{\pm} = \{y: N^{\pm}(f_k, y) < N^{\pm}(f, y) \text{ for some } k \geqq j\} .$$

and let $E_j = E_j^+ \cup E_j^-$.
Since the functions $N^{\pm}$ are integer valued we have

$$\lim_{j\to\infty} \mathscr{L}_n(E_j) = 0$$

where $\mathscr{L}_n$ denotes n dimensional Lebesgue measure. Now

$$\begin{aligned}
&\int_{R^n} | N^+(f_j, y) - N^+(f, y) | dy \\
&\leqq \int_{R^n} N^+(f_j, y)dy - \int_{R^n - E_j^+} N^+(f, y)dy + \int_{E_j^+} (f, y)dy \\
&\leqq \int_{R^n} (N^+(f_j, y) + N^-(f_j, y))dy \\
&\quad - \int_{R^n - E_j} (N^+(f, y) + N^-(f, y))dy + \int_{E_j} N^+(f, y)dy \\
&\leqq \int_{R^n} N(f_j, y)dy - \int_{R^n - E_j} N(f, y)dy + \int_{E_j} N(f, y)dy \\
&= a(f_j) - A(f) + 2\int_{E_j} N(f, y)dy .
\end{aligned}$$

Thus

$$\lim_{j\to\infty} \int_{R^n} | N^\pm(f_j, y) - N^\pm(f, y) | dy = 0 .$$

Now

$$\begin{aligned}
0 &\leqq \int_{R^n} [N^+(f, y) + N^-(f, y) - N(f, y)]dy \\
&\leqq \int_{R^n} | N^+(f, y) - N^+(f_j, y) | dy + \int_{R^n} | N^-(f, y) - N^-(f_j, y) | dy \\
&\quad + \int_{R^n} | N(f, y) - N(f_j, y) | dy .
\end{aligned}$$

Thus, $N(f, y) = N^+(f, y) + N^-(f, y)$ for almost every $y \in R^n$.

Let $n(f, y) = N^+(f, y) - N^-(f, y)$. Then

$$\lim_{j\to\infty} \int_{R^n} | n(f, y) - n(f_j, y) | dy = 0 .$$

Suppose ϕ is a real valued continuous function on R^n with compact support. If $g: Q \to R^n$ is quasi-linear (or of class C^1) then

$$\int_Q \phi(g(x))Jg(x)dx = \int_{R^n} \phi(y)n(g, y)dy .$$

Suppose $\{\bar{f}_j\}$ is a sequence of modifiers of f.

Then, from [3, 3.2], the sequence $\{\bar{f}_j\}$ converges $n-1$ to f and

$$\lim_{j\to\infty} \int_Q | Jf(x) - J\bar{f}_j(x) | dx = 0 .$$

Hence

$$\int_Q \phi(f(x))Jf(x)dx = \lim_{j\to\infty}\int_Q \phi(\bar{f}_j(x))J\bar{f}_j(x)dx$$
$$= \lim_{j\to\infty}\int_{R^n} \phi(y)n(\bar{f}_j, y)dy = \int_{R^n} \phi(y)n(f, y)dy .$$

Thus

$$\lim_{j\to\infty}\int_Q \phi(f_j(x))Jf_j(x)dx = \lim_{j\to\infty}\int_{R^n} \phi(y)n(f_j, y)dy$$
$$= \int_{R^n} \phi(y)n(f, y)dy = \int_Q \phi(f(x))Jf(x)dx$$

and (3) is proved.

References

1. L. Cesari, *Surface Area,* Annals of Mathematics Studies No. 35, Princeton University Press, Princeton, N. J. 1956.

2. C. Goffman and F. C. Liu, *Discontinuous mappings and surface area,* Proc. London Math. Soc., **20** (1970), 237-248.

3. C. Goffman and W. Ziemer, *Higher dimensional mappings for which the area formula holds,* Annals of Math., **92** (1970), 482-488.

4. T. Rado and P. V. Reichelderfer, *Continuous Transformations in Analysis,* Springer-Verlag, Berlin, 1955.

Received October 26, 1971.

UNIVERSITY OF KENTUCKY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 44, No. 2 June, 1973