

Pacific Journal of Mathematics

ARCHIMEDEAN EXTENSIONS OF DIRECTED INTERPOLATION GROUPS

ANDREW M. W. GLASS

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P. F. Conrad has obtained some properties of archimedean extensions (α -extensions) of lattice ordered groups (l -groups). In particular, Conrad proved that every abelian l -group has an \mathcal{A} -closure (an abelian α -extension which has no proper abelian α -extension). D. Khuon proved that every l -group has an α -closure (an α -extension which has no proper α -extension). Using a slightly different definition, Conrad and Bleier defined an α^* -extension of an l -group and proved that every abelian l -group has an α^* -closure and every archimedean l -group has a unique α^* -closure. These results have been extended to another class of l -groups by Glass and Holland (unpublished).

The purpose of this paper is to extend the l -group results to the class of directed interpolation groups. The obvious definitions give rise to some negative results; the situation for abelian \mathcal{P} -groups is more propitious and it is proved that any such group has an \mathcal{A} -closure in this class. However, taking less direct definitions of α -extensions and α^* -extensions gives \mathcal{A} -closures and \mathcal{A}^* -closures in restricted classes of abelian directed interpolation groups.

It is assumed that the reader is familiar with [1], [2], [3], [4], and [5]

1. Definitions and notation. Throughout this paper, additive notation will be used for all groups, abelian or not. \subset will denote strict containment and \subseteq will denote strict containment or equality. On will denote the collection of all ordinals.

If A is a p.o. set and $\alpha, \beta \in A$, then $\alpha \parallel \beta$ will stand for $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$.

If G is a group and $X \subseteq G$, $\langle X \rangle$ will denote the subgroup of G generated by X . If G and H are groups $G \oplus H$ will denote the cartesian sum of G and H . If G and H are p.o. groups, then $G \oplus^{\rightarrow} H$, the *lexicographic sum* of G over H , is the group $G \oplus H$ ordered by: $(g, h) > 0$ if and only if $g > 0$ (in G) or $g = 0$ and $h > 0$ (in H). If $\{G_\alpha: \alpha \in A\}$ is a family of p.o. groups, then $\prod\{G_\alpha: \alpha \in A\}(\Sigma\{G_\alpha: \alpha \in A\})$ will denote the cartesian product (sum) of the family of groups $\{G_\alpha: \alpha \in A\}$ ordered by: $g \geq 0$ if $g_\alpha \geq 0$ (in G_α) for all $\alpha \in A$; $\prod^*\{G_\alpha: \alpha \in A\}$ is the same group as above but ordered by: $g > 0$ if and only if $g_\alpha > 0$ (in G_α) for all $\alpha \in A$.

If G is a p.o. group, G^+ will denote the *positive cone* of $G = \{g \in G: g \geq 0\}$ and G^* will denote the *strictly positive cone* of $G = \{g \in G: g > 0\}$. $R(Z)$ will denote the additive o-group of reals (integers) but $R^+(Z^+)$ will denote the strictly positive reals (integers).

Let G be a p.o. group. The partial order is said to be *dense* if and only if for all $g, h \in G$, if $g < h$ there exists $f \in G$ such that $g < f < h$. The partial order satisfies the *interpolation property* if whenever $g, h, f, k \in G$ and $g, h \leq f, k$, there exists $x \in G$ such that $g, h \leq x \leq f, k$. A p.o. group satisfying the interpolation property is called an *interpolation group*. A directed interpolation group G in which $f \vee g$ (or $f \wedge g$) exists only when $f \leq g$ or $g \leq f$ is said to be an *antilattice*. A directed group G such that for all $f, g, h, k \in G$ whenever $g, h < f, k$, there exists $x \in G$ such that $g, h < x < f, k$ is called a *tight Riesz group*. Let $\bar{P} = \{g \in G: g \in G^+ \text{ or } g \text{ is pseudo-positive}\}$ and let the cone of \leq be \bar{P} . If G has no pseudo-identities, then \leq is said to be a *compatible tight Riesz order* for (G, \leq) .

Let A be a partially ordered set. For each $\alpha \in A$. Let R_α be a partially ordered abelian group. Let K be the cartesian product of $\{R_\alpha: \alpha \in A\}$. The set of all $k \in K$ such that $\{\alpha \in A: k_\alpha \neq 0\}$ satisfies the ascending chain condition (in A) forms abelian group which is denoted by $V(A, R_\alpha)$. For each $v \in V(A, R_\alpha)$, let $M(v) = \{\alpha \in A: v_\alpha \neq 0 \text{ and } v_\beta = 0 \text{ for all } \beta \in A \text{ such that } \beta > \alpha\}$. $V(A, R_\alpha)$ is a partially ordered abelian group under the ordering: $v > 0$ if and only if $v_\alpha > 0$ for all $\alpha \in M(v)$.

2. A naïve approach to α -extensions of directed interpolation groups. Let H be a directed interpolation group and let G be a subgroup of H . G is said to be an interpolation subgroup of H if and only if for all $x, y \in G$ and $z, t \in H$, $x, y \leq z, t$ implies there exists $g \in G$ such that $x, y \leq g \leq z, t$. Note that this condition is equivalent to: $x, y \in G$ and $z, t \in H$ and $x, y \geq z, t$ imply there exists $g \in G$ such that $x, y \geq g \geq z, t$.

It should be observed that if H is an l -group and G an interpolation subgroup of H , then G is an l -subgroup of H . However, let $H = V(A, R_\alpha)$ and $G = V(B, R_\beta)$ where $A = \{\bar{1}, \bar{2}, \bar{3}\}$ ordered by: $\bar{1}, \bar{2} > \bar{3}$ and $\bar{1} \parallel \bar{2}$, $B = \{\bar{1}, \bar{2}\}$ and $R_\alpha = R$ for all $\alpha \in A$. H is a directed interpolation group and G is an l -group under the induced ordering. However, G is not an interpolation subgroup of H as $(0, 0, 1) \leq (1, 0, 0), (0, 1, 0)$ but there is no $g \in G$ such that $(0, 0, 1) \leq g \leq (1, 0, 0), (0, 1, 0)$.

$h_1, h_2 \in H^+$ are said to be α -equivalent if and only if there exist $n_1, n_2 \in \mathbb{Z}^+$ such that $h_1 \leq n_2 h_2$ and $h_2 \leq n_1 h_1$. H is an α -extension of G if and only if every $h \in H^+$ is α -equivalent to some $g \in G^+$ and G is

an interpolation subgroup of H . This coincides with the definition given in [2] for l -groups and is a natural extension to directed interpolation groups. Notice that if K is an α -extension of H and H is an α -extension of G , then K is an α -extension of G . Using the method of [2] and the results of [4] and [5] it is easy to see that H is an α -extension of G if and only if there is a $(1:1)$ map of the convex d -subgroups of G onto those of H which preserves inclusion and maps prime subgroups of G (polars of G) to prime subgroups of H (polars of H).

G is α -closed if and only if G has no proper α -extension. In view of [2] and [6], we cannot hope for there to be a unique α -closure of a directed interpolation group but we can try to prove that *any* path of proper α -extensions eventually terminates. The next two theorems shatter this dream—the second gives a path of proper α -extensions of a certain class of abelian directed interpolation groups which cannot be closed up; the first, more dramatically, proves that *no* path of α -extensions of R can ever be closed up.

THEOREM A. *Any α -extension (in the class of directed interpolation groups) of a dense antilattice is a dense antilattice. Consequently, no dense antilattice other than $\{0\}$ has an α -closure and no abelian dense antilattice other than $\{0\}$ has an α -closure in the class of abelian directed interpolation groups.*

Proof. It is easy to see that any α -extension of an antilattice is an antilattice. Suppose G is a dense directed interpolation group and that H is an α -extension of G . We prove that H is dense.

Assume that $0 < h \in H$ and that there is no $k \in H$ such that $0 < k < h$. Since G is dense, $h \notin G$. We first show that, under these hypotheses, some multiple of h belongs to G . Let g be α -equivalent to h and assume that $m \in \mathbb{Z}^+$ is the least such that $mg \geq h$. Then $g, h \geq 0, h - (m - 1)g$. Since H is an interpolation group, there exists $k \in H$ such that $g, h \geq k \geq 0, h - (m - 1)g$. Thus $h \geq k \geq 0$ and so $h = k$ by the hypothesis and the choice of m . Consequently, $g \geq h$. Let $n \in \mathbb{Z}^+$ be least such that $nh \geq g$ and let $p \in \mathbb{Z}^+$ be greatest such that $g \geq ph$. Now $p \leq n$ and, by hypothesis, $p \neq n$. Hence $0, g - (n - 1)h \leq g - gh, h$ and, as before, $h \leq g - ph$, a contradiction. Thus $nh = g$ for some $n \in \mathbb{Z}^+$ and $g \in G^+$. Choose $g \in G^+$ so that n is minimal. Since G is dense, there exists $g' \in G$ such that $0 < g' < g = nh$. Let $p \in \mathbb{Z}^+$ be least such that $g' \leq ph$. Then $p \leq n$ and $g' - (p - 1)h, 0 \leq g', h$. Hence there exists $k \in H$ such that $g' - (p - 1)h, 0 \leq k \leq g', h$. By hypothesis, $k = h$ and so $h \leq g'$. Let $q \in \mathbb{Z}^+$ be greatest such that $qh \leq g'$. Thus $q \leq p$. If $q < p$, then $g' - (p - 1)h, 0 \leq g' - qh, h$ and, as before, $h \leq g' - qh$. It follows that $(q + 1)h \leq g'$, a contradic-

tion. Consequently, $q = p$ and $g' = ph$. By the choice of n , $p = n$. Hence $g' = ph = nh = g$, a contradiction.

Finally suppose K is an α -closure of a dense antilattice $G \neq \{0\}$. Then K is a dense antilattice. Let L be any abelian trivially ordered group. It is immediate that $K \hat{\oplus} L$ is a dense antilattice which is a proper α -extension of K , a contradiction.

COROLLARY A.1. *R has no α -closure in either the class of directed interpolation groups or the class of abelian directed interpolation groups.*

THEOREM B. *Suppose A is a p.o. set and for each $\alpha \in A$, $R_\alpha \neq \{0\}$ is a subgroup of either R or the trivially ordered additive group of reals. If $V(A, R_\alpha)$ is a directed interpolation group, then there exist $\{G_\beta: \beta \in On\}$ such that $G_0 = V(A, R_\alpha)$ and if $\lambda, \mu \in On$ and $\lambda < \mu$, then G_μ is a proper α -extension of G_λ in the class of abelian directed interpolation groups.*

Proof. Let B be a maximal totally ordered subset of A . If α_0 is a minimal element of B and $R_{\alpha_0} \cong \mathbf{Z}$, let $S_\alpha = R_\alpha$ if $\alpha \neq \alpha_0$ and $S_{\alpha_0} = R$. Then $W = V(A, S_\alpha)$ is a directed interpolation group by Teller's conditions (see [8]) and is an α -extension of $V = V(A, R_\alpha)$ so we may assume that if B has a minimal element α_0 , R_{α_0} is a dense α -group or a subgroup of the trivially ordered additive group of reals. Let $\Gamma = A \cup \{\gamma\}$ where $\gamma \notin A$. Γ is a p.o. set under the ordering: $\gamma_1 < \gamma_2$ if and only if $\gamma_1 < \gamma_2$ in A or $\gamma_1 = \gamma$ and $\gamma_2 \in B$. Let $R_\delta = R_\alpha$ if $\delta \in A$ and R_γ be the trivially ordered additive group of reals. Then $U = V(\Gamma, R_\delta)$ is a directed interpolation group (by Teller's conditions) and an α -extension of V . Continuing in this fashion, the theorem is proved.

Even removing pseudo-identities does not help since $R \boxplus^* R$ is an α -extension of $\{(a, a): a \in R\} \cong R$ in the class of directed interpolation groups without pseudo-identities.

3. α -extensions of abelian \mathcal{P} -groups. Using the results of [3] and the methods of [2], the following generalizations of theorems of [2] are obtained:

THEOREM C.1. *If G is an abelian \mathcal{P} -group, then G has an α -closure in the class of abelian \mathcal{P} -groups. If H is any such α -closure of G and Δ is a plenary subset of $C_1(G)$, then there exists an " ℓ "-isomorphism of H into $V = V(\Delta, R_\delta)$.*

THEOREM C.2. *If A is a p.o. set and $R_\alpha = R$ for all $\alpha \in A$, then*

$V(A, R_\alpha)$ and $F(A, R_\alpha)$ are a -closed in the class of abelian \mathcal{P} -groups. Moreover, $F(A, R_\alpha)$ is an a -closure of $\Sigma(A, R_\alpha)$ in this class.

A p.o. group G is said to be archimedean if and only if for all $f, g \in G$, $nf \leq g$ for all $n \in \mathbb{Z}$ implies $f = 0$.

THEOREM D. *If G is an archimedean abelian \mathcal{P} -group, then so is any a -extension of G in the class of abelian \mathcal{P} -groups.*

Proof. Suppose that an abelian \mathcal{P} -group H is an a -extension of the archimedean abelian \mathcal{P} -group G and $nh \leq k$ for all $n \in \mathbb{Z}$ ($h, k \in H$). If $h, k \geq 0$, then there exist $f, g \in G$ which are a -equivalent to h and k respectively. It is easy to see that $ng' \leq f'$ for all $n \in \mathbb{Z}$ where f', g' are fixed multiples of f, g respectively. Hence $g' = 0$ and so $g = 0$. Thus $h = 0$. Since H is directed, it may be assumed that $k \geq 0$. If $h \leq 0$, the proof is the same as above so assume $h \parallel 0$. There exist $h_1, h_2 \in H$ such that $h = h_1 - h_2$ and h_1, h_2 are pseudo-disjoint. There exist $g_1, g_2 \in G$ such that g_i is a -equivalent to h_i ($i = 1, 2$); say $m_i g_i \geq h_i$ and $n_i h_i \geq g_i$ ($i = 1, 2$). Now $ng_1 \leq nn_1 h_1 = nn_1(h + h_2) \leq k + nn_1 m_2 g_2$ for all $n \in \mathbb{Z}$ and there exists $f \in G$ a -equivalent to k ; say, $k \leq pf$, $p \in \mathbb{Z}^+$. Now $n(g_1 - n_1 m_2 g_2) \leq pf$ for all $n \in \mathbb{Z}$ and hence $g_1 = n_1 m_2 g_2$. It follows that $h_1 \leq m_1 g_1 = m_1 n_1 m_2 g_2 \leq m_1 m_2 n_1 n_2 h_2$ which is impossible since h_1 and h_2 are pseudo-disjoint. Thus H is archimedean.

In [2], Conrad proved that every archimedean abelian \mathcal{P} -group (and hence every integrally closed abelian \mathcal{P} -group) is an l -group. Hence we have shown:

COROLLARY D.1. *Every a -extension of an archimedean l -group in the class of abelian \mathcal{P} -groups is an archimedean l -group.*

In [2], the result was proved in the class of all l -groups not only abelian l -groups. Consequently, that result cannot be captured by the above proof. In view of Example 6.4 of [2], a -closures of archimedean abelian \mathcal{P} -groups are not unique.

K. M. van Meter has proved that every archimedean \mathcal{P} -group is an archimedean l -group and has proved that any a -extension of an archimedean \mathcal{P} -group in the class of \mathcal{P} -groups is an archimedean \mathcal{P} -group (see [9]) and so has proved a stronger theorem than Theorem D. The proof given here is more direct and was discovered independently and at the same time.

In view of Theorem D and Theorem 3.1 of [1]:

COROLLARY D.2. *Every archimedean (abelian) \mathcal{P} -group has a unique a^* -closure in the class of (abelian) \mathcal{P} -groups.*

4. Archimedean extensions of compatible tight Riesz groups.

It is easy to see that G is a tight Riesz group if and only if G is a dense antilattice. By Theorem A, any α -extension of a tight Riesz group is a tight Riesz group. We will restrict our attention to the class of directed interpolation groups without pseudo-identities and confine ourselves to those tight Riesz groups compatible with an l -group. Such partially ordered groups will be called *acceptable tight Riesz groups*. If (G, \leq) is an acceptable tight Riesz group, then (G, \leq) will be the l -group whose positive cone is the positive cone of (G, \leq) together with its pseudo-positive elements. Let $T = \{g \in G: g > 0\}$. Then there exists $\{M_\alpha: \alpha \in A\}$ a collection of prime l -subgroups of (G, \leq) such that $T = G^+ \setminus \bigcup \{M_\alpha: \alpha \in A\}$ (see Theorem 2.6 of [7]). Recall that if (H, \leq) is an l -group which is an α -extension of the l -group (G, \leq) , then there is a (1:1) map ϕ of the prime l -subgroups of (G, \leq) onto the prime l -subgroups of (H, \leq) which preserves containment. Let G be an abelian group and $X \subseteq G$. Let \bar{G} be the divisible closure of G and $\bar{X} = \{y \in G: ny \in X \text{ for some } n \in \mathbb{Z}^+\}$. If (G, \leq) is an l -group, then (\bar{G}, \leq) is an l -group where $h \in \bar{G}^+$ if and only if $nh \in G^+$ for some $n \in \mathbb{Z}^+$ (i.e., $\bar{G} = \bar{G}^+$). Then $\bar{T} = \bar{G}^+ \setminus \bigcup \{\bar{M}_\alpha: \alpha \in A\}$ is the strict positive cone of a compatible tight Riesz group (\bar{G}, \leq) . Moreover, (\bar{G}, \leq) is an α -extension of (G, \leq) and there is a (1:1) map of the prime subgroups of (G, \leq) onto those of (\bar{G}, \leq) which preserves containment. Using these facts, we define what is meant by an \mathcal{A} -extension of an abelian acceptable tight Riesz group. In view of the above remarks, we need only concern ourselves with divisible abelian acceptable tight Riesz groups. If (K, \leq) is an abelian acceptable tight Riesz group and (K, \leq) the corresponding l -group, let $K^+ = \{k \in K: k \geq 0\}$ and $T_K = \{k \in K: k > 0\}$.

Let (G, \leq) and (H, \leq) be divisible abelian acceptable tight Riesz groups. (H, \leq) is an \mathcal{A} -extension of (G, \leq) if and only if (H, \leq) is an \mathcal{A} -extension of (G, \leq) and if $T_G = G^+ \setminus \bigcup \{M_\alpha: \alpha \in A\}$ where each M_α is a prime l -subgroup of (G, \leq) , then $T_H = H^+ \setminus \bigcup \{N_\alpha: \alpha \in A\}$ where N_α is the prime l -subgroup of (H, \leq) corresponding to M_α . It follows at once from the facts concerning \mathcal{A} -extensions of l -groups that:

THEOREM E. *Every abelian acceptable tight Riesz group has an \mathcal{A} -closure (in the class of abelian acceptable tight Riesz groups) which is not necessarily unique. Moreover, any \mathcal{A} -extension of an archimedean abelian acceptable tight Riesz group is archimedean.*

The last fact follows because (G, \leq) is archimedean if and only if (G, \leq) is. The same is not true for integrally closed since if (G, \leq) is $\mathbf{R} \boxplus \mathbf{R}$, then $(G, \leq) = \mathbf{R} \boxplus \mathbf{R}$ is integrally closed whereas (G, \leq) is not. It should be observed that if (H, \leq) is an \mathcal{A} -extension of

(G, \leq) where (H, \leq) and (G, \leq) are abelian divisible acceptable tight Riesz groups, then there is a (1:1) map of the convex d -subgroups of (G, \leq) onto the convex d -subgroups of (H, \leq) which preserves containment and for every $h \geq 0$, there exists a $g \in G$ such that $g \geq 0$ and $g \leq nh$ and $h \leq mg$ for some $m, n \in \mathbb{Z}^+$.

An abelian acceptable tight Riesz group (G, \leq) will be called a *good tight Riesz group* if and only if $T_G = G^+ \setminus \bigcup \{M_\alpha: \alpha \in A\}$ where each M_α is a closed prime subgroup of (G, \leq) . As before, we need only consider divisible good tight Riesz groups.

Let (H, \leq) and (G, \leq) be divisible good tight Riesz groups. (H, \leq) is an a^* -extension of (G, \leq) if and only if (H, \leq) is an a^* -extension of (G, \leq) and if $T_G = G^+ \setminus \bigcup \{M_\alpha: \alpha \in A\}$, then $T_H = H^+ \setminus \bigcup \{N_\alpha: \alpha \in A\}$ where N_α is the closed prime l -subgroup of (H, \leq) corresponding to the closed prime l -subgroup M_α of (G, \leq) .

THEOREM F.1. *Every good tight Riesz group has an a^* -closure (in the class of good tight Riesz groups).*

2. *Every \mathcal{A} -extension (in the class of divisible abelian acceptable tight Riesz groups) of a good tight Riesz group (G, \leq) is a good tight Riesz group which is an a^* -extension of (G, \leq) .*

3. *Every archimedean good tight Riesz group has a unique a^* -closure (in the class of good tight Riesz groups).*

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Received October 1, 1971 and in revised form August 29, 1972.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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