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In this paper, using the Bergman kernel function $K_D(z, \bar{z})$, we give necessary and sufficient conditions that a pseudo-conformal mapping $f(z)$ be starlike or convex in some bounded schlicht domain D for which the kernel function $K_D(z, \bar{z})$ becomes infinitely large when the point $z \in D$ approaches the boundary of D in any way. We also consider starlike and convex mappings from the polydisk or unit hypersphere into C^n.

Generalizing the results obtained by M. S. Robertson [10] using the principle of subordination, T. J. Suffridge has established necessary and sufficient conditions that a function be univalent and map the polydisk or

$$D_p = \left\{z: \left[\sum_{j=1}^{n} |z_j|^p\right]^{1/p} < 1,\ p \geqq 1\right\}$$

onto a starlike or convex domain [11].

Similar problems have been considered by T. Matsuno [8] for the hypershere. In this paper we deal with the same problems in terms of the Bergman kernel function $K_D(z, \bar{z})$, and show the results are equivalent to theorems of Suffridge in case of polydisk or hypersphere.

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1. Preliminaries. We consider bounded schlicht domains D in C^n for which the kernel function becomes infinite everywhere on the boundary ∂D, i.e., it is the union of an increasing sequence of strictly pseudo-convex domains

$$D_t = [z: \varphi_t(z) \equiv K_D(z, \bar{z}) - t < 0, z \in D] \tag{1.1}$$

for some number $t > 0$, where $z = (z_1, \cdots, z_n)'$. (See [3]). First we have

LEMMA 1.1. *If D is a bounded domain, the Bergman kernel function $K_D(z, \bar{z})$ is strictly plurisubharmonic and*

$$1/\omega(D) \leqq K_D(z, \bar{z}) \leqq 1/\pi^n(l(z))^{2n}, \tag{1.2}$$

where $l(z) = \min_{\tau \in \partial D} \rho(\tau, z)$, $\rho(\tau, z) = \max_j \{|\tau_j - z_j|, j = 1, \cdots, n\}$ and $\omega(D)$ signifies the euclidean volume of D.

Proof. The minimum value of the integral $||f||_D^2 = \int_D |f(\zeta)|^2 dv_\zeta$ for functions $f(\zeta) \in \mathscr{L}^2(D)$ satisfying the condition $df(z)/d\zeta \cdot u = 1$, where $u = (u_1, \cdots, u_n)'$ is an arbitrary nonzero column vector, is

$$(1.3) \qquad 1/u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial\zeta^*\partial\zeta} u = \int_D \left| \frac{u^* \dfrac{\partial K_D(\zeta, \bar{z})}{\partial\zeta^*}}{u^* \dfrac{\partial^2 K_D(z, \bar{z})}{\partial\zeta^*\partial\zeta} u} \right|^2 dv_\zeta . \qquad \text{(See [1], [2].)}$$

Here we define partial derivatives of a function $g(\zeta, \bar{\tau})$ as

$$(1.4) \qquad \begin{aligned} \partial^2 g(\zeta, \bar{\tau})/\partial\tau^*\partial\zeta &= (\partial/\partial\bar{\tau}_1, \cdots, \partial/\partial\bar{\tau}_n)' \times (\partial/\partial\zeta_1, \cdots, \partial/\partial\zeta_n) \times g(\zeta, \bar{\tau}) \\ &= \begin{pmatrix} \partial^2/\partial\bar{\tau}_1\partial\zeta_1, \cdots, \partial^2/\partial\bar{\tau}_1\partial\zeta_n \\ \cdots\cdots\cdots \\ \partial^2/\partial\bar{\tau}_n\partial\zeta_1, \cdots, \partial^2/\partial\bar{\tau}_n\partial\zeta_n \end{pmatrix} \times g(\zeta, \bar{\tau}) , \end{aligned}$$

and if $g(\zeta)$ is a function of only ζ, we denote $dg(\zeta)/d\zeta = (\partial/\partial\zeta_1, \cdots, \partial/\partial\zeta_n) \times g(\zeta)$, where the sign $\times$ designates the Kronecker product and the sign * denotes the transposed conjugate matrix. (Cf. [7].)

On the other hand, if we put $f(\zeta) = u^*(\zeta - z)/|u|^2$, then

$$\frac{df(z)}{d\zeta} u = u^* u/|u|^2 = 1 ,$$

therefore

$$(1.5) \qquad \begin{aligned} 1/u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial\zeta^*\partial\zeta} u &\leqq \int_D \left| \frac{u^*(\zeta - z)}{|u|^2} \right|^2 dv_\zeta \\ &\leqq \frac{1}{|u|^2} \int_D |\zeta - z|^2 dv_\zeta \leqq \frac{L^2\omega(D)}{|u|^2} , \end{aligned}$$

where $L = \max_{\tau \in \partial D} |\tau - z|$ and $|u| = (\sum_{j=1}^n |u_j|^2)^{1/2}$.
Thus

$$u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial\zeta^*\partial\zeta} u > 0$$

for all $z \in D$, that is, $K_D(z, \bar{z})$ is strictly plurisubharmonic (see [3]). Next it is well known that the minimum value of the integral $||f||_D^2$ under the condition $f(z) = 1$, $z \in D$, becomes $1/K_D(z, \bar{z})$. Then, for the function $f(\zeta) \equiv 1$, we have

$$(1.6) \qquad 1/K_D(z, \bar{z}) = \int_D |K_D(\zeta, \bar{z})/K_D(z, \bar{z})|^2 dv_\zeta \leqq \int_D dv_\zeta = \omega(D) .$$

Also, using the Cauchy integral formula, we obtain

$$
\begin{aligned}
(1.7)\qquad &\left|\left(\frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})}\right)_{\zeta=z}\right| \\
&\leqq \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{|K_D(\zeta, \bar{z})/K_D(z, \bar{z})|}{r_1\cdots r_n}r_1 d\theta_1\cdots r_n d\theta_n\,,
\end{aligned}
$$

where $\zeta_j - z_j = r_j e^{i\theta_j}$, $0 < r_j < l(z)$, $(j = 1, \cdots, n)$. We get therefore by the Schwarz integral inequality

$$
\begin{aligned}
(1.8)\qquad l^{2n}/2^n &\leqq \frac{1}{(2\pi)^n}\int\cdots\int_{\rho(\zeta,z)<l}\left|\frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})}\right| dv_\zeta \\
&\leqq \frac{1}{(2\pi)^n}\left[(\pi l^2)^n\int\cdots\int_{\rho(\zeta,z)<l}\left|\frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})}\right|^2 dv_\zeta\right]^{1/2}.
\end{aligned}
$$

Then

$$
(1.9)\qquad \pi^{n/2}l^n \leqq \left[\int\int_D\left|\frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})}\right|^2 dv_\zeta\right]^{1/2} = (1/K_D(z, \bar{z}))^{1/2}\,,
$$

hence we have (1.2) from (1.6) and (1.9).

2. Convex mappings. We consider the above mentioned domains D and D_t, and suppose that $\partial K_D(z, \bar{z})/\partial z \neq 0$, $z \neq 0$, in D, and $K_D(0, 0) = \min_{z\in D} K_D(z, \bar{z})$ at only $z = 0$. For a holomorphic univalent function $w = f(z)$ of D, let

$$
(2.1)\qquad \varphi_t(z) = \varphi_t(f^{-1}(w)) \equiv \Phi_t(w),\ t > K_D(0, 0)\,,
$$

and let $\Delta = f(D)$, $\Delta_t = f(D_t)$.
Then we have

$$
(2.2)\qquad \Delta_t = [w\colon \Phi_t(w) < 0,\ w \in \Delta]
$$

corresponding to (1.1). On the boundary $\partial D_t\colon \varphi_t(z) = 0$, the total differential of $\varphi_t(z)$ becomes

$$
(2.3)\qquad d\varphi_t = \frac{\partial\varphi_t}{\partial z}dz + dz^*\frac{\partial\varphi_t}{\partial z^*} = 2\mathscr{R}\left[\frac{\partial\varphi_t}{\partial z}dz\right] = 0\,,
$$

where $dz = (dz_1, \cdots, dz_n)'$. Consequently, since $\partial\varphi_t/\partial z^* = \partial K_D(z, \bar{z})/\partial z^*$ is perpendicular to all tangential vectors dz of the boundary ∂D_t at z, $\partial\varphi_t/\partial z^*$ is a normal vector of ∂D_t at z. And we can derive

$$
(2.4)\qquad \mathscr{R}\left[\frac{\partial\Phi_t}{\partial w}dw\right] = \mathscr{R}\left[\frac{\partial\Phi_t}{\partial z}\left(\frac{dz}{dw}\right)\left(\frac{dw}{dz}\right)dz\right] = \mathscr{R}\left[\frac{\partial\varphi_t}{\partial z}dz\right] = 0\,,
$$

hence $\partial\Phi_t/\partial w^*$ is also a normal vector of the boundary $\partial\Delta_t\colon \Phi_t(w) = 0$ at $w = f(z)$. (See [5], [6].)

We can expand $\Phi_t(w + dw)$ into a Taylor series:

$$(2.5)\quad \begin{aligned} \Phi_t(w+dw) &= \Phi_t(w) + 2\mathscr{R}\left[\frac{\partial \Phi_t}{\partial w}dw\right] \\ &\quad + 2\mathscr{R}\left[\frac{\partial^2 \Phi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2 \Phi_t}{\partial w^* \partial w}dw\right] + 0(|dw|^2)\,, \end{aligned}$$

where $dw^2 = (dw_1, \cdots, dw_n)' \times (dw_1, \cdots, dw_n)'$. (See [3], Chap. IX.) Since

$$\mathscr{R}\left[\frac{\partial \Phi_t}{\partial w}dw\right] = 0$$

at $w \in \partial\Delta_t$, it follows that

$$(2.6)\quad \Phi_t(w+dw) = 2\mathscr{R}\left[\frac{\partial^2 \Phi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2 \Phi_t}{\partial w^* \partial w}dw\right] + 0(|dw|^2)\,.$$

If the point $(w + dw)$ lie always the outside of Δ_t for all $w \in \partial\Delta_t$ and tangential vectors dw at w, i.e., $\Phi_t(w + dw) > 0$, then Δ_t is convex. From (2.6), we must have the following condition in order to consist always $\Phi_t(w + dw) > 0$:

$$(2.7)\quad \mathscr{R}\left[\frac{\partial^2 \Phi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2 \Phi_t}{\partial w^* \partial w}dw\right] > 0\,.$$

Now we can calculate as follows by formulas of matrix derivatives described in [7]:

$$(2.8)\quad \begin{aligned} \frac{\partial^2 \Phi_t}{\partial w^2} &= \frac{\partial}{\partial w}\left(\frac{\partial \varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\right)\left(\left(\frac{dw}{dz}\right)^{-1} \times E\right) \\ &= \frac{\partial^2 \varphi_t}{\partial z^2}\left(\left(\frac{dw}{dz}\right)^{-1} \times \left(\frac{dw}{dz}\right)\right)^{-1} - \frac{\partial \varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\frac{d^2 w}{dz^2}\left(\left(\frac{dw}{dz}\right)^{-1} \times \left(\frac{dw}{dz}\right)^{-1}\right), \end{aligned}$$

$$(2.9)\quad \frac{\partial^2 \Phi_t}{\partial w^2}dw^2 = \left\{\frac{\partial^2 \varphi_t}{\partial z^2} - \frac{\partial \varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\frac{d^2 w}{dz^2}\right\}dz^2\,,$$

$$(2.10)\quad dw^*\frac{\partial^2 \Phi_t}{\partial w^* \partial w}dw = dw^*\left\{\left(\frac{dw}{dz}\right)^{-1*}\frac{\partial^2 \varphi_t}{\partial z^* \partial z}\left(\frac{dw}{dz}\right)^{-1}\right\}dw = dz^*\frac{\partial^2 \varphi_t}{\partial z^* \partial z}dz\,.$$

Then, substituting (2.9) and (2.10) into (2.7), we obtain

$$(2.11)\quad \mathscr{R}\left[\left\{\frac{\partial^2 \varphi_t}{\partial z^2} - \frac{\partial \varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\frac{d^2 w}{dz^2}\right\}dz^2 + dz^*\frac{\partial^2 \varphi_t}{\partial z^* \partial z}dz\right] > 0\,.$$

Thus we have the following Lemma.

LEMMA 2.1. *For a fixed value t, a holomorphic univalent function $w = f(z)$ of D have convex image Δ_t of D_t defined by* (1.1) *if and only if at every point z on the boundary ∂D_t*

$$(2.12)\quad \mathscr{R}\left[\alpha^* \frac{\partial^2 K_D(z, \bar{z})}{\partial z^* \partial z}\alpha + \left\{\frac{\partial^2 K_D(z, \bar{z})}{\partial z^2} - \frac{\partial K_D(z, \bar{z})}{\partial z}\left(\frac{df}{dz}\right)^{-1}\frac{d^2 f}{dz^2}\right\}\alpha^2\right] > 0$$

for all unit vectors α *satisfying*

$$\mathscr{R}\left[\frac{\partial K_D(z, \bar{z})}{\partial z}\alpha\right] = 0 .$$

DEFINITION. We define the class $\mathscr{D}$ of bounded schlicht domains D for which the kernel function $K_D(z, \bar{z})$ becomes infinite everywhere on the boundary ∂D, $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$ only at $z = 0$, $\partial K_D(z, \bar{z})/\partial z \neq 0$, $z \neq 0$, in D, and there is the holomorphic mapping $g(z)$ of D into D satisfying $g(0) = 0$, for some one $z^{(1)}$ of two arbitrary points $z^{(1)}, z^{(2)} (\neq 0)$ in D $g(z^{(1)}) = z^{(2)}$, and $K_D(z, \bar{z}) \geqq K_D(g(z), \overline{g(z)})$.

For example, let D be a minimal domain or representative domain with center at the origin which is the image domain of $E = \{\zeta : |\zeta| = (\sum_{j=1}^n |\zeta_j|^2)^{1/2} < 1\}$ under the biholomorphic mapping $z = \varphi(\zeta)$ satisfying $0 = \varphi(0)$. Then $\det(d\varphi(\zeta)/d\zeta) \equiv \text{const.}$ when D is a minimal, domain and $d\varphi(\zeta)/d\zeta \equiv \text{const.}$ when D is a representative domain (see [4], Theorem 3.1). Hence, for any holomorphic mapping $g(z)$ of D into D satisfying $g(0) = 0$, we have $K_D(z, \bar{z}) \geqq K_D(g(z), \overline{g(z)})$ because $K_E(\zeta, \bar{\zeta}) \geqq K_E(\Phi(\zeta), \overline{\Phi(\zeta)})$ under the holomorphic mapping $\Phi(\zeta) \equiv \varphi^{-1}[g(\varphi(\zeta))]$, $\Phi(0) = 0$, of E into E. Also we have $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$ at only the origin. Moreover, for arbitrary points $z^{(1)}, z^{(2)} \in D$, if $|\varphi^{-1}(z^{(2)})| \leqq |\varphi^{-1}(z^{(1)})|$, then

$$g(z) \equiv \varphi\left(\frac{|\varphi^{-1}(z^{(2)})|}{|\varphi^{-1}(z^{(1)})|} U_2 U_1^* \varphi^{-1}(z)\right)$$

is a holomorphic mapping of D into D satisfying $g(0) = 0$ and $g(z^{(1)}) = z^{(2)}$ where

$$\varphi^{-1}(z^{(1)}) = U_1\begin{pmatrix} |\varphi^{-1}(z^{(1)})| \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \varphi^{-1}(z^{(2)}) = U_2\begin{pmatrix} |\varphi^{-1}(z^{(2)})| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and U_1, U_2 are unitary matrices. And we observe

$$\partial K_D(z, \bar{z})/\partial z = \partial K_E(\zeta, \bar{\zeta})/\partial \zeta \cdot (d\varphi(\zeta)/d\zeta)^{-1} \neq 0,\ z \neq 0 ,$$

because

$$\partial K_E(\zeta, \bar{\zeta})/\partial \zeta = (n+1)\zeta^* K_E(\zeta, \bar{\zeta})/(1 - |\zeta|^2) \neq 0,\ \zeta \neq 0 .$$

THEOREM 2.1. *Let D be a bounded schlicht domain of the class $\mathscr{D}$. Suppose $f: D \to C^n$ is holomorphic, $f(0) = 0$, and $\det(df/dz) \neq 0$ for all $z \in D$. Then f is a univalent map of D onto a convex domain if* *and only if*

$$(2.13)\quad \mathscr{R}\left[\alpha^*\frac{\partial^2 K_D(z,\bar z)}{\partial z^*\partial z}\alpha+\left\{\frac{\partial^2 K_D(z,\bar z)}{\partial z^2}-\frac{\partial K_D(z,\bar z)}{\partial z}\left(\frac{df}{dz}\right)^{-1}\frac{d^2 f}{dz^2}\right\}\alpha^2\right]>0$$

for all unit vectors α *satisfying*

$$\mathscr{R}\left[\frac{\partial K_D(z,\bar z)}{\partial z}\alpha\right]=0 .$$

Proof. The Bergman kernel function $K_D(z, \bar z)$ of this domain D becomes infinite on ∂D. Then we define D_t and Δ_t by (1.1) and (2.2) respectively. If $\Delta = f(D)$ is schlicht and convex, then all Δ_t also become convex, i.e., for any $w^{(1)}, w^{(2)} \in \partial\Delta_t$,

$$(2.14)\qquad w^{(0)}=\tau w^{(2)}+(1-\tau)w^{(1)}\in\Delta_t,\quad 0<\tau<1 .$$

In fact, if we put $z^{(1)} = f^{-1}(w^{(1)})$, $z^{(2)} = f^{-1}(w^{(2)})$, then $K_D(z^{(1)}, \overline{z^{(1)}}) = K_D(z^{(2)}, \overline{z^{(2)}}) = t$. Setting

$$(2.15)\qquad F(z)\equiv\tau f(g(z))+(1-\tau)f(z)$$

where $g(z)$ is a holomorphic mapping of D into D satisfying $g(0) = 0$ and $g(z^{(1)}) = z^{(2)}$, we observe that $F(0) = 0$ and $F(z) \prec f(z)$ because the mapping $f: D \to C^n$ is convex. Hence

$$(2.16)\qquad \psi(z)\equiv f^{-1}(F(z))$$

is a holomorphic mapping of D into D, so we have

$$K_D(z^{(1)},\overline{z^{(1)}})\geqq K_D(\psi(z^{(1)}),\overline{\psi(z^{(1)})})=K_D(f^{-1}(w^{(0)}),\overline{f^{-1}(w^{(0)})}) .$$

Consequently $f^{-1}(w^{(0)}) \in D_t$, so $w^{(0)} \in \Delta_t$. Thus, by Lemma 2.1, (2.13) holds for all $z \in D$. Contrary, if (2.13) is realized for all $z \in$ D, every Δ_t is convex. Therefore we can conclude that the mapped domain Δ is convex.

Particularly if D is a unit hypersphere, then

$$K_D\ (z,\bar z)=\frac{n!}{\pi^n(1-|z|^2)^{n+1}} .$$

Thus we have the following result by Theorem 2.1.

THEOREM 2.2. *Let D be the unit hypersphere and let $f: D \to C^n$ be holomorphic, $f(0) = 0$ and* det $(df/dz) \neq 0$ *for all $z \in D$. Then $f(D)$ is convex if and only if*

$$(2.17)\qquad \mathscr{R}\left[|Az|^2+z^*\left(\frac{df}{dz}\right)^{-1}\frac{d^2 f}{dz^2}(Az\times Az)\right]\geqq 0 ,$$

where

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix}, A_j \geqq 0, j = 1, \cdots, n ,$$

and the equality holds only if $Az = 0$.

Proof. We can compute as follows setting $K = K_D(z, \bar{z})$:

$$\partial K/\partial z = (n+1)\frac{z^*}{1 - |z|^2}K , \tag{2.18}$$

$$\partial^2 K/\partial z^2 = (n+1)(n+2)\frac{(z \times z)^*}{(1 - |z|^2)^2}K , \tag{2.19}$$

$$\partial^2 K/\partial z^* \partial z = (n+1)\frac{(1 - |z|^2)E + (n+2)zz^*}{(1 - |z|^2)^2}K . \tag{2.20}$$

Then, from (2.13), we have

$$\begin{aligned} \mathscr{R}\Big[&(n+2)\{|z^*\alpha|^2 + (z^*\alpha)^2\} \\ &+ (1 - |z|^2)\Big\{1 - z^*\Big(\frac{df}{dz}\Big)^{-1}\frac{d^2 f}{dz^2}\alpha^2\Big\}\Big] > 0 . \end{aligned} \tag{2.21}$$

Since

$$|z^*\alpha|^2 + \mathscr{R}(z^*\alpha)^2 = 0$$

from

$$\mathscr{R}\Big[\frac{\partial K}{\partial z}\alpha\Big] = 0, \text{ i.e., } \mathscr{R}[z^*\alpha] = 0 ,$$

we conclude

$$\mathscr{R}\Big[1 - z^*\Big(\frac{df}{dz}\Big)^{-1}\frac{d^2 f}{dz^2}\alpha^2\Big] > 0 . \tag{2.22}$$

Moreover, under the condition $\mathscr{R}[z^*\alpha] = 0$ it becomes that $z^*\alpha = ip (p \geqq 0, i = \sqrt{-1})$, because both α and $-\alpha$ are satisfy (2.22). Therefore we can put $\alpha = i(Az/|Az|)$ when $Az \neq 0$, where

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix}, A_j \geqq 0, (j = 1, \cdots, n) ,$$

are chosen arbitrarily. Thus we obtain (2.17) from (2.22).

REMARK 1. Suffridge's Theorem 5 [11] shows that

$$F = \frac{df}{dz}\left[A^2 z + \left(\frac{df}{dz}\right)^{-1}\frac{d^2 f}{dz^2}(Az \times Az)\right]\Big/2,\ w = \left(\frac{df}{dz}\right)^{-1} F \in \mathscr{P}_2 ,$$

i.e.,

$$\begin{aligned}\mathscr{R}\sum_{j=1}^{n} w_j |z_j|^2/z_j &= \mathscr{R} z^*\left[A^2 z + \left(\frac{df}{dz}\right)^{-1}\frac{d^2 f}{dz^2}(Az\times Az)\right]\Big/2 \\ &= \mathscr{R}\left[|Az|^2 + z^*\left(\frac{df}{dz}\right)^{-1}\frac{d^2 f}{dz^2}(Az \times Az)\right]\Big/2 \geqq 0 ,\end{aligned}$$

is the necessary and sufficient condition for convexity.

Next, if D is the polydisk $\{z \in C^n : |z_j| < 1, j = 1, \cdots, n\}$, the kernel function $K_D(z, \bar{z})$ becomes $1/\pi^n(1 - |z_1|^2)^2 \cdots (1 - |z_n|^2)^2$. Hence

$$\partial K/\partial z = 2K \cdot z^* Z , \tag{2.23}$$

$$\begin{aligned}\partial^2 K/\partial z^2 &= 4K\cdot(z\times z)^*(Z \times Z) \\ &\quad + 2K\cdot(z \times z)^*(Z \times Z)\begin{pmatrix} \boxed{\begin{matrix}1_0 & & \\ & \ddots & \\ & & 0\end{matrix}} & & & 0 \\ & \boxed{\begin{matrix}0_{1_0} & & \\ & \ddots & \\ & & 0\end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix}0 & & \\ & \ddots & \\ & & 0_1\end{matrix}} \end{pmatrix} ,\end{aligned} \tag{2.24}$$

$$\partial^2 K/\partial z^*\partial z = 4K\cdot Zzz^*Z + 2K\cdot Z^2 , \tag{2.25}$$

where

$$Z = \begin{pmatrix} 1/(1 - |z_1|^2) & & 0 \\ & \ddots & \\ 0 & & 1/(1 - |z_n|^2) \end{pmatrix} .$$

Substituting formally (2.23), (2.24), and (2.25) into (2.13) and setting

$$\mathscr{R}(z^* Z\alpha)^2 + |z^* Z\alpha|^2 = 0 \text{ and } \alpha = i\frac{Z^{-1/2}Az}{|Z^{-1/2}Az|}$$

where

$$Z^{-1/2} = \begin{pmatrix} \sqrt{1 - |z_1|^2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{1 - |z_n|^2} \end{pmatrix} ,$$

in place of the condition

$$\mathscr{R}\left[\frac{\partial K_D(z, \bar{z})}{\partial z}\alpha\right] = 2K \cdot \mathscr{R}[z^* Z\alpha] = 0 ,$$

we arrive at

$$\mathscr{R}\left[|Az|^2 + z^* Z\left(\frac{df}{dz}\right)^{-1} \frac{d^2 f}{dz^2}(Z \times Z)^{-1/2}(Az \times Az)\right] \geqq 0 , \tag{2.26}$$

where the equality holds only if $Az = 0$.

THEOREM 2.3. *Let D be the polydisk and let $f\colon D \to C^n$ be holomorphic, $f(0) = 0$ and $\det(df/dz) \neq 0$ for all $z \in D$. Then f is a univalent map of D onto a convex domain if and only if the condition (2.26) is fulfilled.*

Proof. If f is a convex mapping, then by Suffridge's Theorem 3 [11] $f = T(\varphi_1(z_1), \cdots, \varphi_n(z_n))'$ where T is a nonsingular linear transformation and each $\varphi_j(z_j)$ is a univalent mapping from the unit disk in the plane onto convex domain in the plane. Then we have

$$\left(\frac{df}{dz}\right)^{-1} \frac{d^2 f}{dz^2} = \begin{pmatrix} \varphi_1''(z_1)/\varphi_1'(z_1) 0 \cdots 0 & & & 0 \\ & 0 \varphi_2''(z_2)/\varphi_2'(z_2) 0 \cdots 0 & & \\ & & \ddots & \\ 0 & & & 0 \cdots 0\, \varphi_n''(z_n)/\varphi_n'(z_n) \end{pmatrix} . \tag{2.27}$$

Substituting this into the left side of (2.26), we get

$$\mathscr{R}\left[\sum_{j=1}^{n} A_j^2 |z_j|^2 \{1 + z_j \varphi_j''(z_j)/\varphi_j'(z_j)\}\right] . \tag{2.28}$$

Hence from the hypothesis $\mathscr{R}[1 + z_j\varphi_j''(z_j)/\varphi_j'(z_j)] > 0, j = 1, \cdots, n$, we get the inequality (2.26).

We will prove the converse. Fix $k, 1 \leqq k \leqq n$ and choose $A_k = 1, A_h = 0, h \neq k, 1 \leqq h \leqq n$. From (2.26)

$$\mathscr{R}\left[|z_k|^2 + \frac{z_k^2(1 - |z_k|^2)}{\det J} \sum_{j=1}^{n} \frac{\bar{z}_j}{1 - |z_j|^2} C_j^{k^2}\right] \geqq 0 , \tag{2.29}$$

where $J = df/dz$ and $G_j^{k^2}$ is obtained from $\det J$ by replacing the jth column by the column $\partial^2 f/\partial z_k^2 = (\partial^2 f_1/\partial z_k^1, \cdots, \partial^2 f_n/\partial z_k^2)'$. For $l, 1 \leqq l \leqq n, l \neq k$, setting $|z_j| < 1/2, j \neq l, 1 \leqq j \leqq n$, $(1 - |z_k|^2)/(1 - |z_l|^2)$ tends to infinity when $|z_l| \to 1$. Then we must have always

$$\mathscr{R}\left[\frac{1}{\det J}\frac{z_k^2}{z_l}G_l^{k^2}\right] \geqq 0 \tag{2.30}$$

from the condition (2.29). Here, since it becomes 0 at $z_k = 0$, we see that $G_l^{k^2} \equiv 0$ for each l, $l \neq k$, $1 \leqq l \leqq n$. Next, if we set $A_k = A_l = 1$, $A_m = 0$, $m \neq k, l$, then (2.26) becomes as follows from the above results:

$$\begin{aligned}\mathscr{R}\Bigg[&|z_k|^2 + |z_l|^2 + \frac{|z_k|^2 z_k G_k^{k2}}{\det J} + \frac{|z_l|^2 z_l G_l^{l2}}{\det J} \\ &+ 2\frac{z_k z_l \sqrt{(1-|z_k|^2)(1-|z_l|^2)}}{\det J}\sum_{j=1}^{n}\frac{\bar{z}_j G_j^{kl}}{(1-|z_j|^2)}\Bigg] \geqq 0 .\end{aligned} \tag{2.31}$$

For s, $1 \leqq s \leqq n$, setting

$$|z_h| < 1/2,\ h \neq s,\ 1 \leqq h \leqq n,\ \frac{\sqrt{(1-|z_k|^2)(1-|z_l|^2)}}{1-|z_s|^2}$$

tends to infinity when $|z_s| \to 1$. Then we must have always

$$\mathscr{R}\left[\frac{1}{\det J}\frac{z_k z_l}{z_s}G_s^{kl}\right] \geqq 0 . \tag{2.32}$$

Since it attains to the minimum value 0 at $z_k z_l = 0$, we must have $G_s^{kl} \equiv 0$ for each s. Thus we arrive at the conditions of the Theorem 3 of Suffridge following his methods. So we can conclude that f is a convex mapping.

3. Starlike mappings.

We now consider univalent functions of D which map D onto a starlike domain with respect to 0. First we set up the definition of starlikeness following Suffridge:

DEFINITION. A holomorphic mapping $f: D \to C^n$ is starlike if f is univalent, $f(0) = 0$ and $(1-\tau)f \prec f$ for all $\tau \in I = [0, 1]$.

THEOREM 3.1. *Let D be a bounded schlicht domain for which the kernel function $K_D(z, \bar{z})$ becomes infinite everywhere on the boundary, $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$ at only the origin, and $K_D(z, \bar{z}) \geqq K_D(g(z), \overline{g(z)})$ for any holomorphic mapping $g(z)$ of D into D satisfying $g(0) = 0$. Suppose $f: D \to C^n$ is holomorphic, $f(0) = 0$ and $\det(df/dz) \neq 0$ for all $z \in D$. Then f is starlike if and only if*

$$\mathscr{R}\left[\frac{\partial K_D(z, \bar{z})}{\partial z}\left(\frac{df}{dz}\right)^{-1} f\right] > 0 \tag{3.1}$$

for all $z \in D$, $z \neq 0$.

REMARK 2. Domains which belong to the above mentioned class $\mathscr{D}$ satisfy the conditions of this Theorem.

Proof. If f is starlike, then all image Δ_t are starlike, that is, for all $w^{(1)} \in \partial\Delta_t$ we have $w^{(0)} = (1-\tau)w^{(1)} \in \Delta_t, \tau \in I$. In fact, if we set $z^{(1)} = f^{-1}(w^{(1)})$, $K_D(z^{(1)}, \overline{z^{(1)}}) = t$ and $\psi(z) \equiv f^{-1}((1-\tau)f(z))$, then we obtain

$$K_D(z^{(1)}, \overline{z^{(1)}}) \geqq K_D(\psi(z^{(1)}), \overline{\psi(z^{(1)})}) = K_D(f^{-1}(w^{(0)}), \overline{f^{-1}(w^{(0)})}) \,, \tag{3.2}$$

because $\psi(z)$ is a mapping of D into D and $\psi(0) = 0$. Then it holds that $f^{-1}(w^{(0)}) \in D_t$ which yields $w^{(0)} \in \Delta_t$. Now, since

$$\Phi_t\left(w + \varepsilon\frac{\partial\Phi_t}{\partial w^*}\right) = 2\varepsilon\left|\frac{\partial\Phi_t}{\partial w^*}\right|^2 + 0(\varepsilon^2) > 0$$

when $\varepsilon > 0$ is sufficiently small and $w \in \partial\Delta_t$, $N_w \equiv \partial\Phi_t/\partial w^*$ is the outward normal vector at the boundary point $w \in \partial\Delta_t$. Hence $(1-\tau)w \in \Delta_t (w \in \partial\Delta_t, 0 < \tau \leqq 1)$ implies

$$\cos(-N_w, -w) = \mathscr{R}\left[\frac{\partial\Phi_t}{\partial w}w\right]\Big/\left|\frac{\partial\Phi_t}{\partial w^*}\right||w| > 0 \tag{3.3}$$

which yields (3.1) by virtue of

$$\frac{\partial\Phi_t}{\partial w}w = \frac{\partial K}{\partial z}\left(\frac{df}{dz}\right)^{-1}f(z) \,.$$

Conversely, if (3.1) holds, then we conclude $(1-\tau)w \in \Delta_t$, $w \in \partial\Delta_t$, $0 < \tau < \varepsilon(<1)$ for some $\varepsilon > 0$ by (3.3). Moreover, we can conclude $(1-\tau)w \in \Delta_t$, $w \in \partial\Delta_t$, $0 < \tau \leqq 1$, because, if $(1-\tau_1)w \equiv w^{(1)} \in \partial\Delta_t$ and $(1-\tau)w \in \Delta_t$, $0 < \tau < \tau_1$ for some $\tau_1 < 1$, then $(1-\tau)w^{(1)} \notin \Delta_t$, $w^{(1)} \in \partial\Delta_t$ which is a contradiction. Then the image domain Δ of D becomes starlike.

COROLLARY 3.1. *Let D be the unit hypersphere, and let $f: D \to C^n$ be holomorphic, $f(0) = 0$ and* $\det(df/dz) \neq 0$ *for all $z \in D$. Then $f(z)$ is starlike if and only if*

$$\mathscr{R}\left[z^*\left(\frac{df}{dz}\right)^{-1}f\right] > 0 \tag{3.4}$$

for all $z \in D$, $z \neq 0$.

Proof. Substituting (2.18) into (3.1), we obtain the required result.

REMARK 3. The conditions of Suffridge's Theorem 4 [11]: $f = Jw$, $w \in \mathscr{P}_2$ are the same as (3.4).

REFERENCES

1. S. Bergman, *The Kernel Function and Conformal Mapping*, Mathematical Surveys, Vol. V., Amer. Math. Soc., New York, (1950).
2. B. A. Fuks, *Special chapters of the theory of analytic functions of several complex variables*, Moscow (1963), English Translation AMS (1966).
3. R. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N. J, (1965).
4. Kyong T. Hahn, *Some properties of relative invariants on bounded domains*, Duke Math. J., **34** (1967).
5. S. Higuchi, *On the distribution theorem of holomorphic mappings in several complex variables*, Sci. Rep. Tokyo Kyoiku D., A, Vol. **8** (1963).
6. K. Kikuchi, *On starlike and convex-like domains of pseudo-conformal mappings in several complex variables*, Math. Rep. Tokyo Kyoiku D., **1** (1964), 15-17.
7. ———, *Various m-representative domains in several complex variables*, Pacific J. Math., **33**, No. 3 (1970).
8. T. Matsuno, *Star-like theorems and convex-like theorems in the complex vector space*, Sci. Rep. Tokyo Kyoiku D., Sect. A, **5** (1955).
9. S. Ozaki, I. Ono and T. Umezawa, *General minimum problems and representative domains*, Sci. Rep. Tokyo Kyoiku D., Sect. A, **5** (1955).
10. M. S. Robertson, *Applications of the subordination principle to univalent functions*, Pacific J. Math., **11** (1961), 315-324.
11. T. J. Suffridge, *The principle of subordination applied to functions of several variables*, Pacific J. Math., **33** (1970).

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