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**STARLIKE AND CONVEX MAPPINGS IN SEVERAL COMPLEX  
VARIABLES**

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# STARLIKE AND CONVEX MAPPINGS IN SEVERAL COMPLEX VARIABLES

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**In this paper, using the Bergman kernel function  $K_D(z, \bar{z})$ , we give necessary and sufficient conditions that a pseudo-conformal mapping  $f(z)$  be starlike or convex in some bounded schlicht domain  $D$  for which the kernel function  $K_D(z, \bar{z})$  becomes infinitely large when the point  $z \in D$  approaches the boundary of  $D$  in any way. We also consider starlike and convex mappings from the polydisk or unit hypersphere into  $C^n$ .**

Generalizing the results obtained by M. S. Robertson [10] using the principle of subordination, T. J. Suffridge has established necessary and sufficient conditions that a function be univalent and map the polydisk or

$$D_p = \left\{ z: \left[ \sum_{j=1}^n |z_j|^p \right]^{1/p} < 1, p \geq 1 \right\}$$

onto a starlike or convex domain [11].

Similar problems have been considered by T. Matsuno [8] for the hypersphere. In this paper we deal with the same problems in terms of the Bergman kernel function  $K_D(z, \bar{z})$ , and show the results are equivalent to theorems of Suffridge in case of polydisk or hypersphere.

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**1. Preliminaries.** We consider bounded schlicht domains  $D$  in  $C^n$  for which the kernel function becomes infinite everywhere on the boundary  $\partial D$ , i.e., it is the union of an increasing sequence of strictly pseudo-convex domains

$$(1.1) \quad D_t = \{z: \varphi_t(z) \equiv K_D(z, \bar{z}) - t < 0, z \in D\}$$

for some number  $t > 0$ , where  $z = (z_1, \dots, z_n)'$ . (See [3]). First we have

**LEMMA 1.1.** *If  $D$  is a bounded domain, the Bergman kernel function  $K_D(z, \bar{z})$  is strictly plurisubharmonic and*

$$(1.2) \quad 1/\omega(D) \leq K_D(z, \bar{z}) \leq 1/\pi^n (l(z))^{2n},$$

where  $l(z) = \min_{\tau \in \partial D} \rho(\tau, z)$ ,  $\rho(\tau, z) = \max_j \{|\tau_j - z_j|, j = 1, \dots, n\}$  and  $\omega(D)$  signifies the euclidean volume of  $D$ .

*Proof.* The minimum value of the integral  $\|f\|_D^2 = \int_D |f(\zeta)|^2 dv_\zeta$  for functions  $f(\zeta) \in \mathcal{L}^2(D)$  satisfying the condition  $df(z)/d\zeta \cdot u = 1$ , where  $u = (u_1, \dots, u_n)'$  is an arbitrary nonzero column vector, is

$$(1.3) \quad 1/u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \bar{\zeta}} u = \int_D \left| \frac{u^* \frac{\partial K_D(\zeta, \bar{z})}{\partial \zeta^*}}{u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \bar{\zeta}} u} \right|^2 dv_\zeta. \quad (\text{See [1], [2].})$$

Here we define partial derivatives of a function  $g(\zeta, \bar{\tau})$  as

$$(1.4) \quad \begin{aligned} \partial^2 g(\zeta, \bar{\tau}) / \partial \tau^* \partial \bar{\zeta} &= (\partial / \partial \bar{\tau}_1, \dots, \partial / \partial \bar{\tau}_n)' \times (\partial / \partial \zeta_1, \dots, \partial / \partial \zeta_n) \times g(\zeta, \bar{\tau}) \\ &= \begin{pmatrix} \partial^2 / \partial \bar{\tau}_1 \partial \zeta_1, & \dots, & \partial^2 / \partial \bar{\tau}_1 \partial \zeta_n \\ \dots & \dots & \dots \\ \partial^2 / \partial \bar{\tau}_n \partial \zeta_1, & \dots, & \partial^2 / \partial \bar{\tau}_n \partial \zeta_n \end{pmatrix} \times g(\zeta, \bar{\tau}), \end{aligned}$$

and if  $g(\zeta)$  is a function of only  $\zeta$ , we denote  $dg(\zeta)/d\zeta = (\partial / \partial \zeta_1, \dots, \partial / \partial \zeta_n) \times g(\zeta)$ , where the sign  $\times$  designates the Kronecker product and the sign  $*$  denotes the transposed conjugate matrix. (Cf. [7].)

On the other hand, if we put  $f(\zeta) = u^*(\zeta - z)/|u|^2$ , then

$$\frac{df(z)}{d\zeta} u = u^* u / |u|^2 = 1,$$

therefore

$$(1.5) \quad \begin{aligned} 1/u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \bar{\zeta}} u &\leq \int_D \left| \frac{u^*(\zeta - z)}{|u|^2} \right|^2 dv_\zeta \\ &\leq \frac{1}{|u|^2} \int_D |\zeta - z|^2 dv_\zeta \leq \frac{L^2 \omega(D)}{|u|^2}, \end{aligned}$$

where  $L = \max_{\tau \in \partial D} |\tau - z|$  and  $|u| = (\sum_{j=1}^n |u_j|^2)^{1/2}$ .

Thus

$$u^* \frac{\partial^2 K_D(z, \bar{z})}{\partial \zeta^* \partial \bar{\zeta}} u > 0$$

for all  $z \in D$ , that is,  $K_D(z, \bar{z})$  is strictly plurisubharmonic (see [3]). Next it is well known that the minimum value of the integral  $\|f\|_D^2$  under the condition  $f(z) = 1, z \in D$ , becomes  $1/K_D(z, \bar{z})$ . Then, for the function  $f(\zeta) \equiv 1$ , we have

$$(1.6) \quad 1/K_D(z, \bar{z}) = \int_D |K_D(\zeta, \bar{z})/K_D(z, \bar{z})|^2 dv_\zeta \leq \int_D dv_\zeta = \omega(D).$$

Also, using the Cauchy integral formula, we obtain

$$(1.7) \quad \left| \left( \frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})} \right)_{\zeta=z} \right| \leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} \frac{|K_D(\zeta, \bar{z})/K_D(z, \bar{z})|}{r_1 \dots r_n} r_1 d\theta_1 \dots r_n d\theta_n,$$

where  $\zeta_j - z_j = r_j e^{i\theta_j}$ ,  $0 < r_j < l(z)$ ,  $(j = 1, \dots, n)$ . We get therefore by the Schwarz integral inequality

$$(1.8) \quad \begin{aligned} l^{2n}/2^n &\leq \frac{1}{(2\pi)^n} \int_{\rho(\zeta, z) < l} \dots \int \left| \frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})} \right| dv_\zeta \\ &\leq \frac{1}{(2\pi)^n} \left[ (\pi l^2)^n \int_{\rho(\zeta, z) < l} \dots \int \left| \frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})} \right|^2 dv_\zeta \right]^{1/2}. \end{aligned}$$

Then

$$(1.9) \quad \pi^{n/2} l^n \leq \left[ \int_D \left| \frac{K_D(\zeta, \bar{z})}{K_D(z, \bar{z})} \right|^2 dv_\zeta \right]^{1/2} = (1/K_D(z, \bar{z}))^{1/2},$$

hence we have (1.2) from (1.6) and (1.9).

**2. Convex mappings.** We consider the above mentioned domains  $D$  and  $D_t$ , and suppose that  $\partial K_D(z, \bar{z})/\partial z \neq 0$ ,  $z \neq 0$ , in  $D$ , and  $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$  at only  $z = 0$ . For a holomorphic univalent function  $w = f(z)$  of  $D$ , let

$$(2.1) \quad \varphi_t(z) = \varphi_t(f^{-1}(w)) \equiv \Phi_t(w), \quad t > K_D(0, 0),$$

and let  $\Delta = f(D)$ ,  $\Delta_t = f(D_t)$ .

Then we have

$$(2.2) \quad \Delta_t = \{w: \Phi_t(w) < 0, w \in \Delta\}$$

corresponding to (1.1). On the boundary  $\partial D_t: \varphi_t(z) = 0$ , the total differential of  $\varphi_t(z)$  becomes

$$(2.3) \quad d\varphi_t = \frac{\partial \varphi_t}{\partial z} dz + dz^* \frac{\partial \varphi_t}{\partial z^*} = 2\mathcal{R} \left[ \frac{\partial \varphi_t}{\partial z} dz \right] = 0,$$

where  $dz = (dz_1, \dots, dz_n)'$ . Consequently, since  $\partial \varphi_t/\partial z^* = \partial K_D(z, \bar{z})/\partial z^*$  is perpendicular to all tangential vectors  $dz$  of the boundary  $\partial D_t$  at  $z$ ,  $\partial \varphi_t/\partial z^*$  is a normal vector of  $\partial D_t$  at  $z$ . And we can derive

$$(2.4) \quad \mathcal{R} \left[ \frac{\partial \Phi_t}{\partial w} dw \right] = \mathcal{R} \left[ \frac{\partial \Phi_t}{\partial z} \left( \frac{dz}{dw} \right) \left( \frac{dw}{dz} \right) dz \right] = \mathcal{R} \left[ \frac{\partial \varphi_t}{\partial z} dz \right] = 0,$$

hence  $\partial \Phi_t/\partial w^*$  is also a normal vector of the boundary  $\partial \Delta_t: \Phi_t(w) = 0$  at  $w = f(z)$ . (See [5], [6].)

We can expand  $\Phi_t(w + dw)$  into a Taylor series:

$$\begin{aligned}
 \Phi_t(w + dw) &= \Phi_t(w) + 2\mathcal{R}\left[\frac{\partial\Phi_t}{\partial w}dw\right] \\
 (2.5) \quad &+ 2\mathcal{R}\left[\frac{\partial^2\Phi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2\Phi_t}{\partial w^*\partial w}dw\right] + 0(|dw|^2),
 \end{aligned}$$

where  $dw^2 = (dw_1, \dots, dw_n)' \times (dw_1, \dots, dw_n)'$ . (See [3], Chap. IX.)  
 Since

$$\mathcal{R}\left[\frac{\partial\Phi_t}{\partial w}dw\right] = 0$$

at  $w \in \partial\Delta_t$ , it follows that

$$(2.6) \quad \Phi_t(w + dw) = 2\mathcal{R}\left[\frac{\partial^2\Phi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2\Phi_t}{\partial w^*\partial w}dw\right] + 0(|dw|^2).$$

If the point  $(w + dw)$  lie always the outside of  $\Delta_t$  for all  $w \in \partial\Delta_t$  and tangential vectors  $dw$  at  $w$ , i.e.,  $\Phi_t(w + dw) > 0$ , then  $\Delta_t$  is convex. From (2.6), we must have the following condition in order to consist always  $\Phi_t(w + dw) > 0$ :

$$(2.7) \quad \mathcal{R}\left[\frac{\partial^2\Phi_t}{\partial w^2}dw^2 + dw^*\frac{\partial^2\Phi_t}{\partial w^*\partial w}dw\right] > 0.$$

Now we can calculate as follows by formulas of matrix derivatives described in [7]:

$$\begin{aligned}
 \frac{\partial^2\Phi_t}{\partial w^2} &= \frac{\partial}{\partial w}\left(\frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\right) = \frac{\partial}{\partial z}\left(\frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\right)\left(\left(\frac{dw}{dz}\right)^{-1} \times E\right) \\
 (2.8) \quad &= \frac{\partial^2\varphi_t}{\partial z^2}\left(\left(\frac{dw}{dz}\right)^{-1} \times \left(\frac{dw}{dz}\right)\right)^{-1} - \frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\frac{d^2w}{dz^2}\left(\left(\frac{dw}{dz}\right)^{-1} \times \left(\frac{dw}{dz}\right)^{-1}\right),
 \end{aligned}$$

$$(2.9) \quad \frac{\partial^2\Phi_t}{\partial w^2}dw^2 = \left\{\frac{\partial^2\varphi_t}{\partial z^2} - \frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\frac{d^2w}{dz^2}\right\}dz^2,$$

$$(2.10) \quad dw^*\frac{\partial^2\Phi_t}{\partial w^*\partial w}dw = dw^*\left\{\left(\frac{dw}{dz}\right)^{-1} * \frac{\partial^2\varphi_t}{\partial z^*\partial z}\left(\frac{dw}{dz}\right)^{-1}\right\}dw = dz^*\frac{\partial^2\varphi_t}{\partial z^*\partial z}dz.$$

Then, substituting (2.9) and (2.10) into (2.7), we obtain

$$(2.11) \quad \mathcal{R}\left[\left\{\frac{\partial^2\varphi_t}{\partial z^2} - \frac{\partial\varphi_t}{\partial z}\left(\frac{dw}{dz}\right)^{-1}\frac{d^2w}{dz^2}\right\}dz^2 + dz^*\frac{\partial^2\varphi_t}{\partial z^*\partial z}dz\right] > 0.$$

Thus we have the following Lemma.

**LEMMA 2.1.** *For a fixed value  $t$ , a holomorphic univalent function  $w = f(z)$  of  $D$  have convex image  $\Delta_t$  of  $D_t$  defined by (1.1) if and only if at every point  $z$  on the boundary  $\partial D_t$*

$$(2.12) \quad \mathcal{R} \left[ \alpha^* \frac{\partial^2 K_D(z, \bar{z})}{\partial z^* \partial z} \alpha + \left\{ \frac{\partial^2 K_D(z, \bar{z})}{\partial z^2} - \frac{\partial K_D(z, \bar{z})}{\partial z} \left( \frac{df}{dz} \right)^{-1} \frac{d^2 f}{dz^2} \right\} \alpha^2 \right] > 0$$

for all unit vectors  $\alpha$  satisfying

$$\mathcal{R} \left[ \frac{\partial K_D(z, \bar{z})}{\partial z} \alpha \right] = 0.$$

**DEFINITION.** We define the class  $\mathcal{D}$  of bounded schlicht domains  $D$  for which the kernel function  $K_D(z, \bar{z})$  becomes infinite everywhere on the boundary  $\partial D$ ,  $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$  only at  $z = 0$ ,  $\partial K_D(z, \bar{z})/\partial z \neq 0$ ,  $z \neq 0$ , in  $D$ , and there is the holomorphic mapping  $g(z)$  of  $D$  into  $D$  satisfying  $g(0) = 0$ , for some one  $z^{(1)}$  of two arbitrary points  $z^{(1)}, z^{(2)} (\neq 0)$  in  $D$   $g(z^{(1)}) = z^{(2)}$ , and  $K_D(z, \bar{z}) \geq K_D(g(z), \overline{g(z)})$ .

For example, let  $D$  be a minimal domain or representative domain with center at the origin which is the image domain of  $E = \{\zeta: |\zeta| = (\sum_{j=1}^n |\zeta_j|^2)^{1/2} < 1\}$  under the biholomorphic mapping  $z = \varphi(\zeta)$  satisfying  $0 = \varphi(0)$ . Then  $\det(d\varphi(\zeta)/d\zeta) \equiv \text{const.}$  when  $D$  is a minimal, domain and  $d\varphi(\zeta)/d\zeta \equiv \text{const.}$  when  $D$  is a representative domain (see [4], Theorem 3.1). Hence, for any holomorphic mapping  $g(z)$  of  $D$  into  $D$  satisfying  $g(0) = 0$ , we have  $K_D(z, \bar{z}) \geq K_D(g(z), \overline{g(z)})$  because  $K_E(\zeta, \bar{\zeta}) \geq K_E(\Phi(\zeta), \overline{\Phi(\zeta)})$  under the holomorphic mapping  $\Phi(\zeta) \equiv \varphi^{-1}[g(\varphi(\zeta))]$ ,  $\Phi(0) = 0$ , of  $E$  into  $E$ . Also we have  $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$  at only the origin. Moreover, for arbitrary points  $z^{(1)}, z^{(2)} \in D$ , if  $|\varphi^{-1}(z^{(2)})| \leq |\varphi^{-1}(z^{(1)})|$ , then

$$g(z) \equiv \varphi \left( \frac{|\varphi^{-1}(z^{(2)})|}{|\varphi^{-1}(z^{(1)})|} U_2 U_1^* \varphi^{-1}(z) \right)$$

is a holomorphic mapping of  $D$  into  $D$  satisfying  $g(0) = 0$  and  $g(z^{(1)}) = z^{(2)}$  where

$$\varphi^{-1}(z^{(1)}) = U_1 \begin{pmatrix} |\varphi^{-1}(z^{(1)})| \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \varphi^{-1}(z^{(2)}) = U_2 \begin{pmatrix} |\varphi^{-1}(z^{(2)})| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and  $U_1, U_2$  are unitary matrices. And we observe

$$\partial K_D(z, \bar{z})/\partial z = \partial K_E(\zeta, \bar{\zeta})/\partial \zeta \cdot (d\varphi(\zeta)/d\zeta)^{-1} \neq 0, z \neq 0,$$

because

$$\partial K_E(\zeta, \bar{\zeta})/\partial \zeta = (n+1)\zeta^* K_E(\zeta, \bar{\zeta})/(1-|\zeta|^2) \neq 0, \zeta \neq 0.$$

**THEOREM 2.1.** Let  $D$  be a bounded schlicht domain of the class  $\mathcal{D}$ . Suppose  $f: D \rightarrow C^n$  is holomorphic,  $f(0) = 0$ , and  $\det(df/dz) \neq 0$  for all  $z \in D$ . Then  $f$  is a univalent map of  $D$  onto a convex domain if

and only if

$$(2.13) \quad \Re \left[ \alpha^* \frac{\partial^2 K_D(z, \bar{z})}{\partial z^* \partial z} \alpha + \left\{ \frac{\partial^2 K_D(z, \bar{z})}{\partial z^2} - \frac{\partial K_D(z, \bar{z})}{\partial z} \left( \frac{df}{dz} \right)^{-1} \frac{d^2 f}{dz^2} \right\} \alpha^2 \right] > 0$$

for all unit vectors  $\alpha$  satisfying

$$\Re \left[ \frac{\partial K_D(z, \bar{z})}{\partial z} \alpha \right] = 0.$$

*Proof.* The Bergman kernel function  $K_D(z, \bar{z})$  of this domain  $D$  becomes infinite on  $\partial D$ . Then we define  $D_t$  and  $\Delta_t$  by (1.1) and (2.2) respectively. If  $\Delta = f(D)$  is schlicht and convex, then all  $\Delta_t$  also become convex, i.e., for any  $w^{(1)}, w^{(2)} \in \partial \Delta_t$ ,

$$(2.14) \quad w^{(0)} = \tau w^{(2)} + (1 - \tau) w^{(1)} \in \Delta_t, \quad 0 < \tau < 1.$$

In fact, if we put  $z^{(1)} = f^{-1}(w^{(1)})$ ,  $z^{(2)} = f^{-1}(w^{(2)})$ , then  $K_D(z^{(1)}, \overline{z^{(1)}}) = K_D(z^{(2)}, \overline{z^{(2)}}) = t$ . Setting

$$(2.15) \quad F(z) \equiv \tau f(g(z)) + (1 - \tau)f(z)$$

where  $g(z)$  is a holomorphic mapping of  $D$  into  $D$  satisfying  $g(0) = 0$  and  $g(z^{(1)}) = z^{(2)}$ , we observe that  $F(0) = 0$  and  $F(z) < f(z)$  because the mapping  $f: D \rightarrow C^n$  is convex. Hence

$$(2.16) \quad \psi(z) \equiv f^{-1}(F(z))$$

is a holomorphic mapping of  $D$  into  $D$ , so we have

$$K_D(z^{(1)}, \overline{z^{(1)}}) \geq K_D(\psi(z^{(1)}), \overline{\psi(z^{(1)})}) = K_D(f^{-1}(w^{(0)}), \overline{f^{-1}(w^{(0)})}).$$

Consequently  $f^{-1}(w^{(0)}) \in D_t$ , so  $w^{(0)} \in \Delta_t$ . Thus, by Lemma 2.1, (2.13) holds for all  $z \in D$ . Contrary, if (2.13) is realized for all  $z \in D$ , every  $\Delta_t$  is convex. Therefore we can conclude that the mapped domain  $\Delta$  is convex.

Particularly if  $D$  is a unit hypersphere, then

$$K_D(z, \bar{z}) = \frac{n!}{\pi^n (1 - |z|^2)^{n+1}}.$$

Thus we have the following result by Theorem 2.1.

**THEOREM 2.2.** *Let  $D$  be the unit hypersphere and let  $f: D \rightarrow C^n$  be holomorphic,  $f(0) = 0$  and  $\det(df/dz) \neq 0$  for all  $z \in D$ . Then  $f(D)$  is convex if and only if*

$$(2.17) \quad \Re \left[ |Az|^2 + z^* \left( \frac{df}{dz} \right)^{-1} \frac{d^2 f}{dz^2} (Az \times Az) \right] \geq 0,$$

where

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix}, A_j \geq 0, j = 1, \dots, n,$$

and the equality holds only if  $Az = 0$ .

*Proof.* We can compute as follows setting  $K = K_D(z, \bar{z})$ :

$$(2.18) \quad \partial K / \partial z = (n+1) \frac{z^*}{1 - |z|^2} K,$$

$$(2.19) \quad \partial^2 K / \partial z^2 = (n+1)(n+2) \frac{(z \times z)^*}{(1 - |z|^2)^2} K,$$

$$(2.20) \quad \partial^2 K / \partial z^* \partial z = (n+1) \frac{(1 - |z|^2)E + (n+2)zz^*}{(1 - |z|^2)^2} K.$$

Then, from (2.13), we have

$$(2.21) \quad \mathcal{R} \left[ (n+2) \{ |z^* \alpha|^2 + (z^* \alpha)^2 \} + (1 - |z|^2) \left\{ 1 - z^* \left( \frac{df}{dz} \right)^{-1} \frac{d^2 f}{dz^2} \alpha^2 \right\} \right] > 0.$$

Since

$$|z^* \alpha|^2 + \mathcal{R}(z^* \alpha)^2 = 0$$

from

$$\mathcal{R} \left[ \frac{\partial K}{\partial z} \alpha \right] = 0, \text{ i.e., } \mathcal{R}[z^* \alpha] = 0,$$

we conclude

$$(2.22) \quad \mathcal{R} \left[ 1 - z^* \left( \frac{df}{dz} \right)^{-1} \frac{d^2 f}{dz^2} \alpha^2 \right] > 0.$$

Moreover, under the condition  $\mathcal{R}[z^* \alpha] = 0$  it becomes that  $z^* \alpha = ip(p \geq 0, i = \sqrt{-1})$ , because both  $\alpha$  and  $-\alpha$  are satisfy (2.22). Therefore we can put  $\alpha = i(Az/|Az|)$  when  $Az \neq 0$ , where

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix}, A_j \geq 0, (j = 1, \dots, n),$$

are chosen arbitrarily. Thus we obtain (2.17) from (2.22).

**REMARK 1.** Suffridge's Theorem 5 [11] shows that



$$F = \frac{df}{dz} \left[ A^2 z + \left( \frac{df}{dz} \right)^{-1} \frac{d^2 f}{dz^2} (Az \times Az) \right] / 2, \quad w = \left( \frac{df}{dz} \right)^{-1} F \in \mathcal{S}_2,$$

i.e.,

$$\begin{aligned} \mathcal{R} \sum_{j=1}^n w_j |z_j|^2 / z_j &= \mathcal{R} z^* \left[ A^2 z + \left( \frac{df}{dz} \right)^{-1} \frac{d^2 f}{dz^2} (Az \times Az) \right] / 2 \\ &= \mathcal{R} \left[ |Az|^2 + z^* \left( \frac{df}{dz} \right)^{-1} \frac{d^2 f}{dz^2} (Az \times Az) \right] / 2 \geq 0, \end{aligned}$$

is the necessary and sufficient condition for convexity.

Next, if  $D$  is the polydisk  $\{z \in C^n: |z_j| < 1, j = 1, \dots, n\}$ , the kernel function  $K_D(z, \bar{z})$  becomes  $1/\pi^n (1 - |z_1|^2)^2 \dots (1 - |z_n|^2)^2$ . Hence

$$(2.23) \quad \partial K / \partial z = 2K \cdot z^* Z,$$

$$\partial^2 K / \partial z^2 = 4K \cdot (z \times z)^* (Z \times Z)$$

$$(2.24) \quad + 2K \cdot (z \times z)^* (Z \times Z) \begin{pmatrix} \boxed{\begin{matrix} 1 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 0 \end{matrix}} & & 0 \\ & \boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & & \\ & & 0 & \\ & & & 0 \end{matrix}} & & \\ & 0 & \ddots & \\ & & \boxed{\begin{matrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{matrix}} \end{pmatrix},$$

$$(2.25) \quad \partial^2 K / \partial z^* \partial z = 4K \cdot Z z z^* Z + 2K \cdot Z^2,$$

where

$$Z = \begin{pmatrix} 1/(1 - |z_1|^2) & & 0 \\ & \ddots & \\ 0 & & 1/(1 - |z_n|^2) \end{pmatrix}.$$

Substituting formally (2.23), (2.24), and (2.25) into (2.13) and setting

$$\mathcal{R}(z^* Z \alpha)^2 + |z^* Z \alpha|^2 = 0 \quad \text{and} \quad \alpha = i \frac{Z^{-1/2} A z}{|Z^{-1/2} A z|}$$

where

$$Z^{-1/2} = \begin{pmatrix} \sqrt{1 - |z_1|^2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{1 - |z_n|^2} \end{pmatrix},$$

in place of the condition

$$\mathcal{R}\left[\frac{\partial K_D(z, \bar{z})}{\partial z}\alpha\right] = 2K \cdot \mathcal{R}[z^*Z\alpha] = 0,$$

we arrive at

$$(2.26) \quad \mathcal{R}\left[|Az|^2 + z^*Z\left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2}(Z \times Z)^{-1/2}(Az \times Az)\right] \geq 0,$$

where the equality holds only if  $Az = 0$ .

**THEOREM 2.3.** *Let  $D$  be the polydisk and let  $f: D \rightarrow C^n$  be holomorphic,  $f(0) = 0$  and  $\det(df/dz) \neq 0$  for all  $z \in D$ . Then  $f$  is a univalent map of  $D$  onto a convex domain if and only if the condition (2.26) is fulfilled.*

*Proof.* If  $f$  is a convex mapping, then by Suffridge's Theorem 3 [11]  $f = T(\varphi_1(z_1), \dots, \varphi_n(z_n))'$  where  $T$  is a nonsingular linear transformation and each  $\varphi_j(z_j)$  is a univalent mapping from the unit disk in the plane onto convex domain in the plane. Then we have

$$(2.27) \quad \left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2} = \begin{pmatrix} \varphi_1''(z_1)/\varphi_1'(z_1) & 0 & \dots & 0 & & 0 \\ & 0\varphi_2''(z_2)/\varphi_2'(z_2) & 0 & \dots & 0 & \\ & & \ddots & & & \\ 0 & & & 0 & \dots & 0\varphi_n''(z_n)/\varphi_n'(z_n) \end{pmatrix}.$$

Substituting this into the left side of (2.26), we get

$$(2.28) \quad \mathcal{R}\left[\sum_{j=1}^n A_j^2 |z_j|^2 \{1 + z_j \varphi_j''(z_j)/\varphi_j'(z_j)\}\right].$$

Hence from the hypothesis  $\mathcal{R}[1 + z_j \varphi_j''(z_j)/\varphi_j'(z_j)] > 0$ ,  $j = 1, \dots, n$ , we get the inequality (2.26).

We will prove the converse. Fix  $k$ ,  $1 \leq k \leq n$  and choose  $A_k = 1$ ,  $A_h = 0$ ,  $h \neq k$ ,  $1 \leq h \leq n$ . From (2.26)

$$(2.29) \quad \mathcal{R}\left[|z_k|^2 + \frac{z_k^2(1 - |z_k|^2)}{\det J} \sum_{j=1}^n \frac{\bar{z}_j}{1 - |z_j|^2} G_j^{k^2}\right] \geq 0,$$

where  $J = df/dz$  and  $G_j^{k^2}$  is obtained from  $\det J$  by replacing the  $j$ th column by the column  $\partial^2 f / \partial z_k^2 = (\partial^2 f_1 / \partial z_k^2, \dots, \partial^2 f_n / \partial z_k^2)'$ . For  $l, 1 \leq l \leq n$ ,  $l \neq k$ , setting  $|z_j| < 1/2$ ,  $j \neq l$ ,  $1 \leq j \leq n$ ,  $(1 - |z_k|^2)/(1 - |z_l|^2)$  tends to infinity when  $|z_l| \rightarrow 1$ . Then we must have always

$$(2.30) \quad \Re \left[ \frac{1}{\det J} \frac{z_k^2}{z_l} G_l^{k^2} \right] \geq 0$$

from the condition (2.29). Here, since it becomes 0 at  $z_k = 0$ , we see that  $G_l^{k^2} \equiv 0$  for each  $l, l \neq k, 1 \leq l \leq n$ . Next, if we set  $A_k = A_l = 1, A_m = 0, m \neq k, l$ , then (2.26) becomes as follows from the above results:

$$(2.31) \quad \Re \left[ |z_k|^2 + |z_l|^2 + \frac{|z_k|^2 z_k G_k^{k^2}}{\det J} + \frac{|z_l|^2 z_l G_l^{l^2}}{\det J} + \frac{2z_k z_l \sqrt{(1 - |z_k|^2)(1 - |z_l|^2)}}{\det J} \sum_{j=1}^n \frac{\bar{z}_j G_j^{kl}}{(1 - |z_j|^2)} \right] \geq 0.$$

For  $s, 1 \leq s \leq n$ , setting

$$|z_h| < 1/2, h \neq s, 1 \leq h \leq n, \frac{\sqrt{(1 - |z_k|^2)(1 - |z_l|^2)}}{1 - |z_s|^2}$$

tends to infinity when  $|z_s| \rightarrow 1$ . Then we must have always

$$(2.32) \quad \Re \left[ \frac{1}{\det J} \frac{z_k z_l}{z_s} G_s^{kl} \right] \geq 0.$$

Since it attains to the minimum value 0 at  $z_k z_l = 0$ , we must have  $G_s^{kl} \equiv 0$  for each  $s$ . Thus we arrive at the conditions of the Theorem 3 of Suffridge following his methods. So we can conclude that  $f$  is a convex mapping.

**3. Starlike mappings.** We now consider univalent functions of  $D$  which map  $D$  onto a starlike domain with respect to 0. First we set up the definition of starlikeness following Suffridge:

**DEFINITION.** A holomorphic mapping  $f: D \rightarrow C^n$  is starlike if  $f$  is univalent,  $f(0) = 0$  and  $(1 - \tau)f < f$  for all  $\tau \in I = [0, 1]$ .

**THEOREM 3.1.** Let  $D$  be a bounded schlicht domain for which the kernel function  $K_D(z, \bar{z})$  becomes infinite everywhere on the boundary,  $K_D(0, 0) = \min_{z \in D} K_D(z, \bar{z})$  at only the origin, and  $K_D(z, \bar{z}) \geq K_D(g(z), \overline{g(z)})$  for any holomorphic mapping  $g(z)$  of  $D$  into  $D$  satisfying  $g(0) = 0$ . Suppose  $f: D \rightarrow C^n$  is holomorphic,  $f(0) = 0$  and  $\det(df/dz) \neq 0$  for all  $z \in D$ . Then  $f$  is starlike if and only if

$$(3.1) \quad \Re \left[ \frac{\partial K_D(z, \bar{z})}{\partial z} \left( \frac{df}{dz} \right)^{-1} f \right] > 0$$

for all  $z \in D, z \neq 0$ .

REMARK 2. Domains which belong to the above mentioned class  $\mathcal{D}$  satisfy the conditions of this Theorem.

*Proof.* If  $f$  is starlike, then all image  $\Delta_t$  are starlike, that is, for all  $w^{(1)} \in \partial\Delta_t$  we have  $w^{(0)} = (1 - \tau)w^{(1)} \in \Delta_t$ ,  $\tau \in I$ . In fact, if we set  $z^{(1)} = f^{-1}(w^{(1)})$ ,  $K_D(z^{(1)}, \overline{z^{(1)}}) = t$  and  $\psi(z) \equiv f^{-1}((1 - \tau)f(z))$ , then we obtain

$$(3.2) \quad K_D(z^{(1)}, \overline{z^{(1)}}) \geq K_D(\psi(z^{(1)}), \overline{\psi(z^{(1)})}) = K_D(f^{-1}(w^{(0)}), \overline{f^{-1}(w^{(0)})}) ,$$

because  $\psi(z)$  is a mapping of  $D$  into  $D$  and  $\psi(0) = 0$ . Then it holds that  $f^{-1}(w^{(0)}) \in \Delta_t$  which yields  $w^{(0)} \in \Delta_t$ . Now, since

$$\Phi_t\left(w + \varepsilon \frac{\partial \Phi_t}{\partial w^*}\right) = 2\varepsilon \left| \frac{\partial \Phi_t}{\partial w^*} \right|^2 + 0(\varepsilon^2) > 0$$

when  $\varepsilon > 0$  is sufficiently small and  $w \in \partial\Delta_t$ ,  $N_w \equiv \partial\Phi_t/\partial w^*$  is the outward normal vector at the boundary point  $w \in \partial\Delta_t$ . Hence  $(1 - \tau)w \in \Delta_t$  ( $w \in \partial\Delta_t$ ,  $0 < \tau \leq 1$ ) implies

$$(3.3) \quad \cos(-N_w, -w) = \Re \left[ \frac{\partial \Phi_t}{\partial w} w \right] / \left| \frac{\partial \Phi_t}{\partial w^*} \right| |w| > 0$$

which yields (3.1) by virtue of

$$\frac{\partial \Phi_t}{\partial w} w = \frac{\partial K}{\partial z} \left( \frac{df}{dz} \right)^{-1} f(z) .$$

Conversely, if (3.1) holds, then we conclude  $(1 - \tau)w \in \Delta_t$ ,  $w \in \partial\Delta_t$ ,  $0 < \tau < \varepsilon (< 1)$  for some  $\varepsilon > 0$  by (3.3). Moreover, we can conclude  $(1 - \tau)w \in \Delta_t$ ,  $w \in \partial\Delta_t$ ,  $0 < \tau \leq 1$ , because, if  $(1 - \tau_1)w \equiv w^{(1)} \in \partial\Delta_t$  and  $(1 - \tau)w \in \Delta_t$ ,  $0 < \tau < \tau_1$  for some  $\tau_1 < 1$ , then  $(1 - \tau)w^{(1)} \notin \Delta_t$ ,  $w^{(1)} \in \partial\Delta_t$  which is a contradiction. Then the image domain  $\Delta$  of  $D$  becomes starlike.

COROLLARY 3.1. Let  $D$  be the unit hypersphere, and let  $f: D \rightarrow C^n$  be holomorphic,  $f(0) = 0$  and  $\det(df/dz) \neq 0$  for all  $z \in D$ . Then  $f(z)$  is starlike if and only if

$$(3.4) \quad \Re \left[ z^* \left( \frac{df}{dz} \right)^{-1} f \right] > 0$$

for all  $z \in D$ ,  $z \neq 0$ .

*Proof.* Substituting (2.18) into (3.1), we obtain the required result.

REMARK 3. The conditions of Suffridge's Theorem 4 [11]:  $f = Jw$ ,  $w \in \mathcal{S}_2$  are the same as (3.4).

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