Pacific Journal of Mathematics

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JIMMIE DON LAWSON

Vol. 44, No. 2 June 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 2, 1973

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JIMMIE D. LAWSON

This paper demonstrates that the topology of a compact topological lattice or semilattice can be defined intrinsically, i.e., in terms of the algebraic structure. Properties of various intrinsic topologies are explored.

A variety of ways have been suggested for defining topologies from the algebraic structure of a lattice (see e.g. [4] or [12]). If one is given a topological lattice, a natural question is whether the given topology agrees with one or more of these intrinsic topologies. Some results of this nature may be found in [5] or [13]. In this paper we show that the topology of a compact topological lattice or semilattice can always be defined intrinsically; these results extend to a large class of locally compact lattices.

A *topological lattice* is a lattice L equipped with a Hausdorff topology for which the operations of join and meet are continuous as mappings from $L \times L$ into L. A *topological semilattice* is a (meet) semilattice together with a Hausdorff topology for which the meet operation is continuous.

If A is a subset of a lattice or semilattice, we define

$$L(A) = \{y: y \leqq x \quad \text{for some} \quad x \in A\}$$

and

$$M(A) = \{z: x \leqq z \quad \text{for some} \quad x \in A\}\ .$$

A subset B of a semilattice is an *ideal* if $L(B) = B$. A set A is *convex* if $x, z \in A$ and $x \leqq y \leqq z$ imply $y \in A$. A lattice L is *locally convex* if it has an open base of convex sets. A *closed interval* is a set of the form

$$[a, b] = \{x: a \leqq x \leqq b\}\ .$$

For the definition of undefined lattice properties employed in this paper, the reader is referred to [4].

The topological closure of a set A will be denoted by A^*.

1. Intrinsic topologies. The following intrinsic topologies on a lattice L are considered in this paper.

(1) The interval topology (I). If L has a 0 and 1, the interval topology is defined by taking as a subbase for the closed sets all sets

$\{L(x): x \in L\}$ and all sets $\{M(x): x \in L\}$. If L does not have universal bounds, then a set $K \subset L$ is closed if $K \cap [a, b]$ is closed in the interval topology of the sublattice $[a, b]$ for all a, b with $a \leqq b$.

(2) The order topology (0). A net $\{x_\alpha\}$ in L is said to *order-converge* to x if there exist a monotonic ascending net $\{t_\alpha\}$ with $x = \sup t_\alpha (t_\alpha \uparrow x)$ and a monotonic descending net $\{u_\alpha\}$ with $x = \inf u_\alpha (u_\alpha \downarrow x)$ such that for all $\alpha, t_\alpha \leqq x_\alpha \leqq u_\alpha$. A subset A of L is *closed* in the order topology if $\{x_\alpha\} \subset A$ and x_α order converges to x imply that $x \in A$. Note that if x_α order-converges to x, then for any cofinal subset of the domain directed set it remains true that x_α order-converges to x. Hence the order topology may be defined equivalently by declaring a set U of L open if $x \in U$ and x_α order-converges to x imply x_α is residually in U.

(3) The convex-order topology (CO). A subset U of L is a basic open set for the convex-order topology if (i) U is convex and (ii) if x_α order-converges to $x, x \in U$, then x_α is residually in U. Again, the second condition is equivalent to U being open in the order topology.

We now list some easily derived properties of these intrinsic topologies.

PROPOSITION 1. (1) *The* CO *topology is locally convex.*

(2) *The* 0 *topology is finer than the* CO *topology.*

(3) *Any homomorphism from L to a locally convex lattice that is continuous in the* 0 *topology is continuous in the* CO *topology.*

(4) *If the* 0 *topology is locally convex, then it agrees with the* CO *topology.*

PROPOSITION 2. *The* 0 *topology is finer than the I topology.*

Proof. [4, p. 251].

We shall call a topology on a lattice *agreeable* if (i) the topology is locally convex and (ii) if $t_\alpha \uparrow x$ or $t_\alpha \downarrow x$ then t_α converges to x in the topology.

PROPOSITION 3. *If τ is an agreeable topology on a lattice L, then the* CO *topology is finer than τ.*

Proof. Since τ is locally convex, it suffices to show that if a convex set U is in τ, then it is open in the CO topology. Suppose that x_α is a net that order-converges to $x \in U$. Then there exist $t_\alpha \uparrow x$, $u_\alpha \downarrow x$ such that for all $\alpha, t_\alpha \leqq x_\alpha \leqq u_\alpha$. Since τ is agreeable, t_α and u_α are residually in U, and since U is convex x_α is residually in U.

2. The interval topology in complete lattices. The interval topology has received rather through investigation. In this section we summarize results concerning its relationship to compact topological lattices.

PROPOSITION 4. *Let L be a complete lattice.*

(1) *L is compact in the interval topology.*

(2) *If (L, τ) is a topological lattice, then τ is finer than the interval topology.*

(3) *If L is Hausdorff in the interval topology, then the order and interval topology coincide.*

Proof. (1) This is a result of O. Frink. A proof may be found [4, p. 250].

(2) Since in a topological lattice $M(x)$ and $L(x)$ are closed for each $x \in L$, and these sets are a subbasis for the closed sets of the interval topology, the result follows.

(3) See [3] or [15].

The next theorem contains the central results on compact topological lattices with the interval topology.

THEOREM 5. *The following are equivalent in a compact topological lattice (L, τ):*

(1) *(L, I) is Hausdorff.*

(2) $\tau = 0 = I = \mathrm{CO}$.

(3) *(L, τ) has a basis of open convex sublattices.*

(4) *(L, τ) has a base of neighborhoods at each point of closed intervals.*

(5) *If $y \not\leq x$ then there exists z such that x is in the interior of $L(z)$ and $y \not\leq z$, and dually.*

(6) *Every net has an order-convergent subnet.*

Proof. The equivalence of 3, 4, 5 has been shown by E. B. Davies [6, Theorem 5]. K. Atsumi has shown the equivalence of 1 and 6 [3, Theorem 3]. D. Strauss has shown the equivalence of 1 and 3 [13, Theorem 5]. Conditions 3 and 1 together with part 2 of Proposition 4 imply $\tau = I$. Part 3 of Proposition 4 further implies $I = 0$. Since CO is trapped between I (since I is locally convex) and 0, it also agrees with them. Hence Conditions 3 and 1 imply 2. Condition 2 easily implies Condition 1 since τ is Hausdorff. Hence the six conditions are equivalent.

We remark that if (L, τ) is compact topological lattice of finite breadth, then $\tau = I$ [5]. Hence all the equivalences of Theorem 5 apply to (L, τ). It is known that a finite-dimensional compact connected topological lattice has finite breadth [9].

For complete distributive lattices one obtains a purely algebraic description of lattices which are topological lattices in the interval topology.

THEOREM 6. *Let L be a distributive lattice. The following are equivalent*:

(1) *L is complete and completely distributive.*

(2) *L is complete and (L, I) is Hausdorff.*

(3) *L is complete and L can be embedded in a product of unit intervals (under coordinatewise order) by an lattice isomorphism which preserves all joins and all meets.*

(4) *L admits a topology τ for which (L, τ) is a compact topological lattice with enough continuous lattice homomorphisms into the unit interval (with usual order) to separate points.*

(5) *L admits a topology τ for which (L, τ) is a compact topological lattice with a basis of open convex sublattices.*

Proof. Theorems 4 and 5 of [6] imply the equivalence of Conditions 4 and 5. Strauss has shown the equivalence of Conditions 1 and 2 [13, Theorem 7] and the implication of Condition 3 by Condition 2 [13, Theorem 6]. It is readily seen that Condition 3 implies that L is a closed subset in the product topology of unit intervals (where the unit internal carries its normal topology); hence L is a compact topological lattice in its relative topolopy. Since a product of intervals has a basis of open convex sublattices, the intersection of this basis with L endows L with such a basis. Hence Condition 3 implies Condition 5. That Condition 5 implies Condition 2 follows from Theorem 5 above.

THEOREM 7. *Let B be a Boolean lattice. The following are equivalent*:

(1) *B is complete and completely distributive.*

(2) *B admits a topology τ for which (B, τ) is a compact topological lattice.*

(3) *B is isomorphic with the Boolean lattice of subsets of some set.*

(4) *B is isomorphic to a product of $\{0, 1\}$ with $0 < 1$.*

(5) *B is complete and (B, I) is Hausdorff.*

Proof. By Theorem 6, Conditions 1 and 5 are equivalent and imply Condition 2. Strauss has shown Condition 2 implies Condition 1 [13, Theorem 1].

Tarski has shown that Condition 1 implies Condition 3 (see [14] or [4, p. 119]). If B is isomorphic to all subsets of a set X, then it can be identified with $\{0, 1\}^X$ by a lattice isomorphism. Hence Condition 3 implies Condition 4. Since any product of complete chains is completely distributive [4, p. 120], Condition 4 implies Condition 1.

3. The convex-order topology.

In the preceding section we gave conditions under which a topological lattice had the interval topology and for which all the intrinsic topologies collapsed to this topology. The conditions for a topological lattice to have the order or convex-order topologies are much more general.

THEOREM 8. *Let (L, τ) be a topological lattice with τ a regular, agreeable topology. If each $x \in L$ has a complete neighborhood, then $\tau = \mathrm{CO}$. (A subset is complete if every increasing net in the subset has a sup in the subset, and dually).*

Proof. By Proposition 3, the CO topology is finer than τ.

Conversely, let U be a basic open convex set in the CO topology. If $U \notin \tau$, then there exists x in U and a net $\{x_\alpha\}$ converging to x in (L, τ) such that $x_\alpha \notin U$ for all α.

Let N be a complete neighborhood of x in τ. Let D be the set of all sequences $\{W_n: n = 1, 2, \cdots\}$ satisfying for all n,

(i) $x \in W_n^\circ$, $W_n = W_n^* \subset N$

(ii) $(W_n \vee W_n) \cup (W_n \wedge W_n) \subset W_{n-1}^\circ$.

If $\{W_n\}, \{V_n\} \in D$, we define $\{W_n\} \geqq \{V_n\}$ if $W_n \subset V_n$ for all n. It is straightforward to verify that $(D, \leqq)$ is a directed set. If $\{W_n\} \in D$, let $W = \cap W_n$. Condition (i) implies $x \in W \subset N$ and W is closed. Condition (ii) implies W is a sublattice. Since τ is agreeable, N is complete, and W is closed, W has a largest element w^+ and a smallest element w^-.

If V is an closed neighborhood of x contained in N, then employing the regularity of τ and the continuity of $\vee$ and $\wedge$, one can construct $\{V_n\} \in D$ such that $V = V_1$. Hence $v^+ \in \cap V_n \subset V$. Thus the net $\{w^+: \{W_n\} \in D\}$ is a monotonic decreasing net which converges to x in the τ-topology. It follows from the continuity of the lattice operations that $\{w^+\} \downarrow x$. Dually $\{w^-\} \uparrow x$. Hence residually many of the $\{w^+\}$ and $\{w^-\}$ are in U. Fix $\{W_n\} \in D$ such that $w^+, w^- \in U$.

For each n, pick $x_n \in \{x_\alpha\} \cap W_n$. If $m > n$, then

$$\bigvee_{k=n}^{m} x_k \in \bigvee_{k=n}^{m} W_k \subset \left(\bigvee_{k=n}^{m-2} W_k\right) \vee W_{m-1} \vee W_{m-1}$$
$$\subset \left(\bigvee_{k=n}^{m-3} W_k\right) \vee W_{m-2} \vee W_{m-2} \subset \cdots \subset W_{n-1}\,.$$

Thus for all $m > n$, $y_m = \bigvee_{k=n}^{m} x_k \in W_{n-1}$. Since $W_{n-1} \subset N$, W_{n-1} is closed, N is complete, and the sequence y_m is monotonic increasing, there exists $a_n \in W_{n-1}$ such that $a_n = \sup\{x_k: k \geqq n\}$. The sequence a_n is a decreasing sequence contained in N, and hence converges to $a = \inf\{a_n\}$. Since the sequence $\{a_n\}$ is eventually in each W_n and each W_n is closed, we conclude $a \in W = \cap W_n$. Hence $a \leqq w^+$.

Dually let $b_n = \inf\{x_k: k \geqq n\}$ and $b = \sup\{b_n\}$. Then $w^- \leqq b$. Since $b_n \leqq a_n$ for all n, $w^- \leqq b \leqq a \leqq w^+$. Since U is convex, $a, b \in U$. Since $a_n \downarrow a$ and $b_n \uparrow b$ and $a, b \in U$, there exists m such that a_m, $b_m \in U$. Since $b_m \leqq x_m \leqq a_m$, we have $x_m \in U$. However, this is in contradiction to $x_m \in \{x_\alpha\}$ and $x_\alpha \notin U$ for all α.

The next lemma is a standard and easily proved result about topological lattices (see [7] or [13]).

LEMMA 9. *Let K be a compact subset of a topological lattice. If $\{x_\alpha\}$ is a monotonically increasing (decreasing) net in K, then the net converges to its sup (inf).*

THEOREM 10. *Let L be a topological lattice which is* (i) *compact or* (ii) *locally compact and connected. Then L has the convex order topology.*

Proof. If L is compact, it is well known via the work of Nachbin [10] that L is locally convex. This fact together with Lemma 9 implies the topology on L is agreeable and L is complete. The conclusion then follows from Theorem 8.

If L is locally compact and connected, Anderson has shown L is locally convex [1]. Suppose $u_\alpha \downarrow x$. Let U be a compact neighborhood of x. Since $[x, u_\alpha] = (L \wedge u_\alpha) \vee x$ is connected, if u_α is not residually in U, then cofinally there exists y_α in the boundary of U such that $x \leqq y_\alpha \leqq u_\alpha$. By compactness of U, we can assume by picking subnets if necessary that $\{y_\alpha\}$ converges to some y in the boundary of U.

Fix some α. If $\beta > \alpha$, then $y_\beta \leqq u_\beta \leqq u_\alpha$. Thus $y_\beta \wedge u_\alpha = y_\beta$ for all $\beta > \alpha$ for which y_β is defined. Since $y_\beta \wedge u_\alpha$ converges to $y \wedge u_\alpha$, we have $y \wedge u_\alpha = y$, i.e., $y \leqq u_\alpha$ for all u_α not in U. Since $x = \inf\{u_\alpha\}$, $y \leqq x$. Similarly, since each $y_\alpha \geqq x$, by continuity of $\wedge$, $y \geqq x$. Hence $y = x$. But this is impossible since x is not in the boundary of U. Thus we conclude the topology of L is agreeable. Since L is locally compact, Lemma 9 implies each point has a complete neighborhood. Hence by Theorem 8, L has the convex order topology.

It is a consequence of the preceding theorem that a lattice admits at most one topology for which it is a compact (or locally compact connected) topological lattice, namely the convex order topology. This theorem also allows a nice algebraic condition for continuity of homomorphisms between compact (or locally compact connected) topological lattices. It follows that any isomorphism between such lattices is a homeomorphism.

PROPOSITION 11. *Let L and K be lattices, f a homomorphism from L into K. If $u_\alpha \downarrow x (t_\alpha \uparrow x)$ implies $f(u_\alpha) \downarrow f(x)$ $(f(t_\alpha) \uparrow f(x))$, then f is continuous if L and K are given the convex order topologies.*

Proof. Let U be a basic convex, open set in K. Then $f^{-1}(U)$ is convex in L. Suppose $x \in f^{-1}(U)$ and $\{x_\alpha\}$ order converges to x. Then there exists $u_\alpha \downarrow x$, $t_\alpha \uparrow x$ such that for all α, $u_\alpha \geqq x_\alpha \geqq t_\alpha$. Then $f(u_\alpha) \geqq f(x_\alpha) \geqq f(t_\alpha)$ and by hypothesis $f(u_\alpha) \downarrow f(x)$ and $f(t_\alpha) \uparrow f(x)$. Hence since U is open $f(x_\alpha)$ is eventually in U. Thus x_α is eventually in $f^{-1}(U)$. Hence $f^{-1}(U)$ is open and f is continuous.

It is shown in [13] that if (L, τ) is a topological lattice for which τ is a first countable regular topology for which every point has a σ-complete neighborhood, then τ is finer than the order topology. If further, τ is agreeable, Propositions 2 and 3 show τ is the order topology. Since in the proof of Theorem 10, it was shown that the topology of a locally compact connected or a compact topological lattice is agreeable, it follows that

THEOREM 12. *Let L be a compact or locally compact connected topological lattice which is metrizable. Then L has the order topology.*

The theorem for the compact case appears in [7] and [13]. It is not known whether the theorem remains true without metrizability.

4. Compact semilattices. In this section we give an internal characterization of the topology of a compact semilattice. If S is a semilattice we say I is an *ideal* of S if $L(I) = I$. If A is an ideal in S, define A^+ by $x \in A^+$ if there exists a net x_α in A such that $x_\alpha \uparrow x$.

THEOREM 13. *Let S be a compact topological semilattice. An ideal A of S is closed if and only if $A = A^+$.*

Proof. Suppose A is closed. If $x \in A$, then the constant net x is a monotonic increasing net increasing to x. Hence $A \subset A^+$. If x_α is a net in A and $x_\alpha \uparrow x$, then x_α converges to x in the topology of S (a monotonically increasing net converges to its sup in a compact topological semilattice). Hence $x \in A$. Thus $A = A^+$.

Conversely let $A = A^+$. Let $y \in A^*$. Let D be the set of all sequences $\{W_n: n = 1, 2, \cdots,\}$ satisfying for all n,

(i) $x \in W_n^\circ$, $W_n = W_n^*$

(ii) $W_n \wedge W_n \subset W_{n-1}^\circ$.

If $\{W_n\}, \{V_n\} \in D$, we define $\{W_n\} \geqq \{V_n\}$ if $W_n \subset V_n$ for all n. Then $(D, \leqq)$ is a directed set. If $\{W_n\} \in D$, let $W = \cap W_n$. Then W is closed and is a subsemilattice. Hence W has a minimal element w^-. As in the proof of Theorem 8, $\{w^-: \{W_n\} \in D\}$ is a monotonically increasing net and $w^- \uparrow y$.

Fix a specific w^- associated with a $\{W_n\}$. Since $y \in A^*$, for each n there exists $b_n \in W_n \cap A$. Let $\partial_n = \bigwedge_{m>n} b_m$. Then ∂_n is an increasing sequence, each $\partial_n \in A$ since A is an ideal, and as in the proof of Theorem 8, $\partial_n \uparrow \partial \in W$. Since $A = A^+, \partial \in A$. Since $w^- \leqq \partial$ and A is an ideal, $w^- \in A$. But since the net $\{w^-\} \uparrow y$, we conclude $y \in A$. Hence A is closed.

Theorem 13 makes possible an algebraic description of the closure of an ideal in a compact topological semilattice.

COROLLARY 14. *Let I be an ideal of a compact topological semilattice S. Then $I^* = I^{++}$.*

Proof. Since $I \subset I^*$, we have $I^+ \subset (I^*)^+$. By Theorem 13, $(I^*)^+ = I^*$. Hence $I^+ \subset I^*$. A repetition of the argument with I^+ replacing I shows $I^{++} \subset I^*$.

Let $y \in I^+$ and $x \leqq y$. Then there exists a net $\{y_\alpha\}$ in I such that $y_\alpha \uparrow y$. Then $x \wedge y_\alpha \uparrow x$ and $x \wedge y_\alpha \in I$ for all α. Thus $x \in I^+$; hence we have shown I^+ is an ideal. It is essentially shown in the proof of Theorem 13 that if $y \in I^*$, then $y \in (L(I^+))^+$. Since I^+ is an ideal $L(I^+) = I^+$. Thus $y \in I^{++}$. Hence $I^{++} = I^*$.

A principal application of Theorem 13 is an algebraic or intrinsic method of defining the topology of a compact topological semilattice. It is known that if S is a compact topological semilattice, then the space of all closed ideals S' of S ordered by inclusion and considered as a subspaces of 2^S is a compact distributive topological lattice; furthemore the mapping sending s into $L(s)$ is a topological isomorphism from S into S' (see e.g. [8, Theorem 1.2]). Since the closed ideals of S can be identified algebraically as those ideals for which $I = I^+$ and since the topology of S' can be defined algebraically as the convex-order topology (Theorem 10), the topology of S is determined by its algebraic structure.

THEOREM 15. *Let f be a homomorphism from a compact topological semilattice S onto a compact topological semilattice T. If f has the property that for $x_\alpha \uparrow x, f(x_\alpha) \uparrow f(x)$ and for $y_\alpha \downarrow y, f(y_\alpha) \downarrow f(y)$, then f is continuous.*

The proof of this theorem breaks down conveniently into several steps.

(i) If $t \in T$, $f^{-1}(t)$ has a least element. Since f is a homomorphism $f^{-1}(t)$ is a semilattice. Hence it is a monotonically decreasing net indexed by itself. Since S is compact, the net monotonically decreases to some s. Hence by hypothesis $f(s) = t$. Thus s is a least element for $f^{-1}(t)$.

(ii) If A is an ideal, $f(A)^+ = f(A^+)$. Suppose $y \in f(A)^+$. Then there exists a net $y_\alpha \uparrow y$ where $y_\alpha \in f(A)$ for all α. There exists $w_\alpha \in A$ such that $f(w_\alpha) = y_\alpha$ for each α. There exists x_α, the least element of $f^{-1}(y_\alpha)$; hence $x_\alpha \leqq w_\alpha$. Since A is an ideal, $x_\alpha \in A$. If $\alpha \leqq \beta$, then $f(x_\alpha \wedge x_\beta) = f(x_\alpha) \wedge f(x_\beta) = y_\alpha \wedge y_\beta = y_\alpha$; hence $x_\alpha \wedge x_\beta \in f^{-1}(y_\alpha)$. Since x_α is the least element of $f^{-1}(y_\alpha)$, $x_\alpha = x_\alpha \wedge x_\beta$. Hence the net x_α is increasing. Since S is compact, $x_\alpha \uparrow x$ for some $x \in A^+$. By hypothesis $f(x_\alpha) \uparrow f(x)$, i.e., $y_\alpha \uparrow f(x)$. Thus $p = f(x) \in f(A^+)$. Conversely, let $t = f(s) \in f(A^+)$. Then there exists a net $s_\alpha \uparrow s$, $s_\alpha \in A$ for each α. By hypothesis $f(s_\alpha) \uparrow f(s)$. Hence $t \in f(A)^+$. Thus $f(A)^+ = f(A^+)$.

(iii) f induces a homomorphism $f'\colon S' \to T'$, the lattices of closed ideals of S and T resp. If A is a closed ideal of S, define $f'(A)$ to be $f(A)$. Since f is onto, $f(A)$ is an ideal. Also $f(A)^+ = f(A^+) = f(A)$; hence $f(A)$ is closed, i.e., $f'(A) \in T'$. Always $f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) \subset f(A) \cap f(B)$. Suppose $t \in f(A) \cap f(B)$; then there exists $a \in A$, $b \in B$ such that $f(a) = t = f(b)$. Let x be the least element of $f^{-1}(t)$; then $x \leqq a$, $x \leqq b$. If A and B are ideals, then $x \in A \cap B$. Hence $t = f(x) \in f(A \cap B)$. Thus $f(A \cap B) = f(A) \cap f(B)$.

(iv) f' preserves limits of increasing and decreasing nets.

In S' and T' the limit of a decreasing net is just the intersection. An argument similar to the one just given to show f' preserves finite intersections will show f' also preserves arbitrary intersections. If $\{A_\alpha\}$ is an increasing net in S', then the limit is $(\cup A_\alpha)^*$ and the limit of $f(A_\alpha)$ is $(\cup fA_\alpha))^*$. Now $f((\cup A_\alpha)^*) = f((\cup A_\alpha)^{++}) = (f \cup A_\alpha))^{++}$ (by two applications of (ii)) $= (\cup f(A_\alpha))^{++} = (\cup f(A_\alpha))^*$. Hence f' preserves limits.

(v) The homomorphism f is continuous. Theorems 10 and 11 imply that f' is continuous. Since S and T are embedded in S' and T', f' restricted to their images is continuous. But this restriction of f' is just f.

COROLLARY 16. *Let h be an isomorphism from a compact topological semilattice S onto a compact topological semilattice T. Then h is a homeomorphism. Hence a fixed semilattice admits at most one topology for which it is a compact topological semilattice.*

Proof. Clearly h and h^{-1} preserve limits of increasing and decreasing nets. Hence the conclusion follows from Theorem 15.

For any two compact topologies, the identity mapping must be a homeomorphism. Hence the two agree.

Anderson and Hunter [2] have studied some classes of groups and semigroups in which each automorphism is continuous; this property they call van der Waerden property. Corollary 16 shows that compact semilattices are such semigroups.

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Received October 11, 1971. The research for this paper was supported by NSF grant GP-25014.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 44, No. 2 June, 1973