Pacific Journal of Mathematics

# ACTIONS OF TORUS $T^n$ ON $(n+1)$-MANIFOLDS $M^{n+1}$

Jingyal Pak

Vol. 44, No. 2 June 1973

PACIFIC JOURNAL OF MATHEMATIC
Vol. 44, No. 2, 1973

# ACTIONS OF TORUS $T^n$ ON $(n+1)$-MANIFOLDS $M^{n+1}$

JINGYAL PAK

**Let $\xi$ be a principal $T^l$-bundle over a lens space $L(p, q)$. It is shown here that the total space of $\xi$ can be identified with $L(k, q) \times S_1^1 \times \cdots \times S_l^1$, for some $k \leqq p$. Let $(T^n, M^{n+1})$ be an effective torus action on an orientable $(n+1)$-dimensional manifold. An elementary examination of the parity of dimensions of the slice $S_x$ at $x \in M$ and of the orbit $T^n(x)$, shows that the circle subgroups are the only possible stability groups on $M^{n+1}$. From these two results and the cross-sectioning theorem we can conclude that $T^{n+1}$ and $L(k, q) \times T^{n-2}$ are the only possible types of compact closed orientable $(n+1)$-manifolds which allow $T^n$ actions.**

It is shown in [3] that $T^4$ and $L(p, q) \times S^1$ are the only compact closed orientable 4-manifolds which allow effective $T^3$ actions. The purpose of this note is to show, using an argument similar to that of [3], that $T^{n+1}$ and $L(m, q) \times T^{n-2}$ are the only possible compact closed orientable $(n+1)$-manifolds which allow effective $T^n$ actions for $n \geqq 3$. Here $L(m, q)$ includes the case of $S^2 \times S^1$ and $S^3$. The key lemma used in the proof of this theorem is that every principal $T^l$-bundle over the lens space $L(p, q)$ can be identified with $L(k, q) \times T^l$ for suitable $k \leqq p$. In later papers we intend to work on $T^n$ actions on compact closed non-orientable $(n+1)$-manifolds $M^{n+1}$ and $(n+2)$-manifolds $M^{n+2}$.

Let $G$, a compact Lie group, act on a space $X$. If $x \in X$, $G_x = \{g \in G \mid g(x) = x\}$ will denote the stability group, or isotropy group of $G$ at $x \in X$. $G(x) = \{g(x) \mid g \in G\}$ will be called the orbit of $x \in X$. The orbit space, the set of all orbits, will be denoted by $X/G = X^*$ or $\bar{X}$ with the quotient topology, and the orbit map by $\Pi: X \to X^*$. For each $x \in X$, one can find a certain subset $S_x$ called the slice at $x$ [1, Chapter VIII], with the following properties:

(i) $S_x$ is invariant under $G_x$.

(ii) If $g \in G$, $y, y' \in S_x$, and $g(y) = y'$, then $g \in G_x$.

(iii) There exists a "cell neighborhood" $C$ of $G/G_x$ such that $C \times S_x$ is homeomorphic to a neighborhood of $x$. That is, if $f: C \to G$ is a local cross-section in $G/G_x$ then the map $F: C \times S_x \to X$ defined by $F(c, s) = f(c)s$ is a homeomorphism of $C \times S_x$ onto an open set containing $S_x$ in $X$. The principal orbits are those for which the stability groups are identity. An action is effective if $g(x) = x$ for every $x \in X$ implies $g = e$. We shall assume that $G$ is acting smoothly and effectively on a smooth orientable manifold. By the slice theorem, given in [1, Chapter VIII], it follows that if $T^n$ acts effectively on a compact closed $(n+1)$-manifold $M^{n+1}$, then there exist principal $T^n$ orbits and the orbit space $M/T^n = M^*$ is a compact 1-manifold which we denote by $S^1$ or $[0, 1]$.

LEMMA 1. *Let $(T^n, M^{n+1})$ be a transformation group. Then the circle subgroups of $T^n$ are the only possible nontrivial stability groups on $M^{n+1}$.*

*Proof.* Let $T^i \times F$, $i = 1, \cdots, n$, be a subgroup of $T^n$, where $T^i$ is $i$-dimensional torus subgroup of $T^n$ and $F$ is any finite subgroup of $T^n$ complementary to $T^i$. We assume that if $i = 1$, then $F$ is nontrivial.

First we show that no nontrivial finite subgroup $F$ of $T^n$ can be a stability group. If $M^* = S^1$ then every point in $M^*$ corresponds to a principal orbit, so that we don't have a finite group as a stability group. In any case, if we have a finite stability group $F$ at $x$, then $x$ is isolated. The orbit is $n$-dimensional and the slice is a 1-dimensional interval. Thus $F$ must be $Z_2$ which reverses the orientation (a contradiction, since $M$ is orientable and $T^n$ is connected).

Now cosider the case of $T^i \times F$, $i = 1, \cdots, n$. The orbit will be $(n-i)$-dimensional, and there is an $(n+1)-(n-i) = (i+1)$ dimensional disk slice on which $T^i \times F$ must act as a rotation. But $T^i \times F \not\subset SO(i+1)$ for $i = 1, \cdots n$. Thus there is no point $x \in M$ such that $T^n_x = T^i \times F$ for $i = 1, \cdots n$. This also implies that the fixed point set $F(T^n, M^{n+1}) = \varnothing$ for $n > 1$.

LEMMA 2. *Let $(T^n, M^{n+1})$ be a transformation group. Then the orbit map $\Pi\colon M^{n+1} \to M^*$ has a cross-section.*

*Proof.* If $M^* = S^1$, then the $T^n$-bundle is trivial. If $M^* = [0, 1]$, then the action corresponding over $(0, 1)$ is the trivial $T^n$-bundle, so that we have a cross-section over $(0, 1)$. Now we can extend this cross-section trivially to both ends.

LEMMA 3. *If $M^{n+1}$ is a principal $T^{n-2}$-*bundle over $L(p, q)$, $n \geqq 3$, *then $M^{n+1}$ can be written as $L(k, q) \times T^{n-2}$ for some integer $k \leqq p$.*

*Proof.* By taking a circle subgroup $T_1^1$ of $T^{n-2}$ and the complementary subgroup $T^{n-3}$ to $T_1^1$ in $T^{n-2}$, we can consider $M/T^{n-3}$ as a principal $T_1^1$-bundle over $L(p, q)$. Without loss of generality we can take $T_1^1$ be the first factor of $T^{n-2} = T^1 \times \cdots \times T^1$. But, this bundle is classified by $[L(p, q), K(z, 2)] \cong Z_p$, and (see [5]) for any element $f_i \in [L(p, q), K(z, 2)]$, $i \in Z_p$, the total space of the principal $T_1^1$-bundle determined by $f_i$ is $L(m, q) \times S^1$, where $m = \gcd(i, p)$. Take a circle subgroup $T_2^1$ in $T^{n-3}$ as in the first case and denote the complementary subgroup by $T^{n-4}$. Then $M/T^{n-4}$ is principal $T_2^1$-bundle over $L(m, q) \times S^1$. This bundle is also classified by

$$[L(m, q) \times S^1, K(Z, 2)] \cong H^2(L(m, q) \times S^1, Z) .$$

Let $\xi \in [L(m, q) \times S^1, K(Z, 2)]$ and denote its total space by $E'$. Consider the following diagram:

$$\begin{array}{ccc} E' & \xrightarrow{\Pi'} & L(m, q) \times S^1 \\ \downarrow & & \downarrow \Pi \\ E'' & \xrightarrow{\Pi''} & L(m, q) . \end{array}$$

Here $E''$ is the total space of $\xi$ restricted to $L(m, q) \times t$, where $t$ is any chosen point of $S^1$. Here $\Pi'$ and $\Pi''$ are bundle maps and $\Pi$ is the projection map onto the first coordinate $L(m, q)$. Now $E'$ is the pull-back of $E''$ relative to the projection map $\Pi$, so that we have $E' = E'' \times S^1$. Since $\xi$ restricted to $L(m, q) \times t$ is an element of $[L(m, q), K(Z, 2)] \cong Z_m$ we can consider $f_j \in [L(m, q), K(Z, 2)]$, for some $j \in Z_m$ as representing this bundle element whose total space is $E''$. But $E'' \cong L(d, q) \times S^1$ as before, where $d = \gcd(j, m)$. Hence $E' \cong L(d, q) \times S^1 \times S^1 \cong L(d, q) \times T^2$. Repeating this process a finite number of times we eventually get $M \cong L(k, q) \times T^{n-2}$ for some $k \leqq p$.

THEOREM. *If $T^n$ acts effectively on a compact closed orientable $(n+1)$-manifold $M^{n+1}$, then $M^{n+1}$ must be either $T^{n+1}$ or $L(k, q) \times T^{n-2}$ for $n \geqq 3$.*

*Proof.* If $M^* = S^1$, then every point on $S^1$ corresponds to a principal orbit, and the total space is a $T^n$-bundle over $S^1$. But these bundles are classified by

$$[S^1, K(Z, 2) \times \cdots \times K(Z, 2)] = H^2(S^1, Z + \cdots + Z) = 0 ,$$

so that the bundle is trivial and $M = S^1 \times T^n = T^{n+1}$.

If $M^* = [0, 1]$, then by Lemma 1 there are only two circle subgroups of $T^n$ corresponding to the stability groups at 0 and 1. Let $T_0$ be a subgroup generated by these two circle subgroups. Then any $(n-2)$-dimensional subgroup $T^{n-2}$ of $T^n$ which is complementary to $T_0$ acts freely on $M$. Then $M/T^{n-2}$ is a 3-dimensional orientable manifold $\bar{M}$ and $T_0$ acts on it so that $\bar{M}\backslash T_0 \cong [0, 1]$. But $T_0$ actions on 3-manifolds whose orbit spaces are isomorphic to $[0, 1]$ are classified as lens spaces $L(p, q)$ in [2]. Now, since $T^{n-2}$ acts freely on $M$, $M$ is a principal $T^{n-2}$-bundle over $L(p, q)$. But these bundles can be written as $L(k, q) \times T^{n-2}$ by the Lemma 3.

REMARK. Since the maximal torus subgroup of $SO(m)$ is $T^n$ where $m = 2n$ or $m = 2n + 1$, we see that $(T^n, M^m)$ can have no fixed points unless $m > 2n$ or $m > 2n + 1$. Also we can see from the theorem that a compact simply- connected $(n + 1)$-manifold does not allow effective $T^n$ actions for $n \geqq 3$. Thus extending a result of R. Richardson, Jr. [4] which says that $T^3$ cannot act effectively on the 4-dimensional sphere $S^4$.

## REFERENCES

1. A. Borel, et al., *Seminar on transformation groups*, Ann. of Math. Studies, No. 46, Princeton Univ. Press, Princeton, N. J., 1960.
2. P. Orlik and F. Raymond, *Actions of the torus on 4-manifolds*. 1, Trans. Amer. Math. Soc., **152** (1970), 531-559.
3. J. Pak, *Actions of torus $T^3$ on 4-manifolds $M^4$*, to appear.
4. R. Richardson, Jr., *Groups acting on the 4-sphere*, Illinois J. Math., **5** (1961), 474-485.
5. M. Thornton, *Total spaces of circle bundles over lens spaces*, to appear.

Received October 18, 1971.

WAYNE STATE UNIVERSITY

# PACIFIC JOURNAL OF MATHEMATICS




Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.


The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

# Pacific Journal of Mathematics

Vol. 44, No. 2 June, 1973