

Pacific Journal of Mathematics

***-ACTIONS IN A^* -ALGEBRAS**

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Let U be the open unit disk in the complex plane and f a function defined on U . We show that if A is an infinite dimensional dual B^* -algebra, then f defines a $*$ -action in A if and only if f is continuous at zero and $f(0) = 0$. We also obtain that if A is commutative, then f defines a continuous action in A if and only if f is continuous on U and $f(0) = 0$.

Actions in Banach algebras were introduced and studied recently by Gulick in [1]. Most of her main results were obtained for certain subalgebras of the algebra of all completely continuous operators on a Hilbert space. By using a different approach, we generalize some results in [1].

2. Preliminaries and notation. For any set S in an algebra A , let $L_A(S)$ and $R_A(S)$ denote the left and right annihilators of S in A . A Banach algebra A is called a dual algebra if, for every closed left ideal I and every closed right ideal J , we have $I = L_A(R_A(I))$ and $J = R_A(L_A(J))$. For each element $x \in A$, $Sp_A(x)$ will denote the spectrum of x in A .

Let B be a commutative Banach algebra and X_B its carrier space. For each $x \in B$, we let $x \rightarrow \hat{x}$ be the Gelfand map on B defined by $\hat{x}(\alpha) = \alpha(x)$ for all $\alpha \in X_B$.

All algebras under consideration are over the complex field C . Definitions not explicitly given are taken from Rickart's book [5].

3. Lemmas. In this section, we give two lemmas which are useful in § 4.

LEMMA 3.1. *Let A be an A^* -algebra. If there exists a maximal commutative $*$ -subalgebra B of A which is finite dimensional, then A is finite dimensional.*

Proof. Since B is finite dimensional, B has an identity element e such that $e = \sum_{i=1}^n e_i$, where $\{e_i, i = 1, \dots, n\}$ is the maximal orthogonal family of hermitian minimal idempotents in B . We claim that e is an identity element of A . In fact, for each $a \in A$, let $b = a(1 - e)$. It is straightforward to show that $b^*b \in B$ and $b^*b = 0$. Therefore $b = 0$ and so $a = ae$. Similarly we can show that $a = ea$. Hence e is an identity element of A . Clearly $A = \sum_{i=1}^n \sum_{j=1}^n e_i A e_j$. To complete the proof, it suffices now to show that $e_i A e_j$ is one

dimensional. We may assume $e_i A e_j \neq (0)$. Then there exists an element $x \in A$ such that $e_i x e_j \neq 0$ and so

$$0 \neq (e_i x e_j)(e_i x e_j)^* = e_i x e_j x^* e_i = \lambda e_i,$$

where $\lambda \in C$. Now for each $y \in A$, we have

$$e_i y e_j = \lambda^{-1} e_i x e_j x^* e_i y e_j = \lambda^{-1} e_i x (\lambda' e_j) = \lambda^{-1} \lambda' e_i x e_j,$$

where $\lambda' \in C$. Hence $e_i A e_j$ is one dimensional and this completes the proof.

LEMMA 3.2. *Let A be an A^* -algebra. If the spectrum of every hermitian element of A is finite, then A is finite dimensional.*

Proof. Let B be a maximal commutative $*$ -subalgebra of A . It follows easily from [5, p. 111, Theorem (3.1.6)] that every element of B has a finite spectrum and therefore B is finite dimensional (see [3, p. 376, Lemma 7]). Hence by Lemma 3.1, A is finite dimensional.

4. A^* -algebras and $*$ -actions. In this section, the symbol U denotes the open unit disk in the complex. For a given Banach $*$ -algebra A , we let A_1^* be the set $\{x \in A: xx^* = x^*x \text{ and } Sp_A(x) \subset U\}$. A function f on U is said to define a $*$ -action in A if there exists a mapping $x \rightarrow f'(x)$ of A_1^* into A such that whenever B is a maximal commutative $*$ -subalgebra of A and $x \in B \cap A_1^*$, then $f'(x) \in B$ and $\widehat{f'(x)} = f \circ \widehat{x}$ on the carrier space X_B of B .

THEOREM 4.1. *Let A be an A^* -algebra. Then A is finite dimensional if and only if any function f on U defines a $*$ -action in A .*

Proof. Suppose A is finite dimensional. Let $x \in A_1^*$ and let B be a maximal commutative $*$ -subalgebra of A containing x . Then B is a finite dimensional dual B^* -algebra. Hence the carrier space X_B of B consists of a finite number of elements, say $\alpha_1, \dots, \alpha_n$. Let e_i be the element of B corresponding to the characteristic function of the point $\alpha_i (i = 1, \dots, n)$. Then for each $x \in B$, we have $x = \sum_{i=1}^n \alpha_i(x) e_i$ (see [4, p. 21]). By [5, p. 111, Theorem (3.1.6.)],

$$Sp_B(x) = \{\alpha_i(x); i = 1, \dots, n\}.$$

Let f be any function on U . Define

$$f'(x) = \sum_{i=1}^n f(\alpha_i(x)) e_i.$$

Then it is easy to see that $f'(x) \in B$ and $\widehat{f'(x)} = f \circ \widehat{x}$. Therefore f defines a $*$ -action in A .

Conversely suppose that any function f on U defines a $*$ -action in A . If A were not finite dimensional, then by Lemma 3.2 there would exist an element x in A_1^* such that $Sp_A(x)$ is infinite. Let B be a maximal commutative $*$ -subalgebra of A containing x . Choose $\lambda_n \in Sp_A(x)$ such that $\lambda_n \neq 0$ ($n = 1, 2, \dots$). Let f be any function on U such that $f(\lambda_n) = n$. Since f defines a $*$ -action, there exists some $f'(x) \in B$ such that $f'(\widehat{x}) = f \circ \widehat{x}$. But this means $n = f(\lambda_n) \in Sp_A(f'(x))$, contradicting the boundedness of $Sp_A(f'(x))$. Hence A is finite dimensional and the proof is complete.

THEOREM 4.2. *Let A be an infinite dimensional dual A^* -algebra which is a dense two-sided ideal of a B^* -algebra. If a function f on U defines a $*$ -action in A , then f is continuous at 0 and $f(0) = 0$.*

Rroof. Let B be a maximal commutative $*$ -subalgebra of A . By [4, p. 31, Theorem 19], B is a dual algebra and so its carrier space X_B is discrete. For each $\alpha \in X_B$, let e_α be the element of B corresponding to the characteristic function of α . Then $\{e_\alpha: \alpha \in X_B\}$ is a maximal orthogonal family of hermitian minimal idempotents in A . By Lemma 3.1, B is infinite dimensional and so X_B is infinite. Therefore we can choose a countable subset $\{\alpha_n\}$ of X_B such that the complement $\{\alpha_n\}'$ of $\{\alpha_n\}$ in X_B is infinite.

Let $\{a_n\}$ be a sequence in U such that $a_n \rightarrow 0$. We want to show $f(a_n) \rightarrow f(0) = 0$. By passing to a subsequence, we can assume that $|a_n| \leq (n^2 \|e_{\alpha_n}\|)^{-1}$. Then $x = \sum_{n=1}^\infty e_n e_{\alpha_n}$ is defined in B . Clearly $x \in A_1^*$. Hence there exists some $f'(x) \in B$ such that $f'(\widehat{x}) = f \circ \widehat{x}$ on X_B . By [4, p. 30, Theorem 16], we have

$$(4.1) \quad f'(x) = \sum_\alpha e_\alpha f'(x) e_\alpha = \sum_\alpha \alpha(f'(x)) e_\alpha .$$

Therefore $\alpha(f'(x)) \rightarrow 0$. Since $\alpha_n(x) = a_n$, we have $f(a_n) = \alpha_n(f'(x))$. Thus it follows that $f(a_n) \rightarrow 0$ as $n \rightarrow \infty$. For each $\alpha \in \{\alpha_n\}'$, $\alpha(x) = 0$ and so $\alpha(f'(x)) = f(\alpha(x)) = f(0)$. Since $\{\alpha_n\}'$ is infinite, it follows easily from (4.1) that $\alpha(f'(x)) = 0$ for all $\alpha \in \{\alpha_n\}'$. Hence $f(0) = 0$ and so f is continuous at 0. This completes the proof.

Theorem 4.2 is a generalization of [1, p. 668, Proposition 5.1], since $Cp(1 \leq p < \infty)$ and their $*$ -subalgebras are dual A^* -algebras which are dense two-sided ideals of their completions in the auxiliary norm (see [6]).

We remark that the converse of Theorem 4.2 does not hold as is shown by the following example.

EXAMPLE. Let A be an infinite dimensional proper H^* -algebra. Then A is a dual A^* -algebra which is a dense two-sided ideal of its

completion in an auxiliary norm (see [4, p. 31]). Let B be a maximal commutative $*$ -subalgebra of A and let $\{e_\alpha: \alpha \in X_B\}$ be the maximal orthogonal family of hermitian minimal idempotents given in the proof of Theorem 4.2. Let $\{e_{\alpha_n}: \alpha_n \in X_B\}$ be a countable subset of $\{e_\alpha: \alpha \in X_B\}$ and let $a_n = (n \|e_{\alpha_n}\|)^{-1}$. Then $x = \sum_{n=1}^\infty a_n e_{\alpha_n}$ is defined in B and $\|x\|^2 = \sum_{n=1}^\infty n^{-2}$. Define a function f on U by $f(z) = (\sqrt{n} \|e_{\alpha_n}\|)^{-1}$ if $z = a_n$ and $f(z) = 0$ otherwise. Then f is continuous at 0. If f defines a $*$ -action in A , then there exists an element $f'(x) \in B$ such that $\widehat{f'(x)} = f \circ \hat{x}$. But

$$\|f'(x)\|^2 = \sum_{n=1}^\infty |f(a_n)|^2 \|e_{\alpha_n}\|^2 = \sum_{n=1}^\infty n^{-1}.$$

This is a contradiction. Therefore f does not define a $*$ -action in A .

THEOREM 4.3. *Let A be an infinite dimensional dual B^* -algebra. Then a function f on U defines a $*$ -action in A if and only if f is continuous at 0 and $f(0) = 0$.*

Proof. Suppose f is continuous at 0 and $f(0) = 0$. Let $x \in A_1^*$ and let B be a maximal commutative $*$ -subalgebra of A containing x . By the proof of Theorem 4.2, $x = \sum_{n=1}^\infty \alpha_n(x) e_{\alpha_n}$, where $\alpha_n \in X_B$ and e_{α_n} is the element of B corresponding to the characteristic function of α_n . Since $\alpha_n(x) \rightarrow 0, f(\alpha_n(x)) \rightarrow 0$. For any two positive integers $m, n (m \leq n)$, it follows easily from the commutativity of B that

$$\left\| \sum_{i=m}^n f(\alpha_i(x)) e_{\alpha_i} \right\| = \max \{ |f(\alpha_i(x))| : i = m, \dots, n \}.$$

Therefore $\sum_{n=1}^\infty f(\alpha_n(x)) e_{\alpha_n}$ is defined in B . Now let $f'(x) = \sum_{n=1}^\infty f(\alpha_n(x)) e_{\alpha_n}$. Then $\widehat{f'(x)} = f \circ \hat{x}$. Hence f defines a $*$ -action in A . The converse of the theorem follows from Theorem 4.2 and the proof is complete.

Since the algebra of all completely continuous operators on a Hilbert space is a dual B^* -algebra, Theorem 4.3 generalizes [1, p. 668, Theorem 5.2].

THEOREM 4.4. *Let A be an infinite dimensional commutative dual B^* -algebra and f a function on U . Then f defines a continuous action in A (see [2, p. 109, Definition 5.1]) if and only if f is a continuous function on U and $f(0) = 0$.*

Proof. Suppose f is continuous and $f(0) = 0$. Then by Theorem 4.3, f defines an action in A . Let x_n and $x \in A_1^*$ such that $x_n \rightarrow x$ in A . By the proof of Theorem 4.2, we have

$$x = \sum_{\alpha} \alpha(x_n) e_{\alpha} \quad \text{and} \quad x = \sum_{\alpha} \alpha(x) e_{\alpha},$$

where $\{e_\alpha: \alpha \in X_A\}$ is the maximal orthogonal family of hermitian minimal idempotents in A . Since A is commutative, we have

$$\|x_n - x\| = \sup\{\|\alpha(x_n) - \alpha(x)\|: \alpha \in X_A\}$$

and

$$\|f(x_n) - f(x)\| = \sup\{\|f(\alpha(x_n)) - f(\alpha(x))\|: \alpha \in X_A\}.$$

Therefore it is now easy to see that $f(x_n) \rightarrow f(x)$ in A . Hence f defines a continuous action in A . The converse of the theorem follows from [2, p. 109, Proposition 5.2] and Theorem 4.3.

REMARK. If A is noncommutative, then Theorem 4.4 is not true as is shown in [2, p. 110, Example 5.3].

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