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BOUNDED APPROXIMATION BY RATIONAL FUNCTIONS

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In which it is proved that $H^\infty(\lambda_Q) \cap C(K) = R(K)$.

1. Introduction. Let K be a compact subset of the complex plane C , and let $R(K)$ be the uniform closure in $C(K)$ of the rational functions with poles off K . Denote by Q the set of nonpeak points of $R(K)$, let λ_Q be the area measure $\lambda = dxdy$ restricted to Q , and let $H^\infty(\lambda_Q)$ be the weak-star closure of $R(K)$ in $L^\infty(\lambda_Q)$. Our first goal is to prove the following theorem.

THEOREM 1.1. $H^\infty(\lambda_Q) \cap C(K) = R(K)$.

As an immediate consequence, we can state the following more concrete version of the result: If $f \in C(K)$, and if there is a bounded sequence f_n in $R(K)$ converging pointwise almost everywhere ($dxdy$) to f , then $f \in R(K)$.

Theorem 1.1 will be essentially a corollary of the following theorem of A. M. Davie [5].

DAVIE'S THEOREM. *If $f \in H^\infty(\lambda_Q)$, there is a sequence f_n in $R(K)$ such that $\|f_n\| \leq \|f\|$, and $f_n(q) \rightarrow f(q)$ for almost all ($dxdy$) points $q \in Q$.*

Actually, Davie states explicitly the above result only for those $f \in H^\infty(\lambda_Q)$ which are weak-star limits of bounded sequences in $R(K)$. It follows then from the Krein-Schmulian theorem that the space of such functions is weak-star closed, and so must coincide with $H^\infty(\lambda_Q)$.

We will offer three proofs of Theorem 1.1. The first proof, given in §3, depends also on Vitushkin's description of the functions in $R(K)$. In §§4 and 5, we present two "abstract" proofs of Theorem 1.1, which are dual to each other. All three proofs use a localization procedure.

In §§6 and 7, some extensions of Theorem 1.1 are obtained by the methods of the abstract proof. These are pursued in the setting of uniform algebras.

The first extension is a qualitative form of Theorem 1.1, which is inspired by a theorem of Sarason ([12]; see also [16], [6]). If $h \in C(K)$, then the distance from h to $R(K)$ is defined by

$$d(h, R(K)) = \inf \{\|h - f\| : f \in R(K)\},$$

and the distance $d(h, H^\infty(\lambda_Q))$ is defined similarly. The results of §6,

applied to $R(K)$, will yield the following theorem, which has Theorem 1.1 as an immediate consequence.

THEOREM 1.2. *$d(h, R(K)) = d(h, H^\infty(\lambda_q))$ for all $h \in C(K)$. Equivalently, ball $[R(K)^\perp \cap L^1(\lambda_q)]$ is weak-star dense in ball $R(K)^\perp$.*

B. Cole had observed (unpublished) that a result like Davie's theorem could be used to show that every nonpeak point $q \in Q$ has a representing measure which is absolutely continuous with respect to λ_q . The results of §7, again specialized to the algebra $R(K)$, include the following extension of Cole's result.

THEOREM 1.3. *For each $q \in Q$, the representing measures for q which are absolutely continuous with respect to λ_q are weak-star dense in the set of all representing measures for q .*

Theorem 1.1 can also be derived immediately from Theorem 1.3 and the abstract Hardy space theory. In fact, combining Theorem 1.3 with a theorem from [10, p. 334], we obtain the following.

COROLLARY 1.4. *If $f \in C(K)$, and f lies in the closure of $R(K)$ in $L^1(\nu)$ for all representing measures ν for points $q \in Q$ satisfying $\nu \ll \lambda_q$, then $f \in R(K)$.*

In connection with Theorem 1.3, we mention that it may be impossible to find representing measures for $R(K)$ which are boundedly absolutely continuous with respect to the area measure λ_q .¹ In fact, Wilken [15] has shown that there are compact planar sets K with the following property: If $h \in L^1(\lambda_q)$, then the set of points $q \in Q$ which have representing measures boundedly absolutely continuous with respect to $h d\lambda_q$ has zero area.

In §8 we discuss various other situations in which the abstract results apply.

In §§9, and 10 we characterize the function in $H^\infty(\lambda_q)$ in terms of analytic capacity, using the methods of rational approximation theory. The characterization is parallel to Vitushkin's theorem, and when coupled with Theorem 1.1, it actually yields a sharper form of that theorem. For example, if f is continuous on the plane, then it was known that $f|_K \in R(K)$ whenever there is $a(\delta) \rightarrow 0$ ($\delta \rightarrow 0$) such that for any square S of side δ ;

¹ J. Brennan has pointed out that such an example is also provided by Sinanyan's example [A. M. S. Transl. **74** (1968), Theorem 2.4 on p. 114] of a compact K such that $R(K) \neq C(K)$, while $R(K)$ is dense in $L^p(K, dx dy)$ for all $p < \infty$.

$$\left| \int_{\partial S} f(z) dz \right| \leq a(\delta) \gamma(S \setminus K) ,$$

where γ denotes analytic capacity. We are able to replace the condition by

$$\left| \int_{\partial S} f(z) dz \right| \leq \text{const } \gamma(S \setminus K) .$$

In §11 some applications of the characterization of $H^\infty(\lambda_Q)$ are given.

2. Notation and background. The notation will be as in [7], except that $\Delta(z, \delta)$ will denote the *open* disc with center at z and radius δ . All norms will be supremum norms. All measures are finite regular Borel measures. If f is a bounded function defined on a subset E of C , then the oscillation of f on E is denoted by

$$\text{osc}(f, E) = \sup \{|f(z) - f(w)| : z, w \in E\} .$$

We repeatedly use the following lemma, whose two assertions are dual to one another.

LEMMA 2.1. *If $\tau \in R(K)^\perp$, then its Cauchy transform*

$$\hat{\tau}(z) = \int d\tau(\zeta)/(\zeta - z)$$

vanishes a.e. $(dxdy)$ off Q . If h is a compactly supported bounded Borel function, then

$$H(\zeta) = \iint_{C \setminus Q} \frac{h(z)}{z - \zeta} dxdy$$

is in $R(K)$.

Proof. See [1, p. 171 Corollary 3.3.2].

This lemma implies that any $\tau \in R(K)^\perp$ is supported on \bar{Q} and that $R(K)$ consists of all continuous extensions to K of the functions in $R(\bar{Q})$. The set Q is the smallest subset of K , modulo sets of zero area, for which either assertion of 2.1 is true. It is for that reason that we study convergence on Q rather than on, for instance, the interior K° of K .

If g is a smooth function with compact support, the localization operator T_g is defined by

$$T_g f(\zeta) = g(\zeta) f(\zeta) + \frac{1}{\pi} \iint \frac{f(z)}{z - \zeta} \frac{\partial g}{\partial \bar{z}} dxdy ,$$

where f is any bounded Borel function. We use the following facts: $T_g f$ is analytic wherever f is analytic; $T_g f$ is analytic off $\text{supp } g$, the closed support of g ; and $f - T_g f$ is analytic on the interior of the level set $g^{-1}(1)$. If J is a compact subset of the Riemann sphere disjoint from $\text{supp } g$, then $T_g f \in R(K \cup J)$ whenever $f \in R(K)$ [or, more precisely, whenever $f|_K \in R(K)$]. If g is supported by a disc $\Delta(z_0; \delta)$, then

$$|T_g f| \leq 2\delta \operatorname{osc}(f, \Delta(z_0, \delta)) \left\| \frac{\partial g}{\partial \bar{z}} \right\|.$$

For a discussion of these and other properties of the operator T_g , see [7, VIII. 7.1].

If S is a set of measures, then we write $\tau \ll S$ if $\tau \ll \nu$ for some $\nu \in S$. The second and third proofs of Theorem 1.1 depend on the following equivalent formulation of Davie's theorem, which we state in slightly more generality than necessary.

DAVIE'S THEOREM (alternate version): *Let τ be a measure on K such that $\tau \ll R(K)^\perp$. Then the restriction map $H^\infty(\lambda_Q + |\tau|) \rightarrow H^\infty(\lambda_Q)$ is an isometric isomorphism.*

Proof. Choose $\tau' \in R(K)^\perp$ such that $\tau \ll \tau'$. Let B and B' consist respectively of the pointwise bounded limits of $R(K)$ in $H^\infty(\lambda_Q + |\tau|)$ and $H^\infty(\lambda_Q + |\tau'|)$, and consider the restriction mappings $B' \rightarrow B \rightarrow H^\infty(\lambda_Q)$. Davie's proof of Theorem 1 in [5] shows that the composite map $B' \rightarrow H^\infty(\lambda_Q)$ is an isometry. Since the map $B' \rightarrow B$ is onto, the map $B \rightarrow H^\infty(\lambda_Q)$ is an isometry. The Krein-Schmulian theorem then shows that B is weak-star closed, so that B must coincide with $H^\infty(\lambda_Q + |\tau|)$.

3. First proof of Theorem 1.1. Vitushkin has shown that the following conditions are equivalent, for a continuous function f defined on C :

(i) $f \in R(K)$

(ii) For every $z_0 \in C$, every $\delta > 0$, and every smooth function g supported on $\Delta(z_0; \delta)$,

$$\left| \iint f \frac{\partial g}{\partial \bar{z}} dx dy \right| \leq 2\pi\delta \left\| \frac{\partial g}{\partial \bar{z}} \right\| \operatorname{osc}(f, \Delta(z_0; \delta)) \gamma(\Delta(z_0; \delta) \setminus K).$$

(iii) There exists $r \geq 1$ and a positive function $a(\delta)$ tending to zero with δ , such that

$$\left| \iint f \frac{\partial g}{\partial \bar{z}} dx dy \right| \leq \delta a(\delta) \left\| \frac{\partial g}{\partial \bar{z}} \right\| \gamma(\Delta(z_0; r\delta) \setminus K)$$

whenever $\delta > 0$, $z_0 \in K$, and g is a smooth function supported on $\Delta(z_0; \delta)$.

For a discussion of the analytic capacity γ and of Vitushkin's theorem, see [14] or [7]. The following lemma extends the implication "(i) \Rightarrow (ii)" of Vitushkin's theorem to cover functions in $H^\infty(\lambda_Q)$.

LEMMA 3.1. *Let λ_Q be the area measure on the nonpeak points Q of $R(K)$, and let f be a bounded Borel function on C . If $f \in H^\infty(\lambda_Q)$, then*

$$\left| \iint f(z) \frac{\partial g}{\partial \bar{z}} dx dy \right| \leq 2\pi\delta \operatorname{osc}(f, \Delta(z_0; \delta)) \left\| \frac{\partial g}{\partial \bar{z}} \right\| \gamma(\Delta(z_0; \delta) \setminus K)$$

for any smooth function g supported on a disc $\Delta(z_0; \delta)$.

Proof. Since

$$(T_g f)'(\infty) = \frac{1}{\pi} \iint f(z) \frac{\partial g}{\partial \bar{z}} dx dy,$$

we must study $T_g f$ and find the appropriate estimate for $(T_g f)'(\infty)$.

Let J be the complement of $\Delta(z_0; \delta)$ in the Riemann sphere. Then the nonpeak points of $R(K \cup J)$ all lie on $Q \cup J$. We intend to apply Davie's theorem to the weak-star closure $H^\infty(\lambda_{Q \cup J})$ of $R(K \cup J)$ in $L^\infty(\lambda_{Q \cup J})$.

Let $\{f_n\}$ be a bounded sequence in $R(K)$ which converges pointwise a.e. to f on Q . Then the functions

$$F_n(\zeta) = g(\zeta)f_n(\zeta) + \frac{1}{\pi} \iint_Q \frac{f_n(z)}{z - \zeta} \frac{\partial g}{\partial \bar{z}} dx dy$$

belong to $R(K \cup J)$, because by Lemma 2.1 the difference $f_n - T_g f_n$ belongs to $R(K \cup J)$. Moreover, $\{F_n\}$ converges pointwise boundedly to the function

$$g(\zeta)f(\zeta) + \frac{1}{\pi} \iint_Q \frac{f(z)}{z - \zeta} \frac{\partial g}{\partial \bar{z}} dx dy,$$

so that this function belongs to $H^\infty(\lambda_Q \cup J)$. Again by Lemma 2.1, we find that $T_g f \in H^\infty(\lambda_{Q \cup J})$. By Davie's theorem, there is a sequence $\{h_n\}$ of functions which are analytic on $K \cup J$, such that $\|h_n\| \leq \|T_g f\|$, and $\{h_n\}$ converges pointwise a.e. to $T_g f$ on $Q \cup J$. The inequality

$$|h'_n(\infty)| \leq \|h_n\| \gamma(\Delta(z_0; \delta) \setminus K)$$

leads in the limit to

$$|(T_g f)'(\infty)| \leq \|T_g f\| \gamma(\Delta(z_0; \delta) \setminus K).$$

If we substitute here the expression above for $(T_g f)'(\infty)$ and the

standard estimate (§2) for $\|T_g f\|$, we obtain the estimate of the lemma. That completes the proof.

Now Theorem 1.1 follows immediately from Lemma 3.1 and Vitushkin's theorem. Indeed, if $f \in C(K) \cap H^\infty(\lambda_Q)$, then Lemma 3.1 shows that any bounded continuous extension of f to C satisfies condition (ii) of Vitsushkin's theorem, so that $f \in R(K)$.

4. Second proof of Theorem 1.1. Theorem 1.1 can be easily proved using some functional analysis instead of Vitushkin's theorem. Our original such proof (§5) was simplified by B. Cole, and we present his proof in this section. His key idea is to localize annihilating measures rather than functions. Recall from §2 that each $\tau \ll R(K)^\perp$ is supported on \bar{Q} , while $\text{supp } \lambda_Q = \bar{Q}$.

LEMMA 4.1. *Let $\tau \ll R(K)^\perp$. For each $p \in \text{supp } \tau$ and each $F \in H^\infty(\lambda_\infty + |\tau|)$,*

$$\tau - \text{ess } \limsup_{z \rightarrow p} |F(z)| \leq \lambda_Q - \text{ess } \limsup_{z \rightarrow p} |F(z)|.$$

Proof. Set $c = \lambda_Q - \text{ess } \limsup |F(z)|$. Let $\varepsilon > 0$, and let Δ be a small disc centered at p such that $|F| \leq c + \varepsilon$ a.e. (λ_Q) on $Q \cap \Delta$. Let g be a smooth function supported on Δ such that $g = 1$ near p , and set

$$\tau' = g\tau - \frac{1}{\pi} \frac{\partial g}{\partial \bar{z}} \hat{\tau}(z) dx dy.$$

Then $\tau'^\perp R(K \cap \bar{\Delta})$, and there are several ways to see this. One way is to note that $\hat{\tau}' = g\hat{\tau}$ (see the proof of the Bishop splitting lemma in [7, p. 51]). Another way is to observe that $\int T_g f d\tau = \int f d\tau'$ for all continuous functions f , while $T_g f \in R(K)$ whenever $f \in R(K \cap \bar{\Delta})$.

Since $\hat{\tau} = 0$ a.e. off Q , we have $\tau' \ll \lambda_{Q \cap \Delta} + |\tau|$, and the restriction $F|_\Delta$ is in the weak-star closure of $R(K \cap \bar{\Delta})$ in $L^\infty(\lambda_{Q \cap \Delta} + |\tau'|)$. Applying the alternate version of Davie's theorem (§2) to the algebra $R(K \cap \bar{\Delta})$, we obtain $|F| \leq c + \varepsilon$ a.e. $(\lambda_{Q \cap \Delta} + |\tau'|)$. Since τ coincides with τ' near p , we obtain

$$\tau - \text{ess } \limsup_{z \rightarrow p} |F(z)| = \tau' - \text{ess } \limsup_{z \rightarrow p} |F(z)| \leq c + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we obtain the estimate in 4.1.

Now to prove Theorem 1.1, we suppose that $f \in H^\infty(\lambda_Q) \cap C(K)$. To show that $f \in R(K)$, it suffices to show that $\int f d\tau = 0$ for each $\tau^\perp R(K)$. So fix such a τ , and choose $F \in H^\infty(\lambda_Q + |\tau|)$ such that $F = f$ a.e. $(d\lambda_Q)$. If $p \in \bar{Q}$, then

$$\lambda_Q - \text{ess} \limsup_{z \rightarrow p} |f(z) - f(p)| = 0.$$

By Lemma 4.1,

$$\tau - \text{ess} \limsup_{z \rightarrow p} |f(z) - f(p)| = 0, \quad p \in \text{supp } \tau.$$

It follows that $F = f$ a.e. $(d\tau)$. Since τ is orthogonal to $H^\infty(\lambda_Q + |\tau|)$, we conclude, that $0 = \int F d\tau = \int f d\tau$, as required.

5. Third proof of Theorem 1.1. In this section, we give another functional analytic proof of Lemma 4.1, thereby obtaining a third proof of Theorem 1.1. We begin with two lemmas.

LEMMA 5.1. *If $f \in H^\infty(\lambda_Q)$ is analytic at $p \in \mathbb{C}$, then $[f - f(p)]/(z - p) \in H^\infty(\lambda_Q)$.*

Proof. This is trivial if $p \notin K$, and it follows by a simple limiting argument if $p \in \partial K$. If $p \in K^\circ$, and f is a weak-star limit of the net $\{f_\alpha\}$ in $R(K)$, then $[f - f(p)]/(z - p)$ is the weak-star limit of the net $\{[f_\alpha - f_\alpha(p)]/(z - p)\}$ in $R(K)$. That covers all cases.

LEMMA 5.2. *Let $f \in H^\infty(\lambda_Q)$, and let $p \in K$. Suppose there are $\delta > 0$ and $c > 0$ such that $|f| \leq c$ a.e. $(d\lambda_Q)$ on $\Delta(p; \delta)$. Then there is $f_0 \in H^\infty(\lambda_Q)$ such that $\|f_0\| \leq 9c$, and $f - f_0$ extends to be analytic in a neighborhood of p .*

Proof. Let g be a smooth function supported in $\Delta(p; \delta)$ such that $0 \leq g \leq 1$, $g = 1$ near p , and $|\partial g / \partial \bar{z}| \leq 4/\delta$, and set $f_0 = T_g f$. The properties of T_g and the standard estimate for $T_g f$ show that f_0 has the desired properties.

Alternative proof of Lemma 4.1. Let $\tau \ll R(K)^\perp$, let $F \in H^\infty(\lambda_Q + |\tau|)$ and let $p \in \bar{Q}$. Suppose the assertion of 4.1 is false for F ; that is, suppose

$$\tau - \text{ess} \limsup_{z \rightarrow p} |F(z)| > \lambda_Q - \text{ess} \limsup_{z \rightarrow p} |F(z)|.$$

By replacing F by a constant multiple of some high power of F , we can assume that

$$\begin{aligned} \lambda_Q - \text{ess} \limsup_{z \rightarrow p} |F(z)| &< 1 \\ \tau - \text{ess} \limsup_{z \rightarrow p} |F(z)| &> 100. \end{aligned}$$

Let f be the projection of F onto $H^\infty(\lambda_Q)$. Then there exists $\delta > 0$

such that $|f| < 1$ a.e. $(d\lambda_q)$ on $\mathcal{A}(p; \delta)$. Let f_0 be the function given by Lemma 5.2 and set $f_1 = f_0 + (f - f_0)(p)$. Then $f - f_1$ is analytic at p , and $(f - f_1)(p) = 0$. By Lemma 5.1, there exists $h \in H^\infty(\lambda_q)$ such that $f - f_1 = (z - p)h$. Since $\|f_0\| \leq 9$, we obtain the decomposition

$$(*) \quad f = f_1 + (z - p)h, \quad \|f_1\| \leq 10.$$

Using the alternative version of Davie's theorem (§2), we obtain $F_1, H \in H^\infty(\lambda_q + |\tau|)$ such that

$$F = F_1 + (z - p)H, \quad \|F_1\| \leq 10.$$

Consequently $\tau - \text{ess lim sup}_{z \rightarrow p} |F(z)| \leq 10$. This contradiction establishes the lemma.

The proof of Theorem 1.1 now follows as in §4.

The proof of Lemma 4.1 given here shows that if φ is any complex-valued homomorphism of $H^\infty(\lambda_q)$, and $\varphi(z) = p$ (a point of \bar{Q}), then

$$(**) \quad |\varphi(f)| \leq \lambda_q - \text{ess lim sup}_{z \rightarrow p} |f(z)|, \quad f \in H^\infty(\lambda_q).$$

In abstract terms, this means that the $H^\infty(\lambda_q)$ -convex hull of the fiber over p of the spectrum of $L^\infty(\lambda_q)$ includes the entire fiber over p of the spectrum of $H^\infty(\lambda_q)$.

To verify (**), one proceeds as follows. Suppose $f \in H^\infty(\lambda_q)$ satisfies $\lambda_q - \text{ess lim sup}_{z \rightarrow p} |f(z)| < 1$. Applying φ to the decomposition (*), we obtain $|\varphi(f)| = |\varphi(f_1)| \leq \|f_1\| \leq 10$. The same estimate applies to $\varphi(f^n) = \varphi(f)^n$ for all n , so that in fact $|\varphi(f)| \leq 1$, the estimate (**) now follows easily.

Finally, we mention that the proof of §4 can be regarded in some sense as the adjoint of the preceding proof. Indeed, the measure τ' produced in §4 coincides with $T_g^* \tau$. The proof of §4 is shorter and avoids the estimates for the operator T_g . It does not however seem to yield the additional information concerning the spectrum of $H^\infty(\lambda_q)$.

6. Distance estimates. In this section and the next, we fix a compact space X , a uniform algebra A on X , and a positive measure σ on X . The weak-star closure of A in $L^\infty(\sigma)$ will be noted by $H^\infty(\sigma)$. First we make the following elementary observation.

LEMMA 6.1. *Let τ be a measure on X , such that the restriction map $H^\infty(\sigma + |\tau|) \rightarrow H^\infty(\sigma)$ is an isometry. Then the map is onto, and it is a homeomorphism between the weak-star topologies inherited from $L^\infty(\sigma + |\tau|)$ and $L^\infty(\sigma)$ respectively.*

Proof. An application of the Krein-Schmulian theorem shows that the range of the restriction map is weak-star closed in $L^\infty(\sigma)$, and hence it must coincide with $H^\infty(\sigma)$. Now $H^\infty(\sigma + |\tau|)$ and $H^\infty(\sigma)$ can be regarded as the dual spaces of $L^1(\sigma + |\tau|)/[A^\perp \cap L^1(\sigma + |\tau|)]$ and $L^1(\sigma)/[A^\perp \cap L^1(\sigma)]$ respectively, and the restriction map is the dual of the map

$$hd\sigma + [A^\perp \cap L^1(\sigma)] \rightarrow hd\sigma + [A^\perp \cap L^1(\sigma + |\tau|)] .$$

Since the restriction map is an onto isometry, the predual must also be an onto isometry, and the restriction map is a weak-star homeomorphism.

We will be interested in measures τ which have the following property:

$$(\#) \quad \text{supp } \tau \subseteq \text{supp } \sigma, \text{ and for all } F \in H^\infty(\sigma + |\tau|) \text{ and } x \in \text{supp } \tau , \\ \tau\text{-ess } \limsup_{y \rightarrow x} |F(y)| \leq \sigma\text{-ess } \limsup_{y \rightarrow x} |F(y)| .$$

Recall (see Lemma 4.1) that this property is enjoyed by any measure $\tau \ll R(K)^\perp$, in the case $X = K$, $A = R(K)$, and $\sigma = \lambda_Q$. As a simple consequence of the definition, we obtain the following.

LEMMA 6.2. *If τ satisfies (#), then the restriction map $H^\infty(\sigma + |\tau|) \rightarrow H^\infty(\sigma)$ is an isometry. Moreover, if $h \in C(X)$ and $F \in H^\infty(\sigma + |\tau|)$, then*

$$\|h - F\|_{L^\infty(\sigma + |\tau|)} = \|h - F\|_{L^\infty(\sigma)} .$$

Proof. The first assertion is obvious. To prove the second, let $x \in \text{supp } \tau$. Then

$$\begin{aligned} \tau\text{-ess } \limsup_{y \rightarrow x} |h(y) - F(y)| &= \tau\text{-ess } \limsup_{y \rightarrow x} |h(x) - F(y)| \\ &\leq \sigma\text{-ess } \limsup_{y \rightarrow x} |h(x) - F(y)| \\ &= \sigma\text{-ess } \limsup_{y \rightarrow x} |h(y) - F(y)| . \end{aligned}$$

It follows that $\|h - F\|_{L^\infty(\tau)} \leq \|h - F\|_{L^\infty(\sigma)}$, so that the norm equality asserted by the lemma is valid.

The following theorem is the first abstract version of Theorem 1.1.

THEOREM 6.3. *Let A be a uniform algebra on a compact space X , and let σ be a positive measure on X . Suppose that every $\tau \in A^\perp$ has property (#). Then*

$$d(h, A) = d(h, H^\infty(\sigma)) , \quad \text{all } h \in C(X) .$$

In particular, $H^\infty(\sigma) \cap C(X) = A$.

Proof. Fix $h \in C(X)$, and let $f \in H^\infty(\sigma)$. Choose a measure $\tau \in A^\perp$ such that $\|\tau\| = 1$, and

$$d(h, A) = \int h d\tau .$$

By Lemmas 6.1 and 6.3, there exists $F \in H^\infty(\sigma + |\tau|)$ such that $F = f$ a.e. ($d\sigma$). Since $\tau \perp H^\infty(\sigma + |\tau|)$,

$$\int h d\tau = \int (h - F) d\tau \leq \|h - F\|_{L^\infty(\sigma + |\tau|)} .$$

Applying Lemma 6.2, we obtain

$$d(h, A) \leq \|h - f\|_{L^\infty(\sigma)} .$$

Since this is true for all $f \in H^\infty(\sigma)$, we conclude that

$$d(h, A) \leq d(h, H^\infty(\sigma)) .$$

The reverse inequality is trivial.

The validity of distance estimates as in Theorem 6.3 has a dual formulation in terms of weak-star density of orthogonal measures. Indeed, let A be any closed subspace of $C(X)$, where X is compact, and let σ be a positive measure on X . Then

$$d(h, A) = d(h, H^\infty(\sigma)) , \quad \text{all } h \in C(X) ,$$

if and only if ball $[A^\perp \cap L^1(\sigma)]$ is weak-star dense in ball A^\perp . This assertion follows directly from the following expressions, valid for $h \in C(X)$:

$$d(h, A) = \sup \left\{ \left| \int h d\tau \right| : \tau \in \text{ball } A^\perp \right\}$$

and

$$d(h, H^\infty(\sigma)) = \sup \left\{ \left| \int h g d\sigma \right| : g d\sigma \in \text{ball } [A^\perp \cap L^1(\sigma)] \right\} .$$

If ball $[A^\perp \cap L^1(\sigma)]$ is weak-star dense in ball A^\perp , then the suprema coincide, and the distances are equal. On the other hand, if there exists $\tau \in \text{ball } A^\perp$ which does not lie in the weak-star closure of ball $[A^\perp \cap L^1(\sigma)]$, then the separation theorem for convex sets provides us with an $h \in C(X)$ for which

$$\sup \left\{ \left| \int h g d\sigma \right| : g d\sigma \in \text{ball } [A^\perp \cap L^1(\sigma)] \right\} < \left| \int h d\tau \right| ,$$

and the distance estimate fails.

The preceding remarks, together with Theorem 6.3, serve to complete the proof of Theorem 1.2.

7. Weak-star density of representing measures. In this section, A is again a uniform algebra on a compact space X , and σ is a positive measure on X . It will be convenient to reformulate property (#) of §6.

LEMMA 7.1. *A measure τ on X has property (#) if and only if τ has the following property:*

$$(\#\#) \quad \left\{ \begin{array}{l} \text{If } \mu \in C(X) \text{ is positive and if } F \in H^\infty(\sigma + |\tau|) \\ \text{satisfies } |F| \leq u \text{ a.e. } (d\sigma), \text{ then } |F| \leq u \text{ a.e. } (d\tau). \end{array} \right.$$

Proof. Evidently (#) implies (##). Suppose that (##) is valid. Let $p \in X \setminus \text{supp } \sigma$, and let u be a positive function in $C(X)$ such that $u = 1$ on $\text{supp } \sigma$, while $u(p) < 1$. Applying (##) to the function $f = 1 \in H^\infty(\sigma + |\tau|)$, we see that p cannot belong to the closed support of τ . Hence $\text{supp } \tau \subseteq \text{supp } \sigma$. It is easy to verify that the other requirement of (#) is also valid, so that (##) implies (#).

The maximal ideal space of A will be denoted by M_A . The version which we will require of Cole's theorem on the existence of absolutely continuous representing measures is the following.

LEMMA 7.2. *Let $\varphi \in M_A$. Suppose there is a representing measure τ for φ such that the restriction map $H^\infty(\sigma + \tau) \rightarrow H^\infty(\sigma)$ is an isometry. Then φ has a representing measure which is absolutely continuous with respect to σ .*

Proof. The homomorphism φ is continuous in the weak-star topology of $H^\infty(\sigma + |\tau|)$. By Lemma 6.1, the restriction map is a weak-star homeomorphism, so φ is continuous in the weak-star topology of $H^\infty(\sigma)$. Consequently φ has a complex representing measure, and hence a (positive) representing measure, which is absolutely continuous with respect to σ .

THEOREM 7.3. *Let $\varphi \in M_A$, and let τ be a representing measure for φ . If τ has property (#), then τ lies in the weak-star closure of the set of representing measures for φ which are absolutely continuous with respect to σ .*

Proof. Let $u \in C_R(X)$, and set

$$c = \inf \left\{ \int u d\lambda : \lambda \text{ a representing measure for } \varphi, \lambda \ll \sigma \right\}.$$

By the separation theorem for convex sets, it will suffice to show that $\int u d\tau \geq c$. Replacing u by $u - c$, we can assume that $c = 0$.

According to Lemma 7.2, there is a weak-star continuous extension $\tilde{\varphi}$ of φ from A to $H^\infty(\sigma)$. By the Hoffman-Rossi theorem [11], there is, for each $t > 0$, a function $f_t \in H^\infty(\sigma)$ satisfying

$$|f_t| \leq e^{tu} \text{ a.e. } d\sigma \\ \tilde{\varphi}(f_t) = 1.$$

Suppose $F_t \in H^\infty(\sigma + \tau)$ coincides with f_t a.e. $(d\sigma)$. Property (##) shows that

$$|F_t| \leq e^{tu} \text{ a.e. } d(\sigma + \tau).$$

Now $\int F_t d\tau = \tilde{\varphi}(F_t) = \varphi(f_t) = 1$, so that

$$0 = \left[\int F_t d\tau - 1 \right] / t \leq \int [e^{tu} - 1] d\tau / t.$$

Since $[e^{tu} - 1]/t$ converges uniformly to u as $t \rightarrow 0$, we obtain $0 \leq \int u d\tau$, as required.

Our second main abstract result is the following, which includes Theorem 1.3 as a special case.

COROLLARY 7.4. *Let A be a uniform algebra on a compact space X , and let σ be a positive measure on X . Suppose every measure $\tau << A^\perp$ has property (#). If $\varphi \in M_A$ is a nonpeak point, the set of representing measures for φ which are absolutely continuous with respect to σ is weak-star dense in the set of all representing measures on X for φ .*

Proof. Every representing measure τ for a nonpeak point satisfies $\tau << A^\perp$, so that Theorem 7.3 applies.

8. Property (#) for $R(K)$ and $A(U)$. The purpose of this section is to record some other cases in which the hypotheses of Theorem 6.3 and Corollary 7.4 are met.

If U is an open subset of the complex plane, then $A(U)$ is the algebra of bounded analytic functions on U which extend continuously to \bar{U} . The harmonic measure on ∂U for a point $z \in U$ will be denoted by μ_z . If the components of U are U_1, U_2, \dots , and z_j is any point of U_j , then the measure $\mu = \sum \mu_{z_j} / 2^j$ will be called harmonic measure for U .

THEOREM 8.1. *Every measure $\tau << A^\perp$ has property (#), for each*

of the following choices of compact set X , uniform algebra A on X , and positive measure σ :

- (i) $X = K$, $A = R(K)$, and $\sigma = \lambda_Q$ is the area measure on Q .
- (ii) $X = \partial K$, $A = R(K)$, and σ is the sum of the area measure on $Q \cap \partial K$ and the harmonic measure on ∂K for K° .
- (iii) $X = \bar{U}$, $A = A(U)$, and σ is the area measure on U .
- (iv) X is the closed support of harmonic measure on ∂U , $A = A(U)$, and σ is the harmonic measure on ∂U .

Proof. As already observed, case (i) follows from Lemmas 2.1 and 4.1. The proof which establishes Lemma 4.1 depends only on the fact that the algebra in question is invariant under the T_g operators, and therefore serves to establish the result in case (iii). In passing, we remark that Case (iii) is treated in detail in [6]. Cases (ii) and (iv) are similar, so we consider only case (iv).

So set $X = \text{supp } \mu$, and let τ be a measure on X which satisfies $\tau \ll A(U)^\perp$. Let $f \in H^\infty(\mu + |\tau|)$, and let u be a positive continuous function on X such that $|f| \leq u$ a.e. ($d\mu$). By Lemma 7.1, it suffices to show that $|f| \leq u$ a.e. ($d\tau$).

The map $g \rightarrow \tilde{g}(z)$, defined by

$$\tilde{g}(z) = \int g d\mu_z, \quad z \in U,$$

is an isometry from $L^\infty(\mu)$ to a space of bounded harmonic functions on U (cf. [4], Lemma 2.2). Now the restriction map

$$H^\infty(\mu + |\tau| + \lambda_U) \rightarrow H^\infty(\mu + |\tau|)$$

is an isometric isomorphism, under which the function f corresponds to the element $F \in H^\infty(\mu + |\tau| + \lambda_U)$ defined by $F = f$ a.e. ($\mu + |\tau|$), and $F = \tilde{f}$ a.e. ($d\lambda_U$).

If p is a regular boundary point for U , then

$$\limsup_{U \ni z \rightarrow p} |F(z)| \leq u(p).$$

By the Iversen-Tsuji theorem (cf. [13]), this relation persists for all p in the closure of the set of regular points, that is, for all $p \in \text{supp } \mu$. Consequently there is a positive continuous function v on \bar{U} such that $v = u$ on $\text{supp } \mu$, and $|F| \leq v$ a.e. ($d\lambda_U$). Applying the corresponding result for case (iii), we conclude that $|F| \leq v$ a.e. ($d\tau$), that is, $|f| \leq u$ a.e. ($d\tau$). That completes the proof.

9. Characterization of functions in $H^\infty(\lambda_Q)$. This section and the next are devoted to the following theorem, which is an analogue of Vitushkin's characterization of functions in $R(K)$.

THEOREM 9.1. *Let K be a compact subset of \mathbb{C} . Then the following are equivalent:*

- (i) $f \in H^\infty(\lambda_Q)$
- (ii) *There is a sequence $f_n \in R(K)$ such that $\|f_n\|_K \leq \|f\|_Q$ and $f_n(q) \rightarrow f(q)$ for $dxdy$ -almost all $q \in Q$.*
- (iii) *If g is a smooth function supported on a disc $\Delta(z_0; \delta)$, then*

$$\left| \iint f(z) \frac{\partial g}{\partial \bar{z}} dxdy \right| \leq 2\delta \operatorname{osc}(f, \Delta(z_0; \delta)) \left\| \frac{\partial g}{\partial \bar{z}} \right\| \gamma(\Delta(z_0; \delta) \setminus K).$$

- (iv) *For each $z_0 \in K$, there exist $r \geq 1$ and $c > 0$ such that for $\delta > 0$ sufficiently small,*

$$\left| \iint f(z) \frac{\partial g}{\partial \bar{z}} dxdy \right| \leq c\delta \left\| \frac{\partial g}{\partial \bar{z}} \right\| \gamma(\Delta(z_0; r\delta) \setminus K)$$

for all smooth functions g supported on $\Delta(z_0; \delta)$.

The equivalence of (i) and (ii) in Theorem 9.1 is Davie's theorem. That (i) and (ii) imply (iii) has been established in Lemma 3.1, and (iii) trivially implies (iv). The remaining implication, that (iv) implies (i), is more difficult. It will be treated in the next section. Here we make some observations concerning this theorem and give some corollaries.

The fact that $H^\infty(\lambda_Q) \cap C(K) = R(K)$, and Theorem 9.1, can be combined to strengthen the theorem of Vitushkin cited at the beginning of §3.

COROLLARY 9.2. *Let K be a compact plane set, and let f be a bounded Borel function such that $f|_K \in C(K)$. Suppose that for each $z_0 \in K$, there exist $r \geq 1$ and $c > 0$ such that*

$$\left| \iint f \frac{\partial g}{\partial \bar{z}} dxdy \right| \leq c\delta \left\| \frac{\partial g}{\partial \bar{z}} \right\| \gamma(\Delta(z_0; r\delta) \setminus K)$$

for all $\delta > 0$ sufficiently small, and for every smooth function g supported on $\Delta(z_0; \delta)$. Then $f \in R(K)$.

In a similar vein, we have the following.

COROLLARY 9.3. *Let f be continuous on the Riemann sphere, and let K be a compact set. Assume there is a constant C such that whenever S is an open square having its sides parallel to the coordinate axes,*

$$\left| \int_{\partial S} f dz \right| \leq C \gamma(S \setminus K).$$

Then the restriction of f to K lies in $R(K)$.

Vitushkin has proved a similar theorem but with the hypothesis

$$\left| \int_{\partial S} f dz \right| \leq b(\delta) \gamma(S \setminus K),$$

where S has side δ , and $b(\delta)$ tends to 0 with δ . This result is proved on pp. 177–180 of [14] by verifying that there is $a(\delta)$ tending to 0 with δ such that if g has support $\Delta(z_0; \delta)$, then

$$\left| \iint f \frac{\partial g}{\partial \bar{z}} dx dy \right| \leq a(\delta) \delta \left\| \frac{\partial g}{\partial \bar{z}} \right\| \gamma(\Delta(z_0, 2\delta) \setminus K).$$

The same argument shows that the hypothesis of 9.3 yields condition (iv) of Theorem 9.1, so that the restriction of f to K is in $R(K)$.

A remark is in order concerning the integral in condition (iii) of 9.1. This integral can be carried over C , or over K , or only over Q —the resulting estimates are always equivalent. To see this, let g be a smooth function supported on $\Delta(z_0; \delta)$. If h is any bounded Borel function, then the function

$$H(\zeta) = \iint_{C \setminus Q} \frac{h(z)}{z - \zeta} \frac{\partial g}{\partial \bar{z}} dx dy$$

is analytic off $\Delta(z_0; \delta)$, and H belongs to $R(K)$ by Lemma 2.1. Hence

$$|H'(\infty)| \leq \|H\| \gamma(\Delta(z_0; \delta) \setminus K)$$

Using the estimate

$$\iint_{\Delta(z_0; \delta)} \frac{dx dy}{|z - \zeta|} \leq 2\pi\delta,$$

and evaluating $H'(\infty) = \lim_{\xi \rightarrow \infty} \zeta H(\zeta)$, we obtain

$$(*) \quad \left| \iint_{C \setminus Q} h(z) \frac{\partial g}{\partial \bar{z}} dx dy \right| \leq 2\pi\delta \|h\|_{\Delta(z_0; \delta)} \left\| \frac{\partial g}{\partial \bar{z}} \right\| \gamma(\Delta(z_0; \delta) \setminus K).$$

This inequality shows that the validity of an estimate of the form

$$\left| \iint h(z) \frac{\partial g}{\partial \bar{z}} dx dy \right| \leq c\delta \left\| \frac{\partial g}{\partial \bar{z}} \right\| \gamma(\Delta(z_0; \delta) \setminus K)$$

depends only on the values of h on Q .

10. Completion of the proof of Theorem 9.1. It will be convenient to introduce another condition, which falls between (iii) and (iv) of Theorem 9.1:

(v) There exists $r \geq 1$ and $c > 0$ such that

$$\left| \iint f \frac{\partial g}{\partial \bar{z}} dx dy \right| \leq c \delta \left\| \frac{\partial g}{\partial \bar{z}} \right\| \gamma(\Delta(z_0; r\delta) \setminus K)$$

for all $z_0 \in K$, all $\delta > 0$, and all smooth functions g supported on $\Delta(z_0; \delta)$.

Our next step will be to show that (v) implies (i), so that (i), (ii), (iii) and (v) will be equivalent. Then the equivalence of these conditions with (iv) will follow from Vitushkin's nested disc argument.

In view of the discussion in §9, and in particular the estimate (*) there with h replaced by f , the estimate of (v) is also valid for the function which coincides with f on Q and which vanishes off Q . Since (i) also depends only on the values of f on Q , we can assume that $f = 0$ off Q .

The continuation of the proof that (v) implies (i) will be obtained by modifying appropriately Vitushkin's constructive scheme for approximation. We will only sketch the part of the proof that is exactly analogous to Vitushkin's argument, dwelling longer on the two estimates which make Vitushkin's scheme work in this case. Constants will be denoted by C_1, C_2, \dots , and they will depend at most only on c, r and $\|f\|$.

For each $\delta > 0$, choose a covering $\Delta_{k\delta} = \Delta(z_k; \delta)$ of K by open discs, and choose smooth functions $g_k = g_{k\delta}$ such that

- (a) No point z lies in more than C_1 of the Δ_k
- (b) $g_k = 0$ off Δ_k
- (c) $\Sigma g_k = 1$ near K
- (d) $|\partial g_k / \partial \bar{z}| \leq 4/\delta$.

Define

$$\begin{aligned} f_{k\delta}(\zeta) &= (T_{g_k} f)(\zeta) \\ &= \frac{1}{\pi} \iint \frac{f(z) - f(\zeta)}{z - \zeta} \frac{\partial g_k}{\partial \bar{z}} dx dy. \end{aligned}$$

Then $f_{k\delta}$ is analytic off $\Delta_{k\delta}$, $\|f_{k\delta}\| \leq C_2$, and $f = \Sigma f_{k\delta}$. Using (v) and proceeding as in [6], pp. 174-176, we obtain $F_{k\delta} \in R(K)$ such that $F_{k\delta}$ is analytic off $\Delta(z_{k\delta}; r\delta)$, $F_{k\delta} - f_{k\delta}$ has a triple zero at ∞ , and $\|F_{k\delta}\| \leq C_3$. Set

$$F_\delta = \sum_k F_{k\delta} \in R(K).$$

The estimate on [6], p. 212, shows that $\|F_\delta\| \leq C_4$. It suffices now to show that $F_\delta(z) \rightarrow F(z)$ as $\delta \rightarrow 0$, for almost all $z \in Q$.

Let N be a fixed large integer. Set

$$\begin{aligned} A_\delta(z) &= \sum \{ |F_{k\delta}(z) - f_{k\delta}(z)| : d(z, \Delta_{k\delta}) > N\delta \} \\ B_\delta(z) &= \left| \sum \{ F_{k\delta}(z) : d(z, \Delta_{k\delta}) \leq N\delta \} \right| \\ C_\delta(z) &= \sum \{ |f_{k\delta}(z)| : d(z, \Delta_{k\delta}) \leq N\delta \}. \end{aligned}$$

Then

$$|f(z) - F_\delta(z)| \leq A_\delta(z) + B_\delta(z) + C_\delta(z).$$

The term $A_\delta(z)$ is estimated as on p. 193 of [7], yielding

$$A_\delta(z) \leq \sum_{k=N}^{\infty} C_5 k / (k-1)^3 \leq C_6 / N.$$

Using (a), we see that there are at most C_7 summands involved in $B_\delta(z)$ and $C_\delta(z)$. So we are reduced to estimating individually the summands $|F_{k\delta}(z)|$ and $|f_{k\delta}(z)|$, when $d(z, \Delta_{k\delta}) \leq N\delta$. It is quite simple to estimate $B_\delta(z)$.

LEMMA 10.1. $B_\delta(z) \rightarrow 0$ for all $z \in Q$.

Proof. Let $h_\delta(\zeta) = \Sigma \{F_{k\delta}(\zeta) : d(z, \Delta_{k\delta}) \leq N\delta\}$. Then $h_\delta \in R(K)$, h_δ is analytic off $\Delta(z; (N+1+r)\delta)$, and $h_\delta(\infty) = 0$. Since $|F_{k\delta}| \leq C_3$, we have $|h_\delta| \leq C_3 C_7$. By Schwarz's lemma, h_δ converges uniformly to zero on any set at a positive distance from z . If $h_\delta(z) \not\rightarrow 0$, then the Bishop "1/4 - 3/4 criterion" shows that z is a peak point for $R(K)$. Hence $B_\delta(z) = |h_\delta(z)| \rightarrow 0$ whenever $z \in Q$.

In order to estimate $C_\delta(z)$, we prove the following lemma.

LEMMA 10.2. Let h be a bounded Borel function on C , and let S be a subset of C . Let $z \in S$, let $\delta > 0$, and let g be a smooth function supported on $\Delta(z; \delta)$. Then

$$\begin{aligned} & \frac{1}{\pi} \left| \iint \frac{h(\zeta) - h(z)}{\zeta - z} \frac{\partial g}{\partial \bar{\zeta}} d\bar{\zeta} d\eta \right| \\ & \leq 2\delta \left\| \frac{\partial g}{\partial \bar{z}} \right\| \left\{ \text{osc}(h, \Delta(z; \delta) \cap S) + 2\|h\| \left[\frac{\text{Area}(\Delta(z; \delta) \setminus S)}{\pi\delta^2} \right]^{1/2} \right\}. \end{aligned}$$

Proof. Write the integral as a sum of integrals over $\Delta(z; \delta) \cap S$ and $\Delta(z; \delta) \setminus S$, and use the obvious estimates on each summand, together with the estimate

$$\iint_E \frac{d\zeta d\eta}{|z - \zeta|} \leq 2[\pi \text{Area}(E)]^{1/2}.$$

LEMMA 10.3. Suppose S is a closed subset of Q such that $f|_S$ is continuous. Then $C_\delta(z) \rightarrow 0$ as $\delta \rightarrow 0$, for all points $z \in S$ at which S has full area density.

Proof. Applying the lemma to f , and recalling the definition of $f_{k\delta}$, we find that

$$C_\delta(z) \leq 8C_7 \left\{ \text{osc}(f, \Delta(z; \delta) \cap S) + 2\|f\| \left[\frac{\text{Area}(\Delta(z; \delta) \setminus S)}{\pi\delta^2} \right]^{1/2} \right\}$$

and this tends to zero with δ .

Now we can complete the proof that (v) implies (i). In view of Lusin's theorem and Lemma 10.3, $C_\delta(z)$ tends to zero for $dxdy$ -almost all $z \in Q$. Consequently

$$\limsup_{\delta \rightarrow 0} |f(z) - F_\delta(z)| \leq C_6/N$$

for $dxdy$ -almost all $z \in Q$. Since N can be chosen arbitrarily large, we see that F_δ converges $dxdy$ -almost everywhere on Q to f . Since the F_δ are bounded, f lies in $H^\infty(\lambda_Q)$, as required.

We have shown now that (i), (ii), (iii), and (v) are equivalent, and evidently they imply (iv). To complete the proof, we will use Vitushkin's nested disc argument (cf. [7, p. 218]) to show that (iv) fails whenever (i) fails.

Suppose then that (i) fails. For convenience, we assume that $\|f\| \leq 1$. Since (v) fails, there exist $z_1 \in K$, $\delta_1 > 0$ and a smooth function g_1 supported on $\Delta(z_1; \delta_1)$ such that (taking $r = c = 10$ in (v))

$$\left| \iint f \frac{\partial g_1}{\partial \bar{z}} dxdy \right| > 10\delta_1 \left\| \frac{\partial g_1}{\partial \bar{z}} \right\| \gamma(\Delta(z_1; 10\delta_1) \setminus K).$$

Let $K_1 = \overline{\Delta(z_1; 2\delta_1)} \cap K$. The nonpeak points for $R(K_1)$ are the points in $Q_1 = \Delta(z_1; 2\delta_1) \cap Q$. Applying what we have already proved to $H^\infty(\lambda_{Q_1})$, we see that f violates (iii), so that $f \notin H^\infty(\lambda_{Q_1})$. Hence there exist $z_2 \in K_1$, $\delta_2 > 0$ and a smooth function g_2 supported on $\Delta(z_2; \delta_2)$ such that (taking $r = c = 10^2$, and replacing K by K_1 in (v))

$$\left| \iint f \frac{\partial g_2}{\partial \bar{z}} dxdy \right| > 10^2 \delta_2 \left\| \frac{\partial g_2}{\partial \bar{z}} \right\| \gamma(\Delta(z_2; 10^2 \delta_2) \setminus K_1).$$

Since

$$\left| \iint f \frac{\partial g_2}{\partial \bar{z}} dxdy \right| \leq \pi \delta_2^2 \left\| \frac{\partial g_2}{\partial \bar{z}} \right\|,$$

we obtain

$$\gamma(\Delta(z_2; 10^2 \delta_2) \setminus K_1) > \pi \delta_2 / 10^2.$$

In particular, $\Delta(z_2; 10^2 \delta_2) \setminus K_1$ cannot contain a disc of radius $\pi \delta_2 / 10^2$. Hence $\Delta(z_2; 10^2 \delta_2) \subseteq \Delta(z_1; 2\delta_1 + \delta_2)$, so that $\Delta(z_2; 99\delta_2) \subseteq \Delta(z_1; 2\delta_1)$. Setting $K_2 = \overline{\Delta(z_2; 2\delta_2)} \cap K_1 = \overline{\Delta(z_2; 2\delta_2)} \cap K$, and proceeding in this manner, we construct by induction a sequence of points $z_n \in K$, radii $\delta_n > 0$, and smooth functions g_n supported on $\Delta(z_n; \delta_n)$ such that

- (a) $K_{n-1} = \overline{\Delta(z_{n-1}; \delta_{n-1})} \cap K$
- (b) $\Delta(z_n; (10^n - 1)\delta_n) \subseteq \Delta(z_{n-1}; 2\delta_{n-1})$
- (c) $\left| \iint f \frac{\partial g_n}{\partial \bar{z}} dx dy \right| > 10^n \delta_n \left\| \frac{\partial g_n}{\partial \bar{z}} \right\| \gamma(\Delta(z_n; 10^n \delta_n) \setminus K_{n-1}) .$

From (b) we obtain $(10^n - 1)\delta_{n+1} \leq 2\delta_n$, so that $\delta_{n+1} < 2\delta_n$. It follows that z_n converges to $z \in K$, and that

$$|z - z_n| < \sum_{k=n}^{\infty} |z_{k+1} - z_k| \leq \sum_{k=n}^{\infty} \delta_k \leq 2\delta_n .$$

Hence we have

- (d) g_n is supported on $\Delta(z; 3\delta_n)$.

Moreover from (b) we obtain $\Delta(z; (10^n - 3)\delta_n) \subseteq \Delta(z_{n-1}; \delta_{n-1})$, so that

$$\Delta(z; (10^n - 3)\delta_n) \setminus K = \Delta(z; (10^n - 3)\delta_n) \setminus K_n \subseteq \Delta(z_n; 10^n \delta_n) \setminus K_n .$$

Using (c), and the monotonicity of γ , we obtain

- (e) $\left| \iint f \frac{\partial g_n}{\partial \bar{z}} dx dy \right| > 10^n \delta_n \left\| \frac{\partial g_n}{\partial \bar{z}} \right\| \gamma(\Delta(z; (10^n - 3)\delta_n) \setminus K) .$

From (d) and (e), we see that condition (iv) fails at z . That completes the proof of Theorem 9.1.

11. Approximation of Cauchy transforms. In this section, we will apply Corollary 9.2 to the problem of approximating the Cauchy transform

$$\hat{\nu}(\zeta) = \int \frac{d\nu(z)}{z - \zeta}$$

of a compactly supported measure ν . Since $\hat{\nu}$ is the convolution of ν and the locally integrable function $1/z$, $\hat{\nu}$ is itself locally integrable. Furthermore $\hat{\nu}$ is analytic off the closed support of ν . The discussion in this section will be based on the following simple consequence of Theorem 9.1.

LEMMA 11.1. *Let K be a compact plane set. Let ν be a measure with compact support, such that $|\nu|(K^\circ) = 0$, and such that $\hat{\nu}$ is bounded. If for each $z \in \partial K$, there exists $r \geq 1$ satisfying*

$$\liminf_{\delta \rightarrow 0} \frac{\gamma(\Delta(z; r\delta) \setminus K)}{|\nu|(\Delta(z; \delta))} > 0 ,$$

then $\hat{\nu} \in H^\infty(\lambda_\partial)$.

Proof. Suppose that g is supported on a disc $\Delta(z_0; \delta)$. From Green's formula

$$g(\zeta) = \frac{1}{\pi} \iint \frac{1}{z - \zeta} \frac{\partial g}{\partial \bar{z}} dx dy$$

we obtain $\|g\| \leq 2\delta \|\partial g / \partial \bar{z}\|$. Hence

$$\left| \iint \hat{\nu}(z) \frac{\partial g}{\partial \bar{z}} dx dy \right| = \pi \left| \int g(\zeta) d\nu(\zeta) \right| \leq 2\pi\delta \left\| \frac{\partial g}{\partial \bar{z}} \right\| |\nu|(\Delta(z; \delta)) .$$

Our hypothesis, together with criterion (iv) of Theorem 9.1, yield the desired result.

LEMMA 11.2. *Let K be a compact plane set. Let ν be a measure with compact support, such that $|\nu|(K^0) = 0$, and such that*

$$\sup_{\zeta} \int \frac{d|\nu|(z)}{|z - \zeta|} < \infty .$$

Suppose that for ν -almost all points $z \in \partial K$, there exists $r = r(z) \geq 1$ such that

$$\liminf_{\delta \rightarrow 0} \frac{\gamma(\Delta(z; r\delta) \setminus K)}{|\nu|(\Delta(z; \delta))} > 0 .$$

Then $\hat{\nu} \in H^\infty(\lambda_q)$. If in addition the restriction of $\hat{\nu}$ to K is continuous, then $\hat{\nu} \in R(K)$.

Proof. Let E be the set of $z \in \partial K$ for which there is no $r \geq 1$ satisfying (*). The hypothesis is that $|\nu|(E) = 0$. Let U_n be a sequence of open sets such that $U_n \supseteq U_{n+1} \supseteq E$, and $|\nu|(U_n) \rightarrow 0$. Let ν_n be the restriction of ν to $C \setminus U_n$. Then ν_n satisfies the hypotheses of Lemma 11.1, so that $\hat{\nu}_n \in H^\infty(\lambda_q)$. By the dominated convergence theorem, $\hat{\nu}_n$ converges pointwise to $\hat{\nu}$, so that $\hat{\nu} \in H^\infty(\lambda_q)$. That proves the theorem.

Recall (cf. [2]) that a measure function h is an increasing function of $\delta > 0$ such that $h(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Associated with h is the Hausdorff measure A_h . We will be interested in measure functions satisfying the condition

$$(*) \quad \int_0^1 \frac{h(\delta)}{\delta^2} d\delta < \infty .$$

If $1 < \alpha \leq 2$, then the measure function $h_\alpha(\delta) = \delta^\alpha$ satisfies this condition. The associated measure is the α -dimensional Hausdorff measure A_α .

If h satisfies (*), and ν is any compactly supported measure satisfying

$$(**) \quad |\nu|(\Delta(z; \delta)) \leq h(\delta), \quad \delta > 0, z \in C,$$

then

$$\sup_{\zeta} \int \frac{d|\nu|(z)}{|z - \zeta|} < \infty,$$

and $\hat{\nu}$ is continuous. Moreover, every set E satisfying $\Lambda_k(E) = 0$ will also satisfy $|\nu|(E) = 0$. From Theorem 11.2, we obtain the following.

COROLLARY 11.3. *Suppose the measure function h satisfies (*). Let ν be a compactly supported measure satisfying (**), such that $|\nu|(K^0) = 0$. If*

$$\liminf_{\delta \rightarrow 0} \frac{\gamma(\Delta(z; \delta) \setminus K)}{h(\delta)} > 0$$

for Λ_k -almost all $z \in \partial K$, then $\hat{\nu} \in R(K)$.

This corollary is not the strongest result which can be obtained from the current literature. Indeed, let h be a measure function which satisfies (*), and define

$$\Phi(\delta) = \int_0^\delta \frac{dh(s)}{s}, \quad \delta > 0.$$

A modification of the proof of Vitushkin's instability theorem (see [14, pp. 188–191, and [9, pp. 122–127]) shows that for Λ_k -almost all $z \in C$, one of the following two alternatives holds:

$$(a) \quad \lim_{\delta \rightarrow 0} \frac{\gamma(\Delta(z; \delta) \setminus K)}{h(\delta)} = 0$$

$$(b) \quad \liminf_{\delta \rightarrow 0} \frac{\Phi(2\delta)\gamma(\Delta(z; \delta) \setminus K)}{h(\delta)} \geq 10^{-6}.$$

Moreover in [9] it is shown that if (b) holds Λ_k -almost everywhere on ∂K , then $\hat{\nu} \in R(K)$ for any measure ν on ∂K satisfying (**). Combining these two results, we see that the “lim inf” in Corollary 11.3 can be replaced by “lim sup”.

In closing we mention a new result which follows immediately from Theorem 9.1.

COROLLARY 11.4. (to Theorem 9.1): *Let h be a measure function, and suppose*

$$\liminf_{\delta \rightarrow 0} \frac{\gamma(\Delta(z; \delta) \setminus K)}{h(\delta)} > 0$$

for all $z \in \partial K$. Let $f \in C(K)$. If the modulus of continuity ω_f of f satisfies

$$\omega_f(\delta) \leq h(\delta)/\delta, \quad \delta > 0,$$

then $f \in R(K)$.

It would be interesting to know if the same conclusion holds when the estimate on analytic capacity is assumed to hold only at Δ_h -almost all points of ∂K . When h satisfies the integrability condition (*), this refinement of Corollary 9.4 is known (see [3] or [9]).

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