

# Pacific Journal of Mathematics

**EQUIVARIANT EXTENSIONS OF MAPS**

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# EQUIVARIANT EXTENSIONS OF MAPS

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This paper treats extension and retraction properties in the category  $\mathcal{A}_p$  of compact metric spaces with periodic maps of a prime period  $p$ ; the subspaces and maps in  $\mathcal{A}_p$  are called equivariant subspaces and maps, respectively. The motivation of the paper is the following question: Let  $E$  be a Euclidean space and  $\alpha: E \times E \rightarrow E \times E$  be the involution  $(x, y) \rightarrow (y, x)$ , i.e., the symmetry with respect to the diagonal. Suppose that  $Z$  is a symmetric (i.e., equivariant) closed subset of  $E \times E$  which is an absolute retract; that is,  $Z$  is a retract of  $E \times E$ . When does there exist a symmetric (i.e., equivariant) retraction  $E \times E \rightarrow Z$ ?

This is an extension problem in the category  $\mathcal{A}_p$ . If  $X$  and  $Y$  are spaces in  $\mathcal{A}_p$ ,  $A$  is a closed equivariant subspace of  $X$  and  $f: A \rightarrow Y$  is an equivariant map, then the existence of an extension of  $f$  does not, in general, imply the existence of an equivariant extension. It is shown, however, that if  $A$  contains all the fixed points of the periodic map and  $\dim(X - A) < \infty$ , then a condition for the existence of an extension is also sufficient for the existence of an equivariant extension. In particular, it follows that a finite dimensional space  $X$  in  $\mathcal{A}_p$  is an equivariant ANR (i.e., an absolute neighborhood retract in the category  $\mathcal{A}_p$ ) if and only if it is an ANR and the fixed point set of the periodic map on  $X$  is an ANR. Generally speaking, the paper deals with the question of symmetry in extension and retraction problems.

1. Preliminaries. Suppose that a group  $G$  acts on spaces  $X$  and  $Y$  and that  $A$  is an equivariant subspace of  $X$  (i.e.,  $A$  is stable under the action of  $G$ ). One can then ask for conditions for the existence of an equivariant extension of  $f$ ; or for conditions under which the existence of an extension of  $f$  implies also the existence of an equivariant extension. A general theorem of this type is due to A. Gleason [6] and R. S. Palais [12, p. 19]:

**TIETZE-GLEASON THEOREM.** *Let  $G$  be an orthogonal group acting on a Euclidean space  $E$  by means of orthogonal transformations and let  $G$  act on a normal space  $X$ . Let  $A$  be a closed equivariant subset of  $X$  and let  $f: A \rightarrow E$  be an equivariant map. Then there is an equivariant extension  $g: X \rightarrow E$  of  $f$ .*

This theorem is proved by first extending the map  $f$  to some map  $\bar{f}: X \rightarrow E$  which may not necessarily be equivariant: and then by averaging  $\bar{f}$ , using a Haar measure on  $G$ , to make it equivariant.

Two facts play a crucial role in this proof: one is that  $E$  is convex; and the other is that the action of  $G$  is linear. While the second condition is not necessarily restrictive (in view of results due to Mostow [11]; Copeland and de Groot [2]; Kister and Mann [8]; the action of  $G$  can be linearized), the first condition makes it impossible to apply a theorem of this type to our original problem (these two conditions are, in fact, related: by linearization of the map, the convexity of the space may be distorted).

In this paper we consider actions of  $Z_p$ , the cyclic group of a prime order  $p$ . In other words, we consider the category  $\mathcal{A}_p$  whose objects are periodic homeomorphisms  $a: X \rightarrow X$  of a prime period  $p$  on a space  $X$ ; i.e.,  $a^p = 1$ . An object  $a: X \rightarrow X$  in  $\mathcal{A}_p$  will also be denoted by  $(X, a)$ , or simply by  $X$ , if the periodic map  $a$  is known. A morphism in  $\mathcal{A}_p$  from  $(X, a)$  to  $(Y, b)$  is a map  $f: X \rightarrow Y$  consistent with the periodic maps  $a$  and  $b$ ; it will be called an equivariant map. A subspace  $A$  of  $X$  is said to be equivariant if it is stable under  $a$ , i.e., if  $aA \subset A$ . If  $A$  is an equivariant subspace of  $X$  then the periodic map  $A \rightarrow A$  defined by the restriction of  $a: X \rightarrow X$  of  $A$  will sometimes be denoted by  $a_A: A \rightarrow A$ .

The set of the fixed points of a map  $a: X \rightarrow X$  will be denoted by  $F(a)$ . If  $a: X \rightarrow X$  is a periodic map of a prime period  $p$  then  $F(a) = F(a^q)$ , for every  $q = 1, \dots, p-1$ .

An example of a theorem which carries over to the category  $\mathcal{A}_p$  in a way similar to that of the Tietze theorem is the Dugundji extension theorem. In the category  $\mathcal{A}_p$  it can be stated as follows:

**DUGUNDJI EQUIVARIANT EXTENSION THEOREM** (in the category  $\mathcal{A}_p$ ). *Let  $(X, a)$  be a space in  $\mathcal{A}_p$  such that  $X$  is metrizable and let  $A$  be an equivariant closed subspace of  $X$ . Let  $L$  be a locally convex vector space with a linear periodic map  $b: L \rightarrow L$  of period  $p$  and let  $Q$  be an equivariant convex subspace of  $L$ . Let  $f: A \rightarrow Q$  be an equivariant map. Then  $f$  can be extended to an equivariant map  $g: X \rightarrow Q$ .*

The proof is the same as that of the Tietze-Gleason theorem. By the Dugundji extension theorem there exists an extension  $\bar{f}: X \rightarrow Q$ . We define an equivariant extension  $g$  by

$$g = \frac{1}{p} \sum_{i=1}^p b^{p-i} \circ \bar{f} \circ a^i.$$

In other words, this theorem says that  $(L, b_L)$  is an "absolute extensor" in this category of spaces. One can likewise introduce the definitions of "absolute neighborhood extensor", "absolute retract" and "absolute neighborhood retract" in the category  $\mathcal{A}_p$  or in other similar categories.

Returning to our original problem of the existence of an equivariant retraction, let us now state it as follows (in the case of compact metric spaces):

*Question I.* Let  $Q$  be a Hilbert cube and let  $a: Q \rightarrow Q$  be a periodic map of a prime period  $p$  such that  $a$  is linear with respect to the linear structure on  $Q$ . Let  $Z \subset Q$  be an equivariant closed subspace of  $Q$  which is a retract of  $Q$ . When does there exist an equivariant retraction  $Q \rightarrow Z$ ?

First, it is known that if  $X$  is any separable metric space with a periodic map  $a: X \rightarrow X$  of period  $p$  then there exists an equivariant embedding of  $X$  in a Hilbert cube with a linear, even a distance preserving, map of period  $p$ . Such an embedding is known as a linearization of  $(X, a)$  (see [2], Theorem II). We choose any embedding  $X \subset Q$  and then define an equivariant embedding  $X \rightarrow Q^p$  by

$$x \mapsto (x, ax, \dots, a^{p-1}x).$$

Thus the periodic map becomes a cyclic permutation of the coordinates of  $Q^p$  (in fact, our original case was of the involution  $E \times E \rightarrow E \times E$  of this form). Similarly, if  $\dim X < \infty$ , then  $X$  can be equivariantly embedded in a finite-dimensional cube  $I^n$  with an isometric periodic map.

Returning to Question I, let us assume therefore that there exists an equivariant retraction  $r: Q \rightarrow Z$ . Consider the fixed point sets  $F(a)$  and  $F(a_Z)$  of the map  $a$  on  $Q$  and  $Z$ , respectively. Then  $r$  defines a retraction of  $F(a)$  to  $F(a_Z)$ . But since  $a$  is linear,  $F(a)$  is a compact convex subset of  $Q$  and hence an absolute retract (it is, in fact, homeomorphic to  $Q$  or to a finite-dimensional cube: see [7] and [9]). Therefore the fixed point set  $F(a_Z)$  of  $a$  would have to be an absolute retract; and this need not necessarily be the case, since there is the following example due to E. E. Floyd [4].

*Floyd's example.* There exists a 5-dimensional compact contractible polyhedron  $Z$  with an involution  $a: Z \rightarrow Z$  whose fixed point set  $F(a_Z)$  is not contractible; in fact,  $H_1(F(a_Z)) \neq 0$ .

Similarly, one can construct an example of a compact AR  $Z$  with an involution  $a: Z \rightarrow Z$  such that  $F(a_Z)$  is not an ANR.

Thus, in Question I, the condition that both  $Z$  and  $F(a_Z)$  be AR's is necessary for  $Z$  to be an equivariant retract of  $Q$ ; similarly, the condition that  $Z$  and  $F_a(Z)$  be ANR's is necessary for  $Z$  to be an equivariant neighborhood retract of  $Q$ . Consequently, the question arises as to whether these conditions are also sufficient.

Let us specify our questions as follows:

*Questions.* Let  $Q$  be a Hilbert cube with a linear periodic map  $\alpha: Q \rightarrow Q$  of period  $p$  and let  $Z$  be an equivariant closed subspace of  $Q$ .

*Question I'.* Suppose that both  $Z$  and the fixed point set  $F(\alpha_Z)$  of  $\alpha$  are AR's. Is  $Z$  an equivariant retract of  $(Q, \alpha)$ ?

*Question I''.* Suppose that both  $Z$  and  $F(\alpha_Z)$  are ANR's. Is  $Z$  an equivariant neighborhood retract of an equivariant neighborhood of  $Z$  in  $Q$ ?

The main result of this paper is to show that if  $p$  is prime and the dimension of  $Z$  is finite then the answer to Questions I' and I'' is affirmative. In fact, the following theorem will be proved:

**THEOREM 1.1.** *Let  $X$  be a compact metric space with a periodic map  $\alpha: X \rightarrow X$  of period  $p$  and let  $A$  be an equivariant closed subspace of  $X$  containing all the fixed points, of  $\alpha$  and such that  $\dim(X - A) < \infty$ . Let  $Y$  be a compact metric space with a periodic map  $b: Y \rightarrow Y$  of period  $p$  and let  $f: A \rightarrow Y$  be an equivariant map. Then:*

(i) *If  $Y$  is an AR, then there exists an equivariant extension  $g: X \rightarrow Y$  of  $f$  over  $X$ ;*

(ii) *If  $Y$  is an ANR, then there exists an equivariant extension  $g: U \rightarrow Y$  of  $f$  over an equivariant neighborhood  $U$  of  $A$  in  $X$ .*

We can now use Theorem (1.1) to answer our questions in the finite-dimensional case; in fact, we use part (i) to answer Question I' and part (ii) to answer Question I''. Let us consider, for instance, the case (i) and I'. Since  $\dim Z < \infty$ , there is an equivariant embedding of  $(Z, \alpha_Z)$  in an  $n$ -cube  $I^n$ : that is, an equivariant homeomorphism of  $Z$  onto an equivariant subspace  $Z'$  of  $I^n$  with a periodic map  $b: I^n \rightarrow I^n$  (which can even be assumed isometric). Let us apply Theorem (1.1) to  $X = I^n$ ,  $A = Z' \cup F(b) = Y$  and  $f = 1_A$  (the identity map  $A \rightarrow A$ ). By Theorem (1.1), there exists an equivariant retraction  $r: I^n \rightarrow A$ . Since  $F(b_{Z'})$  is an AR, there exists a retraction  $q: F(b) \rightarrow F(b_{Z'})$ . The retraction  $q$  defines a retraction  $q': A \rightarrow Z'$  by extending via the identity  $Z' \rightarrow Z'$ . The composition  $q' \circ r$  is an equivariant retraction of  $I^n$  to  $Z'$ . Now, since  $I^n$  can be embedded as an equivariant retract of  $Q$ , it follows that  $Z$  is an equivariant retract of  $Q$ .

Similarly, part (ii) of Theorem (1.1) yields an affirmative answer to Question I''.

The proof Theorem (1.1) uses the classical method of replacing  $X - A$  by the nerve of a covering adjusted to the equivariant category.

It works, however, only under the assumption  $\dim(X - A) < \infty$ . It is an open question whether this finite-dimensional assumption in Theorem (1.1) is essential.

The main results of this paper have been announced in [6].

**2. Linearization.** We summarize some results on linearization of periodic maps (see [2]). Given a space  $Z$ , we denote by  $c(p, Z)$  the periodic map of the  $p$ -fold Cartesian product  $Z^p$  defined by  $(z_1, \dots, z_p) \rightarrow (z_p, z_1, \dots, z_{p-1})$  i.e.,  $c(p, Z)$  is a cyclic permutation of the coordinates.

(2.1). If  $Z$  is a vector space then  $(Z^p, c(p, Z))$  is a vector space with the periodic map  $c(p, Z)$  being linear with respect to the product vector space structure.

(2.2). If  $Z$  is a metric space then  $c(p, Z)$  is isometric (i.e., distance preserving) with respect to the product metric in  $Z^p$ .

(2.3). If  $(X, a)$  is an object in  $\mathcal{A}_p$  and there is an embedding  $h: X \rightarrow Z$  of  $X$  in a space  $Z$ , then there is an equivariant embedding of  $(X, a)$  in  $(Z^p, c(p, Z))$  defined by  $x \mapsto (hx, h(ax), \dots, h(a^{p-1}x))$ .

In particular

(2.4). If  $(X, a)$  is an object of  $\mathcal{A}_p$  such that  $X$  is a compact metric space, then there is an equivariant embedding of  $(X, a)$  in  $(Q, c)$  where  $Q$  is a Hilbert cube with an isometric map  $c: Q \rightarrow Q$  of period  $p$ . If  $\dim X < \infty$ , then there is an equivariant embedding of  $(X, a)$  in  $(I^n, c)$ , where  $I^n$  is a finite-dimensional cube with an isometric periodic map  $c: I^n \rightarrow I^n$  of period  $p$ .

Let us also note that if  $(V, a)$  is a vector space with a linear periodic map  $a: V \rightarrow V$  of period  $p$  and  $Z$  is an equivariant convex subset of  $V$  then the equivariant embedding  $h: x \mapsto (hx, h(ax), \dots, h(a^{p-1}x))$  carries  $Z$  onto a convex subset of  $(V^p, c(p, V))$ . In particular, an  $n$ -cube  $(I^n, a)$  with a linear periodic map  $a: I^n \rightarrow I^n$  can be equivariantly embedded as a convex subset of  $(Q, c)$ , where  $Q$  is a Hilbert cube with a linear periodic map  $c: Q \rightarrow Q$ . Using the Dugundji equivariant extension theorem we obtain the following corollary:

(2.5) If  $(I^n, a)$  is an  $n$ -cube with a linear periodic map  $a: I^n \rightarrow I^n$  of period  $p$  then  $(I^n, a)$  can be equivariantly embedded as an equivariant retract of a Hilbert cube  $(Q, c)$  with a periodic map  $c: Q \rightarrow Q$ .

**3. Retracts and extensors in the category  $\mathcal{A}_p$ .** We summarize the definitions and main properties of retracts and extensors in the

category  $\mathcal{A}_p$  of spaces with  $Z_p$ -actions (compare Palais [12], p. 25) which are usually called  $Z_p$ -retracts and  $Z_p$ -extensors. Since the prime integer  $p$  and the group  $Z_p$  is fixed throughout the paper (except where the results are specialized to the case  $p = 2$ ), we shall simply call them equivariant retracts and equivariant extensors.

**DEFINITION 3.1.** An object  $(Y, b)$  of  $\mathcal{A}_p$  is said to be an equivariant absolute extensor (abbreviated to EAE) if given an object  $(X, a)$  of  $\mathcal{A}_p$  such that  $X$  is a metric space, given a closed equivariant subspace  $A$  of  $X$  and given an equivariant map  $f: A \rightarrow Y$ , there is an equivariant extension  $g: X \rightarrow Y$  of  $f$ .

An object  $(Y, b)$  of  $\mathcal{A}_p$  is said to be equivariant absolute neighborhood extensor (EANR) if given  $(X, a)$ ,  $A$  and  $f$  as above, there is an equivariant extension  $g: U \rightarrow Y$  of  $f$  over some equivariant neighborhood  $U$  of  $A$  in  $X$ .

**DEFINITION 3.2.** An object  $(X, a)$  of  $\mathcal{A}_p$  is said to be an equivariant absolute retract (abbreviated to ERA) if  $X$  is a metric space and for any equivariant imbedding  $h: (X, a) \rightarrow (Y, b)$  in an object  $(Y, b)$  of  $\mathcal{A}_p$  such that  $Y$  is a metric space and  $hX$  is closed in  $Y$ , the image  $hX$  in an equivariant retract of  $(Y, b)$ .

An object  $(X, a)$  of  $\mathcal{A}_p$ , where  $X$  is a metric space, is said to be an equivariant absolute neighborhood retract (EANR) if given  $h: (X, a) \rightarrow (Y, b)$  as above, the image  $hX$  is an equivariant neighborhood retract of  $(Y, b)$ .

The following theorems are proved in the same way as in the topological category:

**THEOREM 3.3.** *An equivariant retract of an EAE is an EAE; an equivariant neighborhood retract of an EANR is an EANR.*

**THEOREM 3.4.** *A Hilbert cube  $(Q, c)$  with a linear periodic map  $c: Q \rightarrow Q$  of a prime period  $p$  is an EAE.*

*This is, in fact, a particular case of the Dugundji extension theorem in the category  $\mathcal{A}_p$  (§1).*

**THEOREM 3.5.** *Let  $(X, a)$  be an object of  $\mathcal{A}_p$  such that  $X$  is a compact metric space. Then the following conditions are equivalent:*

- (i)  $(X, a)$  is an EAE.
- (ii)  $(X, a)$  is an EAR.
- (iii)  $(X, a)$  can be equivariantly embedded as an equivariant retract of  $(Q, c)$ , where  $Q$  is a Hilbert cube with an isometric periodic map  $c: Q \rightarrow Q$  of period  $p$ .

*Similarly, the following conditions are equivalent:*

- (iN)  $(X, a)$  is an EANE.

(iiN)  $(X, a)$  is an *EANR*.

(iiiN)  $(N, a)$  can be equivariantly embedded as an equivariant neighborhood retract of  $(Q, c)$ , where  $(Q, c)$  is as above.

Moreover, if  $\dim X < \infty$ , then the Hilbert cube  $(Q, c)$  can be replaced by a finite-dimensional cube  $I^n$  with an isometric involution.

Theorem (3.5) is proved by using the linearization embeddings (§2).

**COROLLARY 3.6.** *The following objects of  $\mathcal{A}_p$  are equivariant absolute retracts:*

- (1) A Hilbert cube  $(Q, c)$  with a linear periodic map  $c: Q \rightarrow Q$ .
- (2) An  $n$ -cube  $(I^n, c)$  with a linear periodic map  $c: Q \rightarrow Q$ .

**4. Equivariant coverings and replacement by polyhedra.** In this section we describe the classical constructions due to Kuratowski [10] and Dugundji [2] which are used in extending maps. We adjust them spaces with periodic maps, but we restrict ourselves to compact metric spaces.

The following notation will be used:  $\text{diam } S$  is the diameter of a subset  $S$  of a metric space  $X$ ;  $B(x, \varepsilon)$  is the open ball in  $X$  of center  $x$  and radius  $\varepsilon$ ; and  $\text{Conv } S$  is the convex hull of a subset  $S$  of a linear space  $L$ .

Let  $\alpha$  be a collection of subsets of  $X$ . If  $U \in \alpha$ , then  $St_\alpha U$  is the union of the members of  $\alpha$  which meet  $U$ . We say that  $\text{Ord } \alpha \leq n$  if every collection of  $n + 1$  members of  $\alpha$  has an empty intersection. If  $X$  is an object of  $\mathcal{A}_p$  with a periodic map  $a: X \rightarrow X$ , let  $\alpha a = \{a^q U \mid U \in \alpha, q = 1, \dots, p-1\}$ ; the collection  $\alpha$  is said to be equivariant if  $\alpha a = \alpha$ .

**COVERING LEMMA 4.1.** *Let  $(X, a)$  be an object of  $\mathcal{A}_p$  such that  $X$  is a compact metric space and let  $A$  be an equivariant closed subspace of  $X$  containing the fixed point set  $F(a)$  of  $a$ . Let  $\alpha$  be an equivariant open cover of  $X - A$ . Then there exists an equivariant countable open cover  $\beta$  of  $X - A$  which is a refinement of  $\alpha$  and satisfies the following conditions:*

- (i)  $\lim_{U \in \beta} (\text{diam } U) = 0$
- (ii) If  $U \in \beta$  then  $\text{Cl } U \subset X - A$ .
- (iii) Every neighborhood of  $A$  in  $X$  contains all but a finite number of elements of  $\beta$ .
- (iv) For every  $U \in \beta$ , the sets  $St_\beta U, a(St_\beta U), \dots, a^{p-1}(St_\beta U)$  are mutually disjoint.
- (v) If  $\dim(X - A) \leq n$  then  $\text{Ord } \beta \leq p(n + 1)$ .



*Proof.* We can assume that  $d$  is an equivariant distance function on  $X$ , i.e., that  $a: X \rightarrow X$  is isometric. Let  $A_0 = X$ ,

$$A_i = \left\{ x \in X \mid d(x, A) < \frac{1}{2^i} \right\}, \quad i = 1, 2, \dots$$

$$C_i = \text{Cl}(A_i) - A_{i+1}, \quad i = 0, 1, \dots$$

The sets  $C_i$  are compact. Since  $p$  is prime, the group  $Z_p$  acts freely on  $X - A$ , i.e., for every  $x \in X - A$ , the orbit  $\{x, ax, \dots, a^{p-1}x\}$  consists of  $p$  distinct points. It follows that, for each  $i = 0, 1, \dots$ , there is a positive number  $\eta_i$  such that

$$(4.2) \quad d(x, a^q x) \geq \eta_i \quad \text{for every } q = 1, \dots, p-1 \quad \text{and } x \in C_i.$$

For each  $i = 0, 1, \dots$ , there is a finite open cover  $\gamma_i$  of  $C_i$  by open balls in  $X$  with centers in  $C_i$  and radii  $r_i > 0$  such that  $\gamma_i$  is a refinement of  $\alpha$  and

$$(4.3) \quad r_i \leq 2^{-i-3}$$

$$(4.4) \quad r_{i+1} \leq \frac{1}{2} r_i$$

$$(4.5) \quad 6\gamma_i \leq \gamma_{i+1}.$$

Let  $\beta_i = \gamma_i \cup a(\gamma_i) \cup \dots \cup a^{p-1}(\gamma_i)$  and  $\beta = \beta_0 \cup \beta_1 \cup \dots$ . Then  $\beta$  is an equivariant countable open cover of  $X - A$  and is a refinement of  $\alpha$  since  $\alpha$  is equivariant. Conditions (i), (ii) and (iii) follow directly from the construction of  $\beta$ .

Let us verify condition (iv). Observe that by (4.3) and (4.4), and since the map  $a$  is isometric, every member of  $\beta$  is an open ball contained in  $A_{j-1} - \text{Cl}(A_{j+1})$ , for some  $j = 1, 2, \dots$ . Thus if a member  $V$  of  $\beta$  meets a member  $U$  of  $\beta_i$  then  $V \in \beta_{i-1} \cup \beta_i \cup \beta_{i+1}$  and, by (4.3) and (4.4),  $U$  and  $V$  are open balls of radii  $\leq r_{i-1} \leq 2^{-i-2}$ .

Since the map  $a: X \rightarrow X$  is isometric, to prove condition (iv) it suffices to show that  $(St_\beta U) \cap (a^q(St_\beta U)) = \emptyset$  for each  $U \in \beta$  and  $q = 1, \dots, p-1$ . Let  $U \in \beta$  and suppose that  $U \in \beta_i$ . Then the center  $x$  of the open ball  $U$  is in  $C_i$ . By the remark above, for every  $y \in St_\beta U$  we have  $d(x, y) < r_i + 2r_{i-1} < 3r_{i-1}$ . If we suppose that  $(St_\beta U) \cap (a^q(St_\beta U)) \neq \emptyset$  for some  $q = 1, \dots, p-1$ , then, since the map  $a$  is isometric, it would follow that  $d(x, a^q x) < 6r_{i-1} \leq \eta_i$ , which contradicts to (4.5) and (4.2).

Suppose now that  $\dim(X - A) \leq n$ . Then the open cover  $\beta$  has an open refinement  $\omega$  of order  $\leq n + 1$ . Since  $C_i$  is compact, there is a finite subcollection  $\omega_i$  of  $\omega$  which covers  $C_i$ . Let  $\beta'_i = \omega_i \cup a(\omega_i) \cup \dots \cup a^{p-1}(\omega_i)$  and  $\beta' = \beta'_0 \cup \beta'_1 \cup \dots$ . Then  $\beta'$  is an equivariant

countable open refinement of  $\beta$  which satisfies the conditions corresponding to (i), (ii) and (iii). Condition (iv) for  $\beta'$  follows from the fact that it holds for  $\beta$  and that  $\beta'$  is a refinement of  $\beta$ .

Let us verify condition (v). Suppose that  $\sigma$  is a subcollection of  $\beta'$  containing  $p$  distinct elements whose intersection is nonempty. For each  $W \in \sigma$ , there is an integer  $q$ ,  $0 \leq q < p$ , such that  $a^q W \in \omega$ ; let  $q(W)$  denote the smallest integer with this property. Then this defines a map  $q: \sigma \rightarrow \{0, \dots, p-1\}$ . Note that  $\text{Card } q^{-1}(j) \leq n+1$ , for each  $j = 0, \dots, p-1$ . For if  $W_0, \dots, W_r$  are distinct elements of  $q^{-1}(j)$ , then  $a^j(W_0), \dots, a^j(W_r) \in \omega$  and  $a^j(W_0) \cap \dots \cap a^j(W_r) \neq \emptyset$  and, consequently,  $r \leq n$ , since  $\text{Ord } \omega \leq n+1$ . Since  $\sigma = q^{-1}(0) \cup \dots \cup q^{-1}(p-1)$ , it follows that  $\text{Card } \sigma \leq p \cdot (n+1)$ .

This completes the proof. Let us observe that conditions (i), (ii) and (iii) imply the following lemma.

**LEMMA 4.6.** *If  $\beta$  is a cover constructed in Lemma (4.1), then for every  $x \in A$  and for every neighborhood  $V$  of  $x$  in  $X$  there exists a neighborhood  $W$  of  $x$  in  $X$  such that if  $U \in \beta$  and  $U \cap W \neq \emptyset$  then  $U \subset V$ .*

(4.7) **LEMMA (Replacement by polyhedra).** *Let  $(X, a)$  be an object of  $\mathcal{A}_p$  such that  $X$  is a compact metric space and let  $A$  be an equivariant closed subspace of  $X$  containing the fixed point set  $F(a)$  of  $a$ . Then there exists an object  $(Z, c)$  in  $\mathcal{A}_p$  such that  $Z$  is a Hausdorff space with a periodic map  $c: Z \rightarrow Z$  and:*

- (i)  $Z$  contains  $A$  as an equivariant subspace.
- (ii)  $Z - A$  has a countable locally finite triangulation  $K$ ,  $|K| = Z - A$ , such that the map  $c$  is simplicial and free on  $K$ .
- (iii) There is an equivariant map of pairs:  $(X, X - A) \rightarrow (Z, |K|)$  which is the identity on  $A$ .
- (iv) There is an equivariant retraction  $r^0: A \cup |K^0| \rightarrow A$ .
- (v) If  $\dim(X - A) \leq n$  then  $\dim K \leq p \cdot (n+1) - 1$ .

*Proof.* Let  $\beta$  be an equivariant open cover of  $X - A$  satisfying the conditions of the Covering Lemma (4.1). Let  $K$  be the nerve of  $\beta$  and  $Z$  be the disjoint set sum of  $A$  and  $|K|$ . Given a member  $U$  of  $\beta$ , we shall also be denoting by  $U$  the vertex of  $K$  corresponding to  $U$ ; and  $St_U U$  will denote the open star of the vertex  $U$  in the complex  $K$ , while  $St_\beta U$  will, as before, denote the union of the members of  $\beta$  intersecting  $U$ .

For a subset  $S$  of  $X$ , let  $\hat{S}$  denote the union of  $A \cap S$  and of the open stars of the vertices corresponding to the members of  $\beta$  which are contained in  $S$ ; i.e.,  $\hat{S} = (A \cap S) \cup (\cup [St_K U | U \subset S])$ . The space  $Z$  is topologized by means of the subbasis consisting of all the open

subsets of  $|K|$  and all the sets of the form  $\hat{U}$ , where  $U$  is an open subset of  $X$ .

Before we proceed with the rest of the proof, we shall establish the following lemma:

**LEMMA 4.8.** *For every  $x \in A$  and every neighborhood  $V$  of  $x$  in  $X$ , there is a neighborhood  $O_V$  of  $x$  in  $Z$  such that if  $y \in O_V \cap (Z - A)$  and  $s$  is an open simplex of  $K$  containing  $y$  then all the vertices of  $s$  (as members of the cover  $\beta$ ) are contained in  $V$ .*

*Proof.* Given a neighborhood  $V$ , we choose a neighborhood  $W$  of  $x$  according to Lemma (4.6). Let  $O_V = \hat{W}$ . Then if  $y \in O_V \cap (Z - A)$  and  $s$  is the carrier of  $y$  in  $K$ , some vertex  $U$  of  $s$  is contained in  $W$ ; and all the other vertices are contained in  $V$  since they meet  $U \subset W$ .

*Continuation of the proof of (4.7).* The fact that  $Z$  is Hausdorff follows readily from lemma (4.8). Since the cover  $\beta$  is equivariant, it follows that  $\alpha: X \rightarrow X$  defines a periodic map on  $K$  which, together with the map  $a$ , define a periodic map  $c: Z \rightarrow Z$  of period  $p$ . The continuity of  $c$  follows from the fact that  $\beta$  is equivariant; and condition (iv) of (4.1) implies that the map  $c$  is free on  $K$ . Thus conditions (i) and (ii) hold.

The map  $\mu: (X, X - A) \rightarrow (Z, Z - A)$  of condition (iii) is defined by the identity on  $A$  and a canonical map  $X - A \rightarrow |K|$  of  $X - A$  into the space of the nerve of the covering  $\beta$  which can be described as follows: if  $x \in X - A$ , then the barycentric coordinates of  $\mu x$  with respect to the vertex  $U$  of  $K$  is

$$\frac{d(x, X - U)}{\sum_{V \in \beta} d(x, X - U)}.$$

Since  $\beta$  is equivariant and the map  $a$  is isometric with respect to  $d$ , it follows that  $\mu: X \rightarrow Z$  is equivariant. The continuity of  $\mu$  follows easily from the definition of the topology, just as in [2].

A retraction  $r^0: A \cup |K^0| \rightarrow A$  may be defined as follows. Let  $A^0$  denote the set of orbits of the map  $c$  on  $K^0$  and let  $\varphi: A^0 \rightarrow K^0$  be any cross-section of the identification map  $K^0 \rightarrow A^0$ . Let  $N^0 = \varphi(A^0)$ . Since  $c$  acts freely on  $K^0$ , it follows that  $K^0$  is the disjoint union  $K^0 = N^0 \cup c(N^0) \cup \dots \cup c^{p-1}(N^0)$ ; thus for each vertex  $V$  of  $K^0$  there is a unique vertex  $U$  of  $N^0$  and a unique integer  $j$ ,  $0 \leq j < p$ , such that  $V = c^j U$ . Given a vertex  $U$  of  $N^0$ , let  $r^0 U$  denote any point of  $A$  such that  $d(U, A) = d(U, r^0 U)$  (such a point exists since  $A$  is compact). If  $V$  is any vertex of  $K^0$ , choose  $U \in N^0$  and an integer  $j$  as

above such that  $V = c^j U$  and define  $r^0 V = a^j(r^0 U)$ . Since the map  $a$  is isometric, we have  $d(V, A) = d(V, r^0 V)$ . Defining  $r^0$  to be the identity on  $A$ , we obtain a retraction  $r^0: A \cup |K^0| \rightarrow A$ . To prove the continuity of  $r^0$ , it suffices to consider the restriction  $r^0|_{(A \cup N^0)}$ , since the sets  $A \cup (a^j N^0)$  are closed and the intersection of any two of them is  $A$ . Thus let  $U$  be a vertex of  $N^0$ , let  $z = r^0 U$  and let  $B = B(z, \varepsilon)$  be an open ball with center  $z$  and radius  $\varepsilon$ . Let  $V = B(z, \varepsilon/3)$  and let  $O_V$  be a corresponding neighborhood of  $x$  in  $Z$  satisfying the assertion of Lemma (4.8). Then  $r^0((A \cup N^0) \cap O_V) \subset B$ . Moreover,  $r^0$  is equivariant by its definition.

Now, if  $\dim(X - A) \leq n$ , then by (4.1), (v),  $\text{Ord } \beta \leq p(n + 1)$  and hence  $\dim K \leq p(n + 1) - 1$ . This proves condition (v).

REMARK. One can easily show that the space  $Z$  is, in fact, compact and metrizable.

**5. Proof of the extension Theorem (1.1).** By (2.4) we can assume that  $Y$  is an equivariant subspace of a Hilbert cube  $Q$  with an isometric periodic map  $b: Q \rightarrow Q$  of period  $p$  such that the map  $Y \rightarrow Y$  is the restriction of  $b$ .

We shall first prove case (ii) of (1.1). Suppose that  $Y$  is an ANR. Then there is an equivariant compact neighborhood  $C$  of  $Y$  in  $Q$  and a (not necessarily equivariant) retraction  $r: C \rightarrow Y$ . Let  $\delta = d(Y, Q - C)$ ; then  $\delta > 0$ . By the uniform continuity of  $r$  there exists a function  $\eta: R_+ \rightarrow R_+$  ( $R_+$  = the set of positive numbers) below the diagonal ( $\eta(\varepsilon) \leq \varepsilon$ ) such that

$$(5.1) \quad \text{If } y \in C \text{ and } d(y, Y) \leq \eta(\varepsilon) \text{ then } d(y, ry) \leq \varepsilon.$$

Consequently,  $d(b^j y, b^j ry) \leq \varepsilon$ , for every  $j = 0, \dots, p - 1$ , since  $b$  is isometric.

Let  $n = p \cdot (\dim(X - A) + 1) - 1$ . Define a sequence of positive numbers  $\varepsilon_0, \dots, \varepsilon_n$  as follows:

$$(5.2) \quad \varepsilon_n = \delta; \quad \varepsilon_{m-1} = \frac{1}{2} \eta\left(\frac{\varepsilon_m}{4}\right) \quad \text{for } 0 < m \leq n.$$

By the uniform continuity of  $f$  there is a  $\xi > 0$  such that if  $x, x' \in A$  and  $d(x, x') \leq \xi$  then  $d(fx, fx') \leq \varepsilon_0$ .

Let  $(Z, c)$  be a space with a periodic map  $c: Z \rightarrow Z$  of period  $p$ , a triangulation  $K$  of  $Z - A$ , an equivariant map  $\mu: X \rightarrow Z$  and an equivariant retraction  $r^0: A \cup |K^0| \rightarrow A$  as provided by the Replacement Lemma (4.7). Let  $L$  be the subcomplex of  $K$  consisting of the simplices  $s$  of  $K$  such that  $d(r^0 U, r^0 V) < \xi$  for all vertices  $U, V$  of  $s$ .

LEMMA 5.3.  $A \cup |L|$  is an equivariant neighborhood of  $A$  in  $Z$ .

*Proof.* Let  $z \in A$ . By the countinuity of  $r^0$  there is a neighborhood  $N$  of  $z$  in  $Z$  such that  $d(r^0 U, z) < \xi/2$  for every  $U \in (A \cup K^0) \cap N$ . Thus  $N \subset A \cup |L|$  since every simplex of  $K$  in  $N$  is in  $L$ .

We shall construct an extension of  $f$  over  $A \cup |L|$ . By induction, we construct a sequence of equivariant maps

$$h_m: A \cup |L^m| \longrightarrow Y$$

such that  $h_m$  extends  $h_{m-1}$  and that the following condition (5.4.m) holds:

$$(5.4.m) \quad \text{diam}(h_m s) \leq \varepsilon_m, \quad \text{for each } m\text{-simplex } s \text{ of } L^m.$$

Define  $h_0: A \cup |L^0| \rightarrow Y$  by  $h_0 U = f r^0 U$ , for each vertex  $U$  of  $L$ . Suppose that  $h_{m-1}: A \cup |L^{m-1}| \rightarrow Y$  is defined so that condition (5.4.m-1) holds. Let  $M = L^m - L^{m-1}$  be the set of the  $m$ -simplices of  $L$ ; and let  $A$  denote the set of orbits of the simplices of  $M$  under the map  $c$ . Let  $\varphi: A \rightarrow M$  be any cross-section of the identification map  $M \rightarrow A$  and  $N = \varphi(A)$ . Since  $c$  acts freely on  $K$ , it follows that  $M$  is the disjoint union  $M = N \cup (cN) \cup \dots \cup (c^{p-1}N)$  and thus for each simplex  $t$  of  $M$  there is a unique simplex  $s$  of  $N$  and a unique integer  $j$ ,  $0 \leq j < p$ , such that  $t = c^j s$ .

By (5.4.m-1) we have for each simplex  $s$  of  $M$

$$\text{diam}(h_{m-1}(\dot{s})) \leq 2\varepsilon_{m-1}$$

and  $2\varepsilon_{m-1} \leq \eta(\varepsilon_m/4) \leq \varepsilon_m/4 \leq \delta$ . Therefore  $\text{Conv}(h_{m-1}(\dot{s})) \subset C$ .

Let  $t$  be a closed  $m$ -simplex of  $M$ . Choose a simplex  $s$  of  $N$  and an integer  $j$ ,  $0 \leq j < p$ , such that  $t = c^j s$ ; thus  $t \in c^j N$ . The map  $h_{m-1}|_{\dot{s}}: \dot{s} \rightarrow Y \subset C$ , where  $\dot{s}$  is the boundary of  $s$ , can be extended in  $\text{Conv}(h_{m-1}(\dot{s}))$  to a map  $u: s \rightarrow C$ . Then both  $u$  and  $r \circ u: s \rightarrow Y$  extend  $h_{m-1}|_{\dot{s}}$ . Note that if  $x \in s$  then by (5.2),

$$d(ux, Y) \leq 2\varepsilon_{m-1} = \eta\left(\frac{\varepsilon_m}{4}\right),$$

and by (5.1),

$$d(rux, ux) \leq \frac{\varepsilon_m}{4}.$$

Therefore

$$(5.5) \quad \text{diam}((ru)s) \leq \text{diam}(us) + 2 \cdot \frac{\varepsilon_m}{4} \leq \frac{\varepsilon_m}{4} + \frac{\varepsilon_m}{2} < \varepsilon_m.$$

Let  $v^{(t)}: t \rightarrow Y$  be defined by

$$(5.6) \quad v^{(t)} = b^j \circ r \circ u \circ c^{p-j}.$$

Then the map  $v^{(t)}$  agrees with  $h_{m-1}$  on  $\dot{t}$ , since  $h_{m-1}$  is equivariant. It follows that the maps  $v^{(t)}$ ,  $t \in M$ , together with  $h_{m-1}$ , define an equivariant map

$$h_m: A \cup |L^{m-1}| \cup |M| = A \cup |L^m| \longrightarrow Y.$$

The fact that  $h_m$  satisfies the inductive condition (5.4.m) follows from the fact that  $v^{(t)}$ ,  $t \in M$ , satisfies it by (5.5), (5.6) and since  $b$  is isometric.

This completes the inductive step of the construction of  $h_m$ . The maps  $h_m$  define a map  $h: A \cup |L| \rightarrow Y$  by  $h|(A \cup |L|) = h_m$ . The map  $h$  is continuous on  $|L|$  since it is defined simplicially there. Hence, it suffices to prove the continuity of  $h$  on  $A$ . Let  $z \in A$  and let  $B(hz, \varepsilon)$  be an open  $\varepsilon$ -ball in  $Y$  with center  $hz = hz$ . Let  $\varepsilon_n = (1/2)\varepsilon$  and let positive numbers  $\varepsilon_0, \dots, \varepsilon_{n-1}$  be constructed as in (5.1). By the continuity of the maps

$$A \cup |K^0| \xrightarrow{r^0} A \xrightarrow{f} Y$$

there is a neighborhood  $G$  of  $z$  in  $Z$  such that  $f(A \cap G) \subset B(fz, \varepsilon/2)$ ,  $G \cap |K|$  is the union of open simplices of  $K$ , and for each simplex  $s$  of  $K$  in  $G$ ,  $fr^0(s^0) \subset B(fz, \varepsilon_0/2)$  (here  $s^0$  denotes the set of the vertices of  $s$ ). Then  $\text{diam}(s^0) < \varepsilon_0$  and, by the construction of (5.1), it follows that  $\text{diam}(hs) < \varepsilon/2$ . Therefore  $hG \subset B(hz, \varepsilon)$ . This completes the proof in case (ii).

In case (i), when  $Y$  is an  $AR$ , there is retraction  $r: Q \rightarrow Y$ , i.e., we may take  $C = Q$ , which is convex. In this case the construction simplifies: we may take  $\varepsilon_n = \infty$  which makes conditions (5.1) and (5.2) vacuous and  $L = K$ . By inductions we can define a map  $h: A \cup |K| \rightarrow Y$  as before. The continuity of  $h$  must, however, be proved as in case (ii), by using the numbers defined in (5.1).

In either case, we have constructed a symmetric map  $h: A \cup |L| \rightarrow Y$ , where  $A \cup |L|$  is a symmetric neighborhood of  $A$  in  $Z$  in case (ii) and  $L = K$  in case (i). Define  $g = h \circ \mu|_{\mu^{-1}(A \cup |L|)}: \mu^{-1}(A \cup |L|) \rightarrow Y$ . Then  $h$  is a symmetric extension of  $f$  over the symmetric neighborhood  $\mu^{-1}(A \cup |L|)$  of  $A$  in  $X$  which in case (i) is the whole of  $X$ .

This completes the proof.

## 6. Equivariant absolute retracts.

**THEOREM 6.1.** *Let  $(X, a)$  be an object of  $\mathcal{A}_p$  such that  $X$  is a compact metric space with  $\dim X < \infty$ . Then:*

- (i)  $X$  is an EAR iff both  $X$  and the fixed point set  $F(a)$  are AR's.
- (ii)  $X$  is an EANR iff both  $X$  and the fixed point set  $F(a)$  are ANR's.

*Proof.* By (2.4) we can assume that  $(X, a)$  is equivariantly embedded in a finite-dimensional cube  $I^n$  with an isometric periodic map  $a: I^n \rightarrow I^n$  of period  $p$  which we still denote by  $a: I^n \rightarrow I^n$ . Let  $F' = F(a_{I^n})$ ; then  $F'(a_X) = F' \cap X$ .

We shall prove case (ii); case (i) is just simpler and was done in §1. If  $X$  is an EANR then there is an equivariant retraction  $r: W \rightarrow X$  of an equivariant neighborhood  $W$  of  $X$  in  $I^n$  to  $X$ . The retraction  $r$  defines a retraction of  $F' \cap W$  to  $F' \cap X$ . Since  $a: I^n \rightarrow I^n$  is isometric,  $F'$  is convex and compact, hence it is an AR (in fact, it is homeomorphic to a cube). Thus both  $X$  and  $F' \cap X$  are ANR's.

Suppose now that both  $X$  and  $F' \cap X$  are ANR's. Then by the Addition Theorem for ANR's ([1], p. 90), it follows that  $F' \cup X$  is an ANR. By the Equivariant Extension Theorem (1.1), the identity  $F' \cup X \rightarrow F' \cup X$  can be extended to an equivariant retraction  $r: U \rightarrow F' \cup X$ , where  $U$  is an equivariant neighborhood of  $F' \cup X$  in  $I^n$ . Since  $F' \cap X$  is an ANR, there is a neighborhood  $V$  of  $F' \cap X$  in  $F'$  and a retraction  $q: V \rightarrow F' \cap X$ . Note that  $V \cup X$  is a neighborhood of  $X$  in  $F' \cup X$ . Let  $U_0 = r^{-1}(V \cap X)$ . Then  $U_0$  is an equivariant neighborhood of  $X$  in  $I^n$  and the map  $U_0 \rightarrow X$  defined by  $x \mapsto qrx$  is an equivariant retraction of  $U_0$  to  $X$ .

Since the cube  $I^n$  with an isometric periodic map period  $p$  is an EAR (see (3.6)), it follows that  $X$  is an EANR.

We thus have an answer to our original question.

**COROLLARY 6.2.** *Let  $E$  be a Euclidean space, let  $F$  be the diagonal of  $E \times E$  and let  $X$  be an equivariant compact subset of  $E \times E$  (with respect to the involution  $(x, y) \rightarrow (y, x)$ ). Then  $X$  is an equivariant retract of  $E \times E$  if and only if  $X$  is a retract of  $E \times E$  and  $F \cap X$  is a retract of  $F$ .*

For, in this case,  $F$  is the fixed point set of the involution  $E \times E \rightarrow E \times E$ .

Just as in Theorem (1.1), it is an open question whether the finite-dimensional assumption in Theorem (6.1) is essential:

**Question 6.3.** Does there exist a space with an involution  $a: X \times X \rightarrow X \times X$  such that both  $X$  and the fixed point set  $F(a)$  are AR's but  $X$  is not an EAR?

More specifically, let  $Q$  be a Hilbert cube and consider the symmetry  $Q \times Q \rightarrow Q \times Q$  with respect to the diagonal  $F$  of  $Q \times Q$ . Let

$X$  be a symmetric subset of  $Q \times Q$  such that  $X$  is a retract of  $Q \times Q$  and  $F \cap X$  is a retract of  $F$ . Does there exist a symmetric retraction of  $Q \times Q$  to  $X$ ?

**7. Equivariant homotopy.** As an application of the previous results, we prove in this section two equivariant homotopy extension theorems.

**DEFINITION 7.1.** If  $(X, a)$  is an object of  $\mathcal{A}_p$  and  $I$  is the unit interval then an equivariant homotopy is an equivariant map  $h: X \times I \rightarrow Y$  to an object  $(Y, b)$  of  $\mathcal{A}_p$ , where the periodic map on  $X \times I$  is  $a \times 1_I: X \times I \rightarrow X \times I$ .

If  $A$  is an equivariant subspace of  $X$  then the maps  $a$  and  $a \times 1_I$  define a periodic map  $(a \times 1_I)_T: T \rightarrow T$ , where  $T = (X \times \{0\}) \cup (A \times I) \subset X \times I$ ; it is the restriction of  $a \times 1_I$ . The following lemma is an equivariant version of the Dowker lemma used in extending homotopies:

**LEMMA 7.2.** *Let  $(X, a)$  be an object of  $\mathcal{A}_p$  such that  $X$  is a metric space, let  $A$  be a closed equivariant subspace of  $X$  and let  $g: (T, (a \times 1_I)_T) \rightarrow (Y, b)$  be an equivariant map, where  $(Y, b)$  is an object  $\mathcal{A}_p$ . If  $g$  can be extended to an equivariant map  $g': U \rightarrow Y$  of an equivariant neighborhood  $U$  of  $T$  in  $X \times I$  then  $g$  can be extended to an equivariant map  $h: X \times I \rightarrow Y$ .*

*Proof.* Choose an equivariant neighborhood  $V$  of  $A$  in  $X$  such that  $(\text{Cl } V) \times I \subset U$  and an equivariant Urysohn function  $u: X \rightarrow I$  which is 1 on  $A$  and 0 on  $X - V$ ; for instance,  $u$  may be defined by using an equivariant distance function  $d$  on  $X$  with the usual formula:

$$ux = \frac{d(x, X - V)}{d(x, A) + d(x, X - V)}.$$

Then we define  $h(x, t) = g'(x, (ux) \cdot t)$ .

**THEOREM 7.3.** *Let  $(X, a)$  be an object of  $\mathcal{A}_p$  such that  $X$  is a metric space, let  $(Y, b)$  be an EANE and let  $f: X \rightarrow Y$  be an equivariant closed subspace of  $X$ . Then any equivariant homotopy of  $f|_A$  can be extended to an equivariant homotopy of  $f$ .*

This follows from (7.2) and (3.1). Similarly, (7.2) and (1.1) yield the following result:

**THEOREM 7.4.** *Let  $(X, a)$  be an object of  $\mathcal{A}_p$  such that  $X$  is a compact metric finite-dimensional space, let  $A$  be a closed equivariant*



subspace of  $X$  containing all the fixed points of  $a$ , let  $(Y, b)$  be an object of  $\mathcal{A}_p$  such that  $Y$  is a compact ANR, and let  $f: X \rightarrow Y$  be an equivariant map. Then any equivariant homotopy of  $f|_A$  can be extended to an equivariant homotopy  $f$ .

The author is indebted to Richard J. Allen who read the manuscript and helped to remove many mistakes.

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Received December 23, 1971 and in revised form March 17, 1972.

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