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**ON SOME MEAN VALUES ASSOCIATED WITH A RANDOMLY
SELECTED SIMPLEX IN A CONVEX SET**

H. GROEMER

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For any convex body K in euclidean n -space denote by $m(K)$ the mean value of the volume of a simplex with vertices at $n + 1$ randomly selected points from K . It is shown that among all convex bodies of given volume the mean value $m(K)$ is minimal if and only if K is an ellipsoid. Actually, a more general result is obtained which shows that the higher order moments of the volume of a randomly selected simplex in a convex set have similar minimal properties.

Throughout this paper R^n denotes euclidean n -space, where n is a given fixed positive integer. A compact convex subset of R^n which has interior points will be called a convex body. The volume of a convex body X will be denoted by $v(X)$. If p_1, p_2, \dots, p_{n+1} are $n + 1$ points of R^n we write $C(p_1, p_2, \dots, p_{n+1})$ to denote the convex hull of the points p_1, p_2, \dots, p_{n+1} . Including various forms of degeneracy, $C(p_1, p_2, \dots, p_{n+1})$ will be called a simplex with vertices at p_1, p_2, \dots, p_{n+1} .

Let K be a given convex body. If x_1, x_2, \dots, x_{n+1} are $n + 1$ points from K the volume of the simplex with vertices at x_1, x_2, \dots, x_{n+1} is given by $v(C(x_1, x_2, \dots, x_{n+1}))$ and, assuming that the points x_1, x_2, \dots, x_{n+1} are variable, the mean value of this volume is defined by

$$(1) \quad m(K) = (1/v(K))^{n+1} \int_{x_1 \in K} \dots \int_{x_{n+1} \in K} v(C(x_1, \dots, x_{n+1})) dx_1 \dots dx_{n+1}.$$

Since $v(C(x_1, x_2, \dots, x_{n+1}))$ is a continuous function in the space $R^{n(n+1)}$ and since the set defined by the $n + 1$ conditions $x_i \in K$ ($i = 1, 2, \dots, n + 1$) is a compact convex set in $R^{n(n+1)}$ it is obvious that $m(K)$ exists for every convex body K .

Blaschke [1], [2] has proved that for convex bodies in R^2 of given volume (i.e., area) the mean value $m(K)$ is minimal if and only if K is an ellipse. See also Klee [11] for the history of this problem. Kingman [10] has conjectured that for any dimension n and fixed volume $v(K)$ the minimum of $m(K)$ is reached if K is a (solid) sphere in R^n . In addition, he pointed out that the higher order moments of the expected volume, i.e., the expressions

$$(2) \quad m_r(K) = (1/v(K))^{n+1} \int_{x_1 \in K} \dots \int_{x_{n+1} \in K} (v(C(x_1, \dots, x_{n+1})))^r dx_1 \dots dx_{n+1}$$

are of interest. The definitions (1) and (2) show that $m_1(K) = m(K)$.

Just as before, it is seen that $m_r(K)$ exists for every convex body K and every $r \geq 0$. It is also clear that $m_r(K)$ is invariant under volume preserving affine transformation.

The main purpose of this paper is to provide a proof of Kingman's conjecture and of a similar but more general statement for the higher order moments. The following theorem contains the precise formulation of our result.

THEOREM. *For any convex body K in R^n and any real number r with $r \geq 1$ the moments $m_r(K)$ satisfy the inequality*

$$m_r(S) \leq m_r(K)$$

where S is a solid sphere in R^n which has the same volume as K . Equality holds if and only if K is an ellipsoid.

Because of $m_1(K) = m(K)$ this theorem has the following corollary as an obvious consequence.

COROLLARY 1. *Among all convex bodies of given volume the mean value $m(K)$ of the volume of a simplex with vertices at $n + 1$ randomly selected points from the convex body K is minimal if and only if K is an ellipsoid.*

Kingman [10] has been able to find an explicit formula for $m(K)$ in the case when K is an ellipsoid of R^n , namely

$$m(K) = 2^n \left(\frac{n+1}{2} \right)^{n+1} \left(\frac{(n+1)^2}{2} \right)^{-1} v(K).$$

Corollary 1 is related to a problem which, in two dimensional space, is frequently referred to as Sylvester's problem (cf. Kendall and Moran [9]). If $n + 2$ points of R^n are selected at random from a convex body K the problem consists of finding the probability, say $P(K)$, that none of these $n + 2$ points is in the interior of their convex hull. A simple calculation shows that (see Kingman [10])

$$P(K) = 1 - \frac{(n+2)m(K)}{v(K)}.$$

It follows that Corollary 1 is equivalent with the following statement.

COROLLARY 2. *For any convex body K of R^n the probability $P(K)$ that the convex hull of $n + 2$ randomly selected points from K con-*

tains none of these points in its interior is maximal if and only if K is an ellipsoid.

Similarly as the proof given by Blaschke for $n = 2, r = 1$ our proof of the above theorem depends on a property of the Steiner symmetrization of a convex body and on a certain characterization of ellipsoids. Since this characterization, which is of independent interest, appears to have been investigated only in the cases $n = 2$ and $n = 3$ (see Bonnesen and Fenchel [4], p. 143) we supply a new proof which imposes no restriction on the dimension or regularity of the convex body (Lemma 2).

First, we prove a lemma which shows that there exist convex bodies which have the desired minimal property with respect to $m_r(K)$.

LEMMA 1. *If r is a given positive number there exists a convex body K_0 in R^n such that $v(K_0) = 1$ and*

$$(3) \quad m_r(K_0) \leq m_r(K)$$

for every convex body K with $v(K) = 1$.

Proof. For every convex body K there exist, according to a theorem of John [8], two ellipsoids E, E' such that $E' \subset K \subset E$ and $v(E) \leq n^r v(E')$. Because of $v(E') \leq v(K)$ this implies $v(E) \leq n^r v(K)$. It follows that to any K with $v(K) = 1$ there is a volume preserving affine transformation σ such that $\sigma K \subset B$, where B is a sphere of volume n^n and center at the origin of the coordinate system. Because of this fact and because of the invariance of m_r under volume preserving affine transformations it is evident that it suffices to prove (3) under the additional assumptions that $v(K) = 1$ and $K \subset B$. Let us denote by \mathcal{K} the class of all convex bodies for which these two conditions are satisfied. If a number μ is defined by

$$\mu = \inf m_r(K) \quad (K \in \mathcal{K})$$

then μ has obviously the property that for every $K \in \mathcal{K}$

$$(4) \quad \mu \leq m_r(K)$$

and that there exists a sequence K_1, K_2, \dots of convex bodies in \mathcal{K} such that

$$(5) \quad \lim_{i \rightarrow \infty} m_r(K_i) = \mu.$$

Because of $K_i \subset B$ the selection theorem of Blaschke can be applied to the class of convex bodies K_i . This justifies the assumption that

the sequence K_1, K_2, \dots converges (in the Hausdorff-Blaschke metric) to some convex set K_0 . Note that $K_i \in \mathcal{K}$ implies $K_0 \in \mathcal{K}$.

The functional m_r is obviously translation invariant, monotone and homogeneous in the sense that $m_r(sK) = s^{n \cdot r} m_r(K)$ for any $s \geq 0$. It is known that such a functional is also continuous (cf. Hadwiger [7], p. 204 and the proof of the continuity of the volume in Blaschke [3], p. 61 or Eggleston [6], p. 72). Therefore, the convergence of K_i to K_0 implies

$$(6) \quad \lim_{i \rightarrow \infty} m_r(K_i) = m_r(K_0).$$

Since (3) is an immediate consequence of (4), (5), and (6) the proof of the Lemma is finished.

K_0 will be referred to as a *minimum body* for m_r . Actually, K_0 does not depend on r if $r \geq 1$; but this cannot be concluded from our proof of Lemma 1.

For the formulation of our next lemma it is convenient to call a subset of R^n *flat* if it is contained in some plane. It should be noted that in this paper a plane is always understood to be a hyperplane. As a further notational simplification the following concept will be used. If K is a convex body and if G is a line in R^n we denote by $\mathcal{S}(K, G)$ the set of midpoints of all line segments of the form $X \cap K$ where X ranges over all lines that are parallel to G and meet K . $\mathcal{S}(K, G)$ will be called a *midpoint set* of K .

LEMMA 2. *A convex body K is an ellipsoid if and only if the midpoint set $\mathcal{S}(K, G)$ is flat for every line G of R^n .*

Proof. If K is a sphere the midpoint set $\mathcal{S}(K, G)$ is obviously flat for every line G . Applying an affine transformation the same result is seen to be true for ellipsoids.

Assume now that for a given convex body K the midpoint set $\mathcal{S}(K, G)$ is flat for every line G . Let H be any plane, and choose a coordinate system in R^n which has the property that H is given by $H = \{(x^1, x^2, \dots, x^n) \mid x^n = 0\}$. Then, if G is a line that is orthogonal to H , the equation of the plane which contains $\mathcal{S}(K, G)$ can be written in the form

$$x^n = a_0 + a_1 x^1 + \dots + a_{n-1} x^{n-1}.$$

The symmetrization of K with respect to the plane H is achieved by mapping each point (p^1, p^2, \dots, p^n) of K onto the point

$$(p^1, p^2, \dots, p^{n-1}, p^n - (a_0 + a_1 p^1 + \dots + a_{n-1} p^{n-1})).$$

This mapping is obviously an affine transformation. Hence, one can

conclude that every symmetrization is a volume preserving affine transformation, provided that the midpoint set $\mathcal{P}(K, G)$ is flat for every line G of R^n .

The convex body obtained from K by symmetrization with respect to a plane H will be denoted by $\tilde{K}(H)$.

It is known (see Danzer, Laugwitz, and Lenz [5]) that there is an ellipsoid, say L , which contains K and has smallest possible volume. It is also known (see Hadwiger [7], p. 170) that there is a sequence of planes, say H_1, H_2, \dots , in R^n such that the sequence of convex bodies which is defined by $K_1 = K$, $K_{i+1} = \tilde{K}_i(H_i)$ ($i = 1, 2, \dots$) contains a subsequence that converges to a sphere S . It follows that there are volume preserving affine transformations $\sigma_1, \sigma_2, \dots$ such that the sequence $\sigma_1 K, \sigma_2 K, \dots$ converges to S . If $K = L$ the proof of the lemma is obviously finished. Let us assume that $K \neq L$. In this case we have

$$(7) \quad v(K) = v(S) < v(L) .$$

Since the sequence $\sigma_1 K, \sigma_2 K, \dots$ converges to S there exists for any positive ε an index h such that

$$(8) \quad \sigma_h K \subset S^\varepsilon .$$

Here, S^ε denotes the parallel domain of S , which, in this case is a sphere of radius $r + \varepsilon$ if S has radius r . Because of (7) ε can be taken so small that

$$(9) \quad v(S^\varepsilon) < v(L) .$$

(8) implies that the ellipsoid $\sigma_h^{-1} S^\varepsilon$ contains K , and (9) shows that $v(\sigma_h^{-1} S^\varepsilon) < v(L)$. However, according to the definition of L it is impossible that an ellipsoid which contains K has smaller volume than L . It follows that the trivial case $K = L$ is the only possibility.

LEMMA 3. *Let G_1, G_2, \dots, G_{n+1} be $n + 1$ distinct lines in R^n which are of the form $G_k = \{(c_k^1, c_k^2, \dots, c_k^{n-1}, z_k) \mid -\infty < z_k < \infty\}$. Assume that to each G_k there corresponds an interval I_k of the form $I_k = \{(c_k^1, c_k^2, \dots, c_k^{n-1}, z_k) \mid |z_k - p_k| \leq l_k\}$ where $l_k > 0$. Write $z = (z_1, z_2, \dots, z_{n+1})$, $p = (p_1, p_2, \dots, p_{n+1})$, $e = (1, 1, \dots, 1)$, $c^j = (c_1^j, c_2^j, \dots, c_{n+1}^j)$ and*

$$D(z) = \frac{1}{n} \det(e, c^1, c^2, \dots, c^{n-1}, z) .$$

Finally, if r is a given real number with $r \geq 1$ write

$$(10) \quad M(p) = \int_{|z_k - p_k| \leq l_k} |D(z)|^r dz .$$

Then, if the numbers c_k^1 and the interval lengths l_k are fixed, $M(p)$ attains its absolute minimum value exactly for those vectors p for which all the midpoints $(c_k^1, c_k^2, \dots, c_k^{n-1}, p_k)$ of the intervals I_k ($k = 1, 2, \dots, n+1$) are contained in some plane of R^n .

Proof. Since $D(z)$ is a linear function of z (10) can be written in the form

$$(11) \quad M(p) = \int_{|u_k| \leq l_k} |D(u) + D(p)|^r du$$

where $u = (u_1, u_2, \dots, u_{n+1})$ and $u = z - p$. If p varies over the total R^{n+1} the linear function $D(p)$ takes on any value between $-\infty$ and ∞ . Therefore, a comparison of (11) with the function

$$(12) \quad F(y) = \int_{|u_k| \leq l_k} |D(u) + y|^r du$$

shows that $M(p)$ and $F(y)$ have the same greatest lower bound. If all y -values for which $F(y)$ is (absolutely) minimal are known, the set of all vectors p for which $M(p)$ is minimal are found by solving the linear equation

$$(13) \quad y = D(p)$$

for each such known y -value.

Now, to investigate the minimum value of $F(y)$ we note that $D(u) = -D(-u)$ implies

$$\int_{|u_k| \leq l_k} |D(u) + y|^r du = \int_{|u_k| \leq l_k} |D(u) - y|^r du.$$

This, together with the definition (12), shows that

$$(14) \quad \begin{aligned} & F(y) - F(0) \\ &= \frac{1}{2} \int_{|u_k| \leq l_k} (|D(u) + y|^r + |D(u) - y|^r - 2|D(u)|^r) du. \end{aligned}$$

Since for a fixed value of r ($r \geq 1$) the function $|\zeta|^r$ is convex it follows that the integrand in (14), say $T(u, y)$, has the property that for all values of u and y

$$(15) \quad T(u, y) \geq 0.$$

(The convexity of the function $|\zeta|^r$, i.e., the relation $|(\zeta_1 + \zeta_2)/2|^r \leq (|\zeta_1|^r + |\zeta_2|^r)/2$, is a special case of Hölder's inequality $|\alpha a + \beta b| \leq (|\alpha|^p + |\beta|^p)^{1/p} (|a|^q + |b|^q)^{1/q}$, namely the case $\alpha = \beta = 1/2$, $a = \zeta_1$, $b = \zeta_2$, $p = r/r - 1$, $q = r$). In addition to (15) it is clear that for $y \neq 0$

$$(16) \quad T(0, y) = 2 |y|^r > 0.$$

Because of the continuity of $T(u, y)$ as a function in u (16) implies that for a given value of y with $y \neq 0$ the inequality

$$(17) \quad T(u, y) > 0$$

holds not only for $u = 0$ but for a whole interval with center at $u = 0$. From (14), (15), and (17) it follows that for any $y \neq 0$

$$F(y) > F(0).$$

Hence, $F(y)$ attains an absolute minimum value at $y = 0$ and nowhere else. This result in conjunction with (13) shows that $M(p)$ is minimal if and only if $D(p) = 0$. Since $D(p)$ is the volume of a simplex with vertices at the points $(c_k^1, c_k^2, \dots, c_k^{n-1}, p_k)$ we find finally that these points are contained in a plane if and only if $M(p)$ is minimal.

Proof of the Theorem. Since it has already been pointed out that $m_r(K)$ is a homogeneous function of K it suffices to prove the Theorem under the assumption $v(K) = 1$.

As before, let H be the plane $\{(x^1, x^2, \dots, x^n) \mid x^n = 0\}$. Assume that G_1, G_2, \dots, G_{n+1} are $n + 1$ given lines which are orthogonal to H and have the property that each G_k intersects K in a line segment I_k of positive length l_k . The midpoint of I_k will again be denoted by $(c_k^1, c_k^2, \dots, c_k^{n-1}, p_k)$. Under these assumptions the number $M(p)$ can be defined by (11). However, since in this case the vector p is completely determined if K and G_1, G_2, \dots, G_{n+1} are given we write now $M(K; G_1, G_2, \dots, G_{n+1})$ instead of $M(p)$. Let $\tilde{K} = \tilde{K}(H)$ be the convex body which is obtained from K by symmetrization with respect to the plane H . Since all the segments $\tilde{K} \cap G_k$ have midpoints that are contained in a plane, namely H , Lemma (3) shows that

$$(18) \quad M(\tilde{K}; G_1, G_2, \dots, G_{n+1}) \leq M(K; G_1, G_2, \dots, G_{n+1})$$

where equality holds if and only if the midpoints of the segments $K \cap G_k$ are already contained in some plane. Assume now that K is a minimum body for m_r and that K is not an ellipsoid. Then Lemma 2 shows that there is a line G such that the midpoint set $\mathcal{S}(K, G)$ is not flat. This implies obviously that $\mathcal{S}(K, G)$ contains $n + 1$ points which are not contained in a plane of R^n . A simple continuity argument shows further that one may assume that the line segments corresponding to these $n + 1$ midpoints have positive lengths. A suitable selection of the coordinate system permits us to assume that the plane $H = \{(x^1, x^2, \dots, x^n) \mid x^n = 0\}$ is orthogonal to G . Hence, if G_1, G_2, \dots, G_{n+1} is any system of $n + 1$ lines that are parallel to G

and meet K in intervals of positive lengths one obtains (18) and the additional information that strict inequality holds for at least one such system of $n + 1$ lines.

Denote now by K_H the projection of K onto the plane H . Further, if w_k is a point of K_H denote by $G(w_k)$ the line which is orthogonal to H and contains w_k . Using the definitions (2) and (10) an obvious rearrangement of the order of integration shows that

$$(19) \quad m_r(K) = \int_{w_1 \in K_H} \cdots \int_{w_{n+1} \in K_H} M(K; G(w_1), G(w_2), \dots, G(w_{n+1})) dw_1 \cdots dw_{n+1}.$$

(Since the integrand has been defined only if the intervals $K \cap G(w_k)$ have positive lengths and if the points w_k are distinct, a set of measure 0 has been neglected.) Because of (18) with strict inequality for at least one system w_1, w_2, \dots, w_{n+1} and because of the continuity of the integrand in (19) (considered as a function of w_1, w_2, \dots, w_{n+1}) the equation (19) implies that

$$m_r(\tilde{K}(H)) < m_r(K).$$

This contradicts the assumption that K be a minimum body for m_r . Therefore, only ellipsoids can be minimal bodies. Because of Lemma 1 and since m_r is invariant under volume preserving affine transformations it follows that any sphere S of unit volume is a minimal body, that

$$m_r(S) < m_r(K)$$

if K is not an ellipsoid, and that

$$m_r(S) = m_r(K)$$

if K is an ellipsoid. Hence, the Theorem is proved.

It might be worth noting that essentially the same method of proof can be used to establish a similar theorem with the higher order moments replaced by more general types of functions.

REFERENCES

1. W. Blaschke, *Über affine Geometrie XI: Lösung des "Vierpunktproblems" von Sylvester aus der Theorie der geometrischen Wahrscheinlichkeiten*, Ber. Verh. sächs. Akad. Leipzig, Bd. **69** (1971), 436-453.
2. ———, *Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie*, Springer, Berlin, 1923.
3. W. Blaschke, Kreis, und Kugel, *Walter de Gruyter*, Berlin, 1956.
4. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, *Ergebn. d. Math.* Bd. **3**, Springer, Berlin, 1934.

5. L. Danzer, D. Laugwitz, and H. Lenz, *Über das Löwnersche Ellipsoid und sein Analogon unter dem einem Eikörper einbeschriebenen Ellipsoiden*, Arch. d. Math., **8** (1957), 214-219.
6. H. G. Eggleston, *Convexity*, Cambr. Tracts in Math., vol. **47**, Cambridge University Press, Cambridge 1958.
7. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Grundle. d. math. Wiss., Bd. **93**, Springer, Berlin-Göttingen-Heidelberg, 1957.
8. F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and essays presented to R. Courant, pp. 187-204, Interscience, New York, 1948.
9. M. G. Kendall and P.A.P. Moran, *Geometrical Probability*. Griffin, London, 1963.
10. J. F. C. Kingman, *Random secants of a convex body*, J. Appl. Prob., **6** (1969), 660-672.
11. V. Klee, *What is the expected volume of a simplex whose vertices are chosen at random from a given convex body?* Amer. Math. Monthly, **76** (1969), 286-288.

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