Pacific Journal of Mathematics

CENTRAL 2-SYLOW INTERSECTIONS

MARCEL HERZOG

Vol. 45, No. 2 October 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 2, 1973

CENTRAL 2-SYLOW INTERSECTIONS

MARCEL HERZOG

Let G be a finite group. A subgroup D of G is called a 2-*Sylow intersection* if there exist distinct Sylow 2-subgroups S_1 and S_2 of G such that $D = S_1 \cap S_2$. An involution of G is called *central* if it is contained in a center of a Sylow 2-subgroup of G. A 2-Sylow intersection is called *central* if it contains a central involution. The aim of this work is to determine all non-abelian simple groups G which satisfy the following condition
B: the 2-rank of all central 2-Sylow intersections is not higher than 1, under the additional assumption that the centralizer of a central involution of G is solvable.

In 1964, M. Suzuki [5] determined all simple groups with all 2-Sylow intersections being trivial (i.e. of rank 0). Using a recent fusion theorem by E. Shult [3, p. 62] the author proved [4] that no additional simple groups are involved if Suzuki's condition is weakened to read: all central 2-Sylow intersections are trivial (i.e. no central involution is contained in a 2-Sylow intersection).

This paper is a step toward the characterization of all simple groups G which satisfy Condition B (in short $G \in B$). We will prove the following

THEOREM. *Let G be a non-abelian simple group. Suppose that $G \in B$ and the centralizer of a central involution z in G is solvable. Then G is isomorphic to one of the following groups*:
(i) $\mathrm{PSL}(2, q)$, $q = 2^n > 2$;
(ii) $\mathrm{Sz}(q)$, $q = 2^n \geqq 8$;
(iii) $\mathrm{PSU}(3, q)$, $q = 2^n > 2$ *and*
(iv) $\mathrm{PSL}(2, q)$, $q \equiv 3$ or $5 \pmod 8$, $q > 5$.

A finite group G is of 2-rank n if an elementary abelian 2-subgroup of G of maximal order contains 2^n elements. The 2-length of G is denoted by $1_2(G)$. The maximal power of 2 dividing $|G|$ is denoted by $|G|_2$. An involution z of G is called isolated if it belongs to a Sylow 2-subgroup S of G and $z^g \in S$ implies $z^g = z$. The maximal normal subgroup of G of odd order is denoted by $0(G)$. Finally the groups Q_8, S_3 and S_4 are the ordinary quarternion group, the symmetric group on 3 letters and the symmetric group on 4 letters, respectively.

2. Properties of groups satisfying Condition B.

LEMMA 1. *Let* $G \in B$, $H \subseteqq G$.

(i) *If* $|H|_2 = |G|_2$ *then* $H \in B$.
(ii) *If* $H \lhd G$ *and* $|G/H|_2 = |G|_2$ *then* $G/H \in B$.

Proof. (i) is obvious. If H is a normal subgroup of G of odd order, then the S_2-subgroups of $\bar{G} = G/H$ are of the form $SH/H = \bar{S} \cong S$, where S is an S_2-subgroup of G. Let S_1 and S_2 be S_2-subgroups of G such that $\bar{S}_1 \cap \bar{S}_2$ is a central 2-Sylow intersection of 2-rank at least 2. Since H is of odd order, there exists a 2-subgroup D of G, such that $\bar{S}_1 \cap \bar{S}_2 = DH/H = \bar{D} \cong D$. It is clear that there exist $h_1, h_2 \in H$ such that $D \subseteqq S_1^{h_1} \cap S_2^{h_2}$. If zH is a central involution of SH/H, $z \in S$, then $[z, s] \in S \cap H = 1$ for all $s \in S$, hence $z \in Z(S)$. Thus D contains a central involution of G and as $G \in B$ and the 2-rank of D is at least 2, it follows that $S_1^{h_1} = S_2^{h_2}$, $\bar{S}_1 = \bar{S}_2$ and $\bar{D}$ is not a 2-Sylow intersection of G. Thus $\bar{G} \in B$.

LEMMA 2. *Let* $G \in B$, $H \subseteqq G$ *and suppose that the following assumptions hold*:
(i) H *is solvable*;
(ii) $|H|_2 = |G|_2$ *and*
(iii) $0_2(H)$ *contains a central involution of* G.
Then $1_2(H) = 1$, *unless* $0_2(\bar{H}) \cong Q_8$ *and* $\bar{H}/0_2(\bar{H}) \cong S_3$, *where* $\bar{H} = H/0(H)$.

Proof. By Lemma 1 H and $\bar{H}$ satisfy Condition B and $0_2(\bar{H})$ obviously contains a central involution of $\bar{H}$. If $0_2(\bar{H})$ is cyclic or generalized quaternion (but not ordinary quaternion), then $\mathrm{Aut}(0_2(\bar{H}))$ is a 2-group and therefore $\bar{H}/C(0_2(\bar{H}))$ is a 2-group. As $\bar{H}$ is solvable, $C(0_2(\bar{H})) \subseteqq 0_2(\bar{H})$ and consequently $\bar{H}$ is a 2-group, hence $1_2(H) = 1$.

If $0_2(\bar{H})$ is of 2-rank at least 2, then $\bar{H} \in B$ forces $\bar{H}$ to be 2-closed, hence $1_2(H) = 1$.

Suppose, finally, that $0_2(\bar{H}) \cong Q_8$. Then $\bar{H}/C(0_2(\bar{H}))$ is isomorphic to a subgroup of S_4 and if $\bar{H}$ is not 2-closed then obviously 24 divides the order of $\bar{H}/C(0_2(\bar{H}))$. Thus $\bar{H}/C(0_2(\bar{H})) \cong S_4$ and $\bar{H}/0_2(\bar{H}) \cong S_3$.

LEMMA 3. *Let* $G \in B$ *and suppose that* S *and* S_1 *are* S_2-*subgroups of* G. *Let* $z \in Z(S)$ *be an involution*, $g \in G$, *and suppose that* $z^g \in S_1$. *Then* $z^g \in Z(S_1)$.

Proof. Suppose that z^g is not central in S_1. Then $S_1 \cap C_G(z^g)$ contains z^g and a central involution of S_1. Let T be an S_2-subgroup of $C_G(z^g)$ containing $S_1 \cap C_G(z^g)$; as $C_G(z^g) \supseteqq S^g$, T is an S_2-subgroup of G. Since the 2-rank of $D = S_1 \cap T$ is at least 2 and D contains a central involution of G, it follows from our assumptions that $S_1 = T$, hence $z^g \in Z(S_1)$, a contradiction.

LEMMA 4. *Let $G \in B$ and suppose that $|\Omega_1(Z(S))| = 2$, where S is an S_2-subgroup of G. Then $\Omega_1(Z(S)) \subseteq Z^*(G)$, where $Z^*(G)/0(G) = Z(G/0(G))$.*

Proof. Let $z \in \Omega_1(Z(S))$; then by Lemma 3 z is an isolated involution in G. It follows then by the Z^*-theorem of Glauberman [2] that $\Omega_1(Z(S)) \subseteq Z^*(G)$.

LEMMA 5. *Let $G \in B$, S be an S_2-subgroup of G and $G = 0(G)S$. Suppose that $|\Omega_1(Z(S))| > 2$ and S is not normal in G. Then the 2-rank of G is at most 2.*

Proof. Let G be a counterexample of minimal order. Then S contains an elementary abelian subgroup A of order 8 such that $|A: Z(S) \cap A| \leqq 2$. Let $H = 0(G)$ and $C = C_S(H)$. Then $C \lhd S$ and consequently $C \lhd SH = G$. As G is not 2-closed and $G \in B$, we have $A \not\subset C$. Consider AH; A is not normal in AH and $|A \cap A^h| \leqq 2$ for all $h \in H - N(A)$, as otherwise $G \notin B$. Thus AH is a counterexample and by the minimality of G, $G = AH$.

Let P be a Sylow p-subgroup of H, such that $A \subseteq N(P)$ and $A \not\subset C(P)$; then again by the minimality of G, $G = AP$. As by a theorem of Burnside A does not centralize $P/\Phi(P)$, it follows by Lemma 1 (ii) and the minimality G that $\Phi(P) = 1$, P is elementary abelian. Since A acts on P in a completely reducible way, it follows again by the minimality of G that A acts irreducibly of P and $A/C_A(P)$ acts faithfully and irreducibly on P. Thus $A/C_A(P)$ is a cyclic group and $C_A(P)$ is a normal subgroup of G of 2-rank 2. As $C_A(P)$ contains a central involution and $G \in B$, it follows that G is 2-closed, a contradiction.

3. Proof of the theorem.

Let $H = C_G(z)$. If H is 2-closed then by Lemma 3 z belongs to a unique Sylow 2-subgroup of G. Therefore by Theorem C of [4] G is isomorphic to one of the groups in (i)-(iii).

Suppose now that H is not 2-closed. Let $\bar{H} = H/0(H)$ and suppose that $0_2(\bar{H}) \cong Q_8$ and $\bar{H}/0_2(\bar{H}) \cong S_3$. Then obviously

$$(*) \qquad \text{2-rank } H = \text{2-rank } G = 2.$$

Otherwise it follows by Lemma 2 that $1_2(H) = 1$, hence $0_{2',2}(H) = SL$, where $L = 0_{2'}(H)$ and S is an S_2-subgroup of G. Since H is not 2-closed, S is not normal in $0_{2',2}(H)$. As G is simple, it follows by Lemma 4 that $|\Omega_1(Z(S))| > 2$ and Lemma 5 then yields (*) again.

Thus in all cases 2-rank $G = 2$ and by the classification theorem of Alperin, Brauer and Gorenstein [1] only three types of 2-groups could occur as a Sylow subgroup S of a group not mentioned in (i)-(iv):

(a) dihedral of order 8 at least,

(b) quasi-dihedral, or

(c) wreathed.

In all of these cases $Z(S)$ is cyclic, hence by Lemma 4 G is non-simple, a contradiction. The proof of the theorem is complete.

References

1. J. L. Alperin, R. Brauer and D. Gorenstein, *Finite simple groups of 2-rank two*, to appear.

2. G. Glauberman, *Central elements of core-free groups*, J. Algebra, **4** (1966), 403-420.

3. ———, *Global and local properties of finite groups*, Finite Simple Groups, Academic Press, 1971.

4. M. Herzog, *On 2-Sylow intersections*, Israel J. Math., **11** (1972), 326-327.

5. M. Suzuki, *Finite groups of even order in which Sylow 2-groups are independent*, Ann. Math., **80** (1964), 58-77.

Received December 6, 1971. This paper was written while the author was a visiting professor in Aarhus University, Denmark.

TEL-AVIV UNIVERSITY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 45, No. 2 October, 1973