

# Pacific Journal of Mathematics

**CONCERNING DENTABILITY**

M. EDELSTEIN

## CONCERNING DENTABILITY

MICHAEL EDELSTEIN

**It is shown that  $c_0$  contains a closed and bounded convex body which is dentable but fails to have extreme points. On the other hand, there exists a strictly convex, closed, symmetric, convex body which fails to be dentable. (Thus dentability is, in general, unrelated to extremal structure.)**

1. In [2], Rieffel introduced the notion of dentability for a subset  $K$  of a Banach space  $X$ . Rephrased, it reads:

1.1.  $K$  is dentable if, for every  $\varepsilon > 0$ , there is an  $x \in K$  and an  $f \in X^*$  such that some hyperplane determined by  $f$  separates  $x$  from  $K_\varepsilon = K \sim \overline{B(x, \varepsilon)}$ , where  $B(x, \varepsilon)$  is the ball of radius  $\varepsilon$  about  $x$ .

One of the questions asked by Rieffel [Ibid., p. 77] is whether a closed and bounded convex set exists in some Banach space which is dentable but has no strongly exposed points. We answer this question by exhibiting a dentable symmetric closed convex body in  $c_0$  which has no extreme points at all. To further show that the connection between dentability and extreme structure can be quite tenuous, we also exhibit in  $c_0$  a strictly convex body which (in spite of the fact that each boundary part is exposed) is not dentable.

Another question of Rieffel, namely, whether each weakly compact subset of a Banach space is dentable has recently been answered in the affirmative by Troyanski [3]. The example of the unit ball in the conjugate Banach space  $m$  is used by us (Proposition 3) to show that, in contrast to the above, a weak\*-compact set need not be dentable.

2. Dentability properties of certain subsets of  $c_0$  and  $m$ .

**PROPOSITION 1.** *There is a dentable closed and bounded convex body in  $c_0$  which has no extreme point.*

*Proof.* For  $n = 1, 2, \dots$  set  $B_n = B((2 - 2^{1-n})e_n, 2^{1-n})$ , where  $e_n = \{x_i\} \in c_0$  with  $x_n = 1$ ,  $x_i = 0$  for  $i \neq n$ . Let  $C_n = (-B_n) \cup B_n$  and  $C = \overline{\text{co}}(\bigcup_{n=1}^{\infty} C_n)$ . We claim that  $C$  has the desired properties.

(i)  $C$  has no extreme points.

Suppose, for a contradiction, that  $C$  has an extreme point

$$y = (y_1, y_2, \dots).$$

Clearly,  $\|y\| > 1$  (since  $\bar{C}_1$  contains the unit ball) and without restriction of generality we may assume that  $\|y\| = y_k$  for some  $k$ . Let  $\{u^{(m)}\}$  be a sequence in  $co\{\bigcup_{n=1}^\infty C_n\}$  converging to  $y$  with

$$(1) \quad \|u^{(m)} - y\| < \min(y_k - 1, 2^{-k-2}) \quad (m = 1, 2, \dots).$$

Write

$$(2) \quad u^{(m)} = \sum_{i=1}^l \lambda_i u^{(mi)}$$

with  $u^{(mi)} \in C_i$ ,  $\lambda_i \geq 0$  ( $i = 1, 2, \dots, l$ ), and  $\sum_{i=1}^l \lambda_i = 1$ . It is clear from the definition of the  $B_i$  that, for  $i > k$ ,  $u_k^{(mi)} \leq 2^{1-i} \leq 2^{-k}$ , where  $u_k^{(mi)}$  is the  $k$ th coordinate of  $u^{(mi)}$ .

Thus, by (1),

$$1 < u_k^{(m)} = \sum_{i=1}^k \lambda_i u_k^{(mi)} + \sum_{i=k+1}^l \lambda_i u_k^{(mi)} \leq 2 \sum_{i=1}^k \lambda_i + 2^{-k} \left(1 - \sum_{i=1}^k \lambda_i\right).$$

It follows that

$$(3) \quad \sum_{i=1}^k \lambda_i > \frac{1 - 2^{-k}}{2 - 2^{-k}} > \frac{1}{2} - \frac{1}{2^{k+1}} \geq \frac{1}{4}.$$

Now let  $j$  be a positive integer with the property that  $|y_j| < 2^{-k-3}$ . To show that  $y$ , contrary to assumption, cannot be an extreme point, we exhibit two points  $\bar{y}$  and  $\underline{y}$  in  $C$  such that  $\bar{y}_j > y_j > \underline{y}_j$  with all other coordinates of these points equal. To this end define  $\{\bar{u}^{(m)}\}$  and  $\{\underline{u}^{(m)}\}$  as follows.

Using (2), set

$$\bar{u}_n^{(mi)} = \underline{u}_n^{(mi)} = u_n^{(mi)}$$

for  $m = 1, 2, \dots, j; n \neq j, i = 1, 2, \dots, l;$

$$\bar{u}_j^{(mi)} = -\underline{u}_j^{(mi)} = \begin{cases} 2^{-k} & \text{for } i \leq k \\ 0 & \text{for } i > k \end{cases}.$$

It follows from (3) that

$$\bar{u}_j^{(m)} = -\underline{u}_j^{(m)} \geq 2^{-k-2}.$$

Thus,  $\bar{u}_j^{(m)} \geq y_j + 2^{-k-3}$  and  $\underline{u}_j^{(m)} \leq y_j - 2^{-k-3}$ . It is now obvious that  $\{\bar{u}^{(m)}\}$  and  $\{\underline{u}^{(m)}\}$  converge to points  $\bar{y}$  and  $\underline{y}$ , respectively, having the desired properties. This completes the proof that  $C$  has no extreme points.

(ii)  $C$  is dentable.

Let  $\varepsilon > 0$  be given and choose  $n$  so that  $2^{2-n} < \varepsilon$ . We show that  $\overline{co}(C \sim B)$  where  $B = B(2e_n, \varepsilon)$  does not contain  $2e_n \in C$ .

To this end, consider the set  $H^{(n)} = \{x \in co(\bigcup_{n=1}^{\infty} C_n) : x_n \geq 2 - 2^{-n}\}$ . Any member  $h$  of  $H^{(n)}$  can be represented in the form  $h = \sum_{i=1}^m \lambda_i x^i$  with  $\lambda_i \geq 0$ ,  $\sum_{i=1}^m \lambda_i = 1$  and  $x_i \in C_i$ ,  $i = 1, 2, \dots, m$ ;  $m \geq n$ . Now, by definition,  $h_n = \sum_{i=1}^m \lambda_i x_n^i \geq 2 - 2^{-n}$ . On the other hand,

$$\begin{aligned} h_n &= \lambda_n x_n^n + \sum_{i \neq n} \lambda_i x_n^i \leq \lambda_n x_n^n + (1 - \lambda_n) \\ &= \lambda_n (x_n^n - 1) + 1 \leq \lambda_n + 1. \end{aligned}$$

It follows that  $\lambda_n \geq 1 - 2^{-n}$ . Consequently,

$$\|2e_n - h\| \leq 2^{2^{-n}} \quad (h \in H^{(n)}),$$

for  $|(2e_n)_n - h_n| \leq |2 - (2 - 2^{-n})| = 2^{-n}$  and, for  $k \neq n$ ,

$$(2e_n - h)_k = |\sum \lambda_i x_k^i| \leq 1 - \lambda_n \leq 2^{2^{-n}}.$$

Thus  $B(2e_n, \varepsilon)$  contains  $H^{(n)}$  and clearly,  $\overline{C} \sim \overline{H^{(n)}}$  is convex with  $2e_n \notin \overline{C \sim H^{(n)}}$ . We have shown that  $C$  is dentable completing thereby the proof of the proposition.

**PROPOSITION 2.** *In  $c_0$  there exists a symmetric, closed and bounded convex body which is strictly convex and fails to be dentable.*

*Proof.* Let

$$C = \left\{ x \in c_0 : \|x\| + \left( \sum_{n=1}^{\infty} 2^{-n} x_n^2 \right)^{1/2} \leq 1 \right\}.$$

It is well-known (cf. [1, p. 362]) that  $C$  defines an equivalent strictly convex norm and, therefore, only the nondentability has to be shown. We note that for  $x = (x_1, x_2, \dots, x_n, \dots) \in \text{bdry} C$ , we have  $\|x\| \geq 1/2$  so that for such an  $x$  there is an integer  $m$  with  $|x_m| = \|x\| \geq 1/2$ . Let  $1/4 > \varepsilon > 0$  and choose  $0 < \delta < \varepsilon/2$  small enough so that  $\|x\| = \|x'\| + \delta$  if  $x'$  is the vector obtained from  $x$  by replacing each coordinate  $x_i$ , with  $|x_i| = \|x\|$ , by  $|x_i| - \delta$ . Next, let  $k$  be large enough so that  $|x_k| < \delta$  and

$$\left( \sum_{n \neq k} 2^{-n} x_n^2 + \frac{1}{2^{k+4}} \right)^{1/2} \leq \left( \sum_{n=1}^{\infty} 2^{-n} x_n^2 \right)^{1/2} + \delta.$$

To prove nondentability, it clearly suffices to exhibit  $u, v \in C$  such that  $\|(u + v)/2 - x\| < \delta$  and  $\|u - v\| \geq 1/2$ . To this end, set  $u_i = v_i = x_i$  for those  $i \neq k$  for which  $|x_i| < \|x\|$ ;  $u_k = -v_k = 1/4$ ; and  $u_j = v_j = x_j - \delta |x_j|/|x_j|$ , otherwise. Since  $\|u\| = \|v\| = \|x\| - \delta$  and

$$\left( \sum_{n=1}^{\infty} 2^{-n} u_n^2 \right)^{1/2} = \left( \sum_{n=1}^{\infty} 2^{-n} v_n^2 \right)^{1/2} \leq \left( \sum_{n=1}^{\infty} 2^{-n} x_n^2 \right)^{1/2} + \delta,$$

we have  $u, v \in C$ . Also,  $\|(u + v)/2 - x\| < \delta$ , since  $|((u + v)/2 - x)_k| = |x_k| < \delta$ , and, for all coordinates  $j \neq k$  at which  $u, v$  and  $x$  are distinct, we have  $|((u + v)/2 - x)_j| = \delta$ . Finally,

$$\|u - v\| = \|u_k - v_k\| = \frac{1}{2}.$$

PROPOSITION 3. *The unit ball in  $m$  is not dentable.*

*Proof.* Let  $0 < \varepsilon < 1/4$  and  $x = (x_1, x_2, \dots) \in m$  with  $\|x\| \leq 1$ . Either (i) there is an integer  $k$  with  $|x_k| \leq 1/4$ , or (ii) for every index  $j$ ,  $|x_j| > 1/4$ .

In case (i), define  $\bar{x}$  and  $\underline{x}$  by setting

$$\begin{aligned}\bar{x} &= \left(x_1, x_2, \dots, x_k + \frac{1}{4}, \dots\right) \\ \underline{x} &= \left(x_1, x_2, \dots, x_k - \frac{1}{4}, \dots\right)\end{aligned}$$

so that  $(1/2)(\bar{x} + \underline{x}) = x$  and  $\|\bar{x} - \underline{x}\| = 1/2 > \varepsilon$ .

In case (ii), define

$$x^{(i)} = \left(x_1, x_2, \dots, x_i - \frac{x_i}{4|x_i|}, \dots\right) \quad (i = 1, 2, \dots),$$

so that  $\|x - x^{(i)}\| = 1/4$ .

Now,  $x \in \overline{co}\{x^{(i)}: i = 1, 2, \dots\}$ . For,

$$\left(x - \frac{1}{j} \sum_{n=1}^j x^{(n)}\right)_k = \begin{cases} 0, & \text{if } k > j \\ \frac{1}{j} \left(x_k - \frac{x_k}{4|x_k|}\right) & \end{cases}$$

showing that  $(1/j) \sum_{n=1}^j x^{(n)} \rightarrow x$ . Thus, the dentability condition fails, proving the proposition.

#### REFERENCES

1. G. Köthe, *Topological Vector Spaces I*, Berlin-Heidelberg-New York, 1969.
2. M. A. Rieffel, *Dentable subsets of Banach spaces with application to a Radon-Nikodym theorem*, Proc. Conf. Functional Anal., Thompson Book Co., Washington, 1967 pp. 71-77.
3. S. L. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math., **37** (1971), 173-179.

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AMERICAN MATHEMATICAL SOCIETY  
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Allan Francis Abrahamse, <i>Uniform integrability of derivatives on <math>\sigma</math>-lattices</i> .....	1
Ronald Alter and K. K. Kubota, <i>The diophantine equation <math>x^2 + D = p^n</math></i> .....	11
Grahame Bennett, <i>Some inclusion theorems for sequence spaces</i> .....	17
William Cutler, <i>On extending isotopies</i> .....	31
Robert Jay Daverman, <i>Factored codimension one cells in Euclidean <math>n</math>-space</i> .....	37
Patrick Barry Eberlein and Barrett O'Neill, <i>Visibility manifolds</i> .....	45
M. Edelstein, <i>Concerning dentability</i> .....	111
Edward Graham Evans, Jr., <i>Krull-Schmidt and cancellation over local rings</i> .....	115
C. D. Feustel, <i>A generalization of Kneser's conjecture</i> .....	123
Avner Friedman, <i>Uniqueness for the Cauchy problem for degenerate parabolic equations</i> .....	131
David Golber, <i>The cohomological description of a torus action</i> .....	149
Alain Goulet de Rugy, <i>Un théorème du genre "Andô-Edwards" pour les Fréchet ordonnés normaux</i> .....	155
Louise Hay, <i>The class of recursively enumerable subsets of a recursively enumerable set</i> .....	167
John Paul Helm, Albert Ronald da Silva Meyer and Paul Ruel Young, <i>On orders of translations and enumerations</i> .....	185
Julien O. Hennefeld, <i>A decomposition for <math>B(X)^*</math> and unique Hahn-Banach extensions</i> .....	197
Gordon G. Johnson, <i>Moment sequences in Hilbert space</i> .....	201
Thomas Rollin Kramer, <i>A note on countably subparacompact spaces</i> .....	209
Yves A. Lequain, <i>Differential simplicity and extensions of a derivation</i> .....	215
Peter Lorimer, <i>A property of the groups <math>\text{Aut PU}(3, q^2)</math></i> .....	225
Yasou Matsugu, <i>The Levi problem for a product manifold</i> .....	231
John M.F. O'Connell, <i>Real parts of uniform algebras</i> .....	235
William Lindall Paschke, <i>A factorable Banach algebra without bounded approximate unit</i> .....	249
Ronald Joel Rudman, <i>On the fundamental unit of a purely cubic field</i> .....	253
Tsuan Wu Ting, <i>Torsional rigidities in the elastic-plastic torsion of simply connected cylindrical bars</i> .....	257
Philip C. Tonne, <i>Matrix representations for linear transformations on analytic sequences</i> .....	269
Jung-Hsien Tsai, <i>On <math>E</math>-compact spaces and generalizations of perfect mappings</i> .....	275
Alfons Van Daele, <i>The upper envelope of invariant functionals majorized by an invariant weight</i> .....	283
Giulio Varsi, <i>The multidimensional content of the frustum of the simplex</i> .....	303