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This note was motivated by a paper of P. D. Taylor, which contains a simple proof of Rockafellar's basic theorem that the subdifferential map ∂f of a lower semicontinuous proper convex function f on a Banach space is maximal monotone. Taylor based his proof on a theorem which can be considered as a sharpening (for the epigraph of a convex function) of a result (Lemma 1.1) concerning support points and functions of convex sets due to Brøndsted-Rockafellar and Phelps. It is shown that Taylor's theorem can be generalized somewhat, using related methods. (It is shown, by an example, that there is a limitation on the extent of generalization possible.) The theorem follows from a slightly technical result (Proposition 1.3) which admits a dual version (Proposition 2.2). As an application of Proposition 2.2, a short proof of Rockafellar's theorem relating the graph of $(\partial f^*)^{-1}$ to that of ∂f is given. The methods of this paper yield a generalization (Corollary 1.9) of one of the density results of Bishop-Phelps.

1. Let E be a topological vector space and E^* its dual. The natural pairing between these spaces will be denoted by $\langle x, x^* \rangle$ for $x \in E$ and $x^* \in E^*$.

We recall some standard definition and facts about convex sets and functions. For more detail see Moreau [6] or Rockafellar [9]. If $f: E \rightarrow [\infty, \infty]$ is a function, then its epigraph (or "supergraph") $\text{epi } f$ is $\{(x, r) \in E \times R \mid f(x) \leq r\}$. Recall f is convex if and only if $\text{epi } f$ is convex and that f is lower semicontinuous (l.s.c.) if and only if $\text{epi } f$ is closed in $E \times R$. If $\text{epi } f$ is nonempty and contains no vertical lines, i.e., sets of the form $\{(x, r) \mid r \in R\}$ where $x \in E$, then f is called *proper*. The natural projection of $\text{epi } f$ onto E is called the *effective domain* of f and is written $\text{dom } f$; thus $\text{dom } f = \{x \in E \mid f(x) < \infty\}$.

If g is a function on a set X we write $\sup g(X)$ in place of $\sup \{g(x) \mid x \in X\}$. If C is a closed convex nonempty subset of E , a *support point* for C is a point $x \in C$ for which there exists an element $x^* \in E^* \setminus \{0\}$ such that $\langle x, x^* \rangle = \sup x^*(C)$. Such an element x^* is called a *support functional* for C .

We identify $(E \times R)^*$ and $E^* \times R$ in the obvious way, so that the pairing between $E^* \times R$ and $E \times R$ is given by $\langle (x, r), (x^*, s) \rangle = \langle x, x^* \rangle + rs$ for $(x, r) \in E \times R$ and $(x^*, s) \in E^* \times R$.

If f is a convex function on E , a *subgradient* for f at a point $x \in \text{dom } f$ is an element $x^* \in E^*$ such that $(x^*, -1)$ is a support func-

tional of $\text{epi } f$ at $(x, f(x))$. The collection (possibly empty) of all subgradients at x is denoted $\partial f(x)$. In this way, a set valued map $\partial f: E \rightarrow E^*$ is obtained.

A *monotone set* G in $E \times E^*$ is a subset of $E \times E^*$ for which $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*) \in G$ and $(y, y^*) \in G$. A monotone set is called *maximal* if it is not properly contained in any other monotone subset of $E \times E^*$. By Zorn's lemma any monotone set is contained in a maximal monotone set.

If f is a convex function on E , then $\text{gr } \partial f = \{(x, x^*) \mid x^* \in \partial f(x)\}$ is a monotone subset of $E \times E^*$. Rockafellar [11, 10] showed that $\text{gr } \partial f$ is actually maximal monotone if f is a l.s.c. proper convex function and E is a Banach space.

If N is a closed subspace of E , then N^\perp will denote the annihilator of N in E^* , i.e., $N^\perp = \{n^* \in E^* \mid \langle n, n^* \rangle = 0 \text{ for each } n \in N\}$. If f is a convex function on E , then the *conjugate* of f is the function $f^*: E^* \rightarrow R \cup \{\infty\}$ defined by $f^*(x^*) = \sup \{\langle x, x^* \rangle - f(x) \mid x \in E\}$.

We recall that the following three statements are equivalent [9]: $z^* \in \partial f(z)$, $z \in \partial f^*(z^*)$, and $f^*(z^*) + f(z) = \langle z, z^* \rangle$.

If $C \subset E$, the *indicator function* ψ_C for C is defined for each $x \in E$ by $\psi_C(x) = 0$ if $x \in C$ and by $\psi_C(x) = \infty$ if $x \notin C$.

If $C \subset E^*$, the *support function* S_C for C is defined for each $x \in E$ by $S_C(x) = \sup \{\langle x, x^* \rangle \mid x^* \in C\}$.

If C is a nonempty convex subset of E , then 0^+C will denote the *asymptotic cone* of C , i.e., $0^+C = \{y \in E \mid x + \lambda y \in C \text{ for each } \lambda \geq 0 \text{ and } x \in C\}$.

The following lemma of Phelps [7] is a geometric formulation of the lemma of Brøndsted-Rockafellar [2].

LEMMA 1.1. *Let C be a closed convex subset of the Banach space E . Suppose $x \in C$, $x^* \in E^*$ and $\varepsilon > 0$ satisfy*

$$\sup x^*(C) \leq \langle x, x^* \rangle + \varepsilon.$$

Then for any $k > 0$ there exist $w \in C$ and $w^ \in E^*$ satisfying*

$$\langle w, w^* \rangle = \sup w^*(C), \|x - w\| \leq \varepsilon/k \quad \text{and} \quad \|x^* - w^*\| \leq k.$$

We remark that Lemma 1 of [12] and the result [1, Theorem 2] which inspired it are easy consequences of the above lemma.

We show next that the lemma yields a short proof of [1, Theorem 2].

THEOREM 1.2. [*Bishop-Phelps*]. *Suppose that C and X are subsets of a Banach space E , that C is closed and convex and that X is bounded and nonempty. If $\varepsilon > 0$ and if $x^* \in E^*$ is such that*

$$\sup x^*(C) < \inf x^*(X) ,$$

then there exist $w^* \in E^*$ and $w \in C$ such that

$$\|x^* - w^*\| \leq \varepsilon \quad \text{and} \quad \langle w, w^* \rangle = \sup w^*(C) < \inf w^*(X) .$$

Proof. Let $\delta = \inf x^*(X) - \sup x^*(C)$; then $\delta > 0$ and we can assume $2\varepsilon - \delta < 0$. Choose $z \in C$ such that $\sup x^*(C) \leq \langle z, x^* \rangle + \varepsilon$. Since X is bounded, there is a number N such that

$$N - 1 > \sup \{\|z - x\| \mid x \in X\} .$$

Let $k = \varepsilon/N$ in Lemma 1.1; then there exist $w^* \in E^*$ and $w \in C$ such that

$$\sup w^*(C) = \langle w, w^* \rangle, \|w^* - x^*\| \leq \varepsilon/N \quad \text{and} \quad \|w - z\| \leq N .$$

Since $N > 1$, we have $\varepsilon/N < \varepsilon$, so it only remains to show that

$$\langle w, w^* \rangle = \sup w^*(C) < \inf w^*(X) .$$

First notice that

$$\langle x, x^* \rangle - \langle z, x^* \rangle \leq (\varepsilon/N) \|x - z\| + \langle x, w^* \rangle - \langle z, w^* \rangle$$

for any $x \in X$ (since $\|w^* - x^*\| \leq \varepsilon/N$). This implies that

$$(1) \quad \langle z, w^* \rangle - \langle z, x^* \rangle + \inf x^*(X) \leq \varepsilon + \inf w^*(X) .$$

Similarly, $\sup w^*(C) = \langle w, w^* \rangle \leq \langle z, w^* \rangle + (\varepsilon/N) \|w - z\| - \langle z, x^* \rangle + \langle w, x^* \rangle$ which combines with (1) to yield

$$\begin{aligned} \langle w, w^* \rangle &\leq \varepsilon + \inf w^*(X) - \inf x^*(X) + \varepsilon + \sup x^*(C) \\ &\leq 2\varepsilon - \delta + \inf w^*(X) \\ &< \inf w^*(X) . \end{aligned}$$

Proposition 1.3 (below) yields a slightly generalized version of Taylor's theorem [12]. The proof of the proposition, although technical, is conceptually very simple: one separates $\text{epi } f$ from an appropriate subset of a given linear variety and then uses Lemma 1.1 to obtain a supporting hyperplane of the desired type. In order to facilitate this idea, we introduce some notation.

Suppose that N is a closed subspace of the Banach space E , that B_N is the unit ball of N , and that f is a convex function on E . For each $\varepsilon > 0$, we let

$$S(f, \varepsilon, N) = \{z^* \in E^* \mid \sup (z^*, -1)(\text{epi } f) \leq \inf (z^*, -1)(B_N \times \{-\varepsilon\})\} .$$

Thus, $S(f, \varepsilon, N)$ is the projection onto E^* of those functionals in $E^* \times \{-1\}$ which separate the convex set $\text{epi } f$ from the convex set

$B_N \times \{-\varepsilon\}$. It can also be described by

$$\{z^* \in E^* \mid f^*(z^*) + \|z^*\|_N \leq \varepsilon\} \quad \text{where} \quad \|z^*\|_N = \sup \{|\langle n, z^* \rangle| \mid n \in B_N\}.$$

PROPOSITION 1.3. *Let f be a l.s.c. proper convex function on a Banach space E and let N be a closed subspace of E . Suppose that $f(0) = 0$ and that $z^* \in S(f, \varepsilon, N)$. Then for each $k \in (0, 1)$, there exist $w \in E$ and $w^* \in E^*$ such that*

$$\begin{aligned} w^* \in \partial f(w), \quad |\langle w, w^* \rangle| &\leq (\varepsilon/k) \left(\frac{1+k}{1-k} \right), \quad \|w^*\|_N \leq \frac{k+\varepsilon}{1-k} \\ \|w\| &\leq \varepsilon/k, \quad |f(w)| \leq \varepsilon/k, \quad \|z^* - w^*\| \leq \frac{k(1+\|z^*\|)}{1-k}. \end{aligned}$$

Proof. Since $z^* \in S(f, \varepsilon, N)$ and $(0, 0) \in \text{epi } f$, it follows that, $0 \leq \sup(z^*, -1)(\text{epi } f) \leq -\|z^*\|_N + \varepsilon$ and hence that $\sup(z^*, -1)(\text{epi } f) \leq \varepsilon$ and $\|z^*\|_N \leq \varepsilon$. By Lemma 1.1, for any $k > 0$ there exist $G \in (E \times R)^*$ and $(w, f(w)) \in \text{epi } f$ such that

$$\sup G(\text{epi } f) = G(w, f(w)), \quad \|(w, f(w))\| \leq \varepsilon/k \quad \text{and} \quad \|(z^*, -1) - G\| \leq k.$$

Thus $|G(0, -1) - 1| \leq k$ and since $k < 1$ we have $0 < 1 - k < G(0, -1)$. Let $\alpha = (G(0, -1))^{-1}$; then there is an element $w^* \in E^*$ such that for each $y \in E$

$$\langle y, w^* \rangle = \alpha G(y, 0).$$

If $y \in \text{dom } f$, then

$$\begin{aligned} \langle y, w^* \rangle - f(y) &= \alpha G(y, 0) - f(y) \\ &= \alpha G(y, f(y)) \\ &\leq \alpha G(w, f(w)) \\ &= \langle w, w^* \rangle - f(w) \end{aligned}$$

and therefore $w^* \in \partial f(w)$.

Since $(w, f(w)) \in \text{epi } f$, we have $\langle w, z^* \rangle - f(w) \leq \varepsilon$; thus

$$\begin{aligned} 0 \leq G(w, f(w)) &\leq \|(w, f(w))\| k + \langle w, z^* \rangle - f(w) \\ &\leq \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

and therefore

$$\begin{aligned} |\langle w, w^* \rangle| &= |\alpha G(w, 0)| = |\alpha G(w, f(w)) + f(w)| \\ &\leq \frac{2\varepsilon}{1-k} + \frac{\varepsilon}{k} = \frac{\varepsilon}{k} \left(\frac{1+k}{1-k} \right). \end{aligned}$$

Because $\|z^* - G(0, -1)w^*\| \leq k$, the triangle inequality yields

$$\begin{aligned}
\|z^* - w^*\| &\leq k + (1 - G(0, -1))\|w^*\| \\
&\leq k + k\alpha(k + \|z^*\|) \\
&\leq k\left(1 + \frac{k + \|z^*\|}{1 - k}\right) \\
&= \frac{k}{1 - k}(1 + \|z^*\|).
\end{aligned}$$

Clearly,

$$\|w^*\|_N = \alpha\|G\|_{N \times \{0\}} \leq \frac{k + \|z^*\|_N}{1 - k} \leq \frac{k + \varepsilon}{1 - k}$$

and the proof is complete.

COROLLARY 1.4. *Suppose that f is a l.s.c. proper convex function on a Banach space E , that $f(0) = 0$, and that N is a closed subspace. Then the following two statements are equivalent:*

(1) *For each $\varepsilon > 0$ the set $S(f, \varepsilon, N)$ is nonempty.*

(2) *For each $\varepsilon > 0$ there exist $w \in E$ and $w^* \in E^*$ such that $w^* \in \partial f(w)$, $|\langle w, w^* \rangle| \leq \varepsilon$, $\|w\| \leq \varepsilon$, $\|w^*\|_N \leq \varepsilon$ and $|f(w)| \leq \varepsilon$.*

Proof. (1) \rightarrow (2). We can suppose that $0 < \varepsilon < 1$. Choose $\delta > 0$ so that $(\delta^{1/2} + \delta)/(1 - \delta^{1/2}) < \varepsilon$. Let $x^* \in S(f, \delta, N)$ and apply Proposition 1.3 with $k = \delta^{1/2}$. Since $\delta^{1/2} < (\delta^{1/2} + \delta)/(1 - \delta) < \varepsilon$ assertion (2) follows.

(2) \rightarrow (1). Let $\varepsilon > 0$ and apply (2) with $\varepsilon/3$. Since $w^* \in \partial f(w)$ we have $f^*(w^*) = \langle w, w^* \rangle - f(w)$, so that

$$\begin{aligned}
f^*(w^*) + \|w^*\|_N &\leq \langle w, w^* \rangle - f(w) + \|w^*\|_N \\
&\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}$$

and therefore $w^* \in S(f, \varepsilon, N)$.

It is now easy to prove the more general version of the theorem of Taylor [12] referred to earlier. It is readily seen that (1) of Corollary 1.4 is equivalent to (1)' For each $\varepsilon > 0$ we have

$$0 \notin \text{cl}(B_N \times \{-\varepsilon\} - \text{epi } f),$$

which is in turn equivalent to saying that there exists a hyperplane in $E \times R$ which *strictly* separates $\text{epi } f$ from $B_N \times \{-\varepsilon\}$. This suggests that a generalization of Taylor's theorem is not possible for *arbitrary* closed subspaces N and the example following Theorem 1.5 shows that this is indeed the case. In [12], the space N is assumed to be one dimensional.

THEOREM 1.5. *Suppose f is a l.s.c. proper convex function on a Banach space E and that N is a reflexive subspace of E . Suppose also that $y \in \text{dom } f$ and $y^* \in E^*$ are such that for each $n \in N$*

$$f(y + n) \geq f(y) + \langle n, y^* \rangle .$$

Then for each $\varepsilon > 0$ there exist $z \in E$ and $z^ \in E^*$ satisfying*

$$z^* \in \partial f(z), \|z^* - y^*\|_N \leq \varepsilon, \|y - z\| \leq \varepsilon \quad \text{and} \quad |\langle y - z, z^* \rangle| \leq \varepsilon .$$

Proof. Let $h = f(y + \cdot) - f(y) - \langle \cdot, y^* \rangle$; then h is a l.s.c. proper convex function on E , $h(0) = 0$ and $h \geq 0$ on N . Hence the weakly closed convex set $\text{epi } h$ is disjoint from the weakly compact convex set $B_N \times \{-\varepsilon\}$ for every $\varepsilon > 0$. By the separation theorem, there is a $G \in (E \times R)^*$ such that

$$\sup G(\text{epi } h) < \inf G(B_N \times \{-\varepsilon\}) .$$

Since $(0, 0) \in \text{epi } h$, we have $G(0, -1) > 0$. Thus for any $\varepsilon > 0$, the set $S(h, \varepsilon, N)$ is nonempty. By applying Corollary 1.4, with $\delta = \varepsilon/(1 + \|y^*\|)$ in place of ε , we obtain $w \in E$ and $w^* \in E^*$ such that

$$w^* \in \partial h(w), |\langle w, w^* \rangle| \leq \delta, \|w\| \leq \delta \quad \text{and} \quad \|w^*\|_N \leq \delta .$$

Let $z^* = w^* + y^*$ and $z = w + y$; it is easy to check that z^* and z satisfy the conclusion of the theorem.

We refer the reader to [12] for the application of this result to an easy proof of Rockafellar's theorem that for any l.s.c. proper convex function f on a Banach space, the subdifferential map ∂f is maximal monotone.

The following is an example of a l.s.c. proper convex function F and a subspace N of codimension 1 for which $F(0) = 0$ and $F(n) \geq 0$ for all $n \in N$, but (1) of Corollary 1.4 does not hold. This shows that Theorem 1.5 is not valid for arbitrary closed subspaces N .

Let

$$E = l^\infty \quad \text{and} \quad N = \{x = \{x_i\} \in l^\infty \mid x_1 = 0\} .$$

Define $F: l^\infty \rightarrow R \cup \{\infty\}$ by

$$F(x) = \liminf_{\varepsilon \rightarrow 0} \{f(y) \mid \|y - x\| < \varepsilon\}$$

where, for $y = \{y_i\} \in l^\infty$,

$$f(y) = - \left(\prod_{i=1}^n \{y_i\} \right)^{1/n} \quad \text{if} \quad y_i \in [0, 1]$$

for each i and $y_i = 0$ for almost all i (n is the smallest positive integer such that $i > n$ implies $y_i = 0$); $f(y) = \infty$ otherwise. We have using [11, p. 27] that f is convex and hence F is a l.s.c. proper convex function. Also, $F(0) = 0$ and $F(n) \geq 0$ for each $n \in N$. Let

$$y_k = \{\overbrace{\varepsilon^k, \dots, \varepsilon^k}^k, \overbrace{1, 1, 1, \dots, 1}^{k^2 - k}, 0 \dots$$

and $m_k = y_k - \varepsilon^k e_1$ where $e_1 = \{1, 0 \dots\}$. Then $F(y_k) = -\varepsilon$ for each k , and $m_k - y_k \rightarrow 0$. Thus $0 \in \text{cl}(B_N \times \{-\varepsilon\} - \text{epi } F)$ and the discussion preceding Theorem 1.5 shows that the conclusion of Theorem 1.5 does not hold.

We remark that if f is continuous at some point of N , then as has been noted by Ioffe [5, Theorem 1] $\partial f|_N = \gamma \circ \partial f$ on N where $\gamma: E^* \rightarrow E^*/N^\perp$ is the canonical projection. Our example shows that continuity cannot be weakened to lower semicontinuity even if one supposes N has codimension one, where [1, Lemma 4] applies. Since $0 \in \partial F|_N(0)$, we have to show that if $y^* \in \partial F(0)$, then $y^* \notin N^\perp$. Suppose $y^* \in \partial F(0)$ i.e., $\langle x, y^* \rangle \leq F(x)$ for each $x \in E$. Take k large enough so that $|\langle m_k - y_k, y^* \rangle| < \varepsilon/2$; then for large enough k ,

$$\begin{aligned} \langle m_k, y^* \rangle &< \varepsilon/2 + \langle y_k, y^* \rangle \\ &\leq \varepsilon/2 + F(y_k) \\ &= -\varepsilon/2 \end{aligned}$$

and so $y^* \notin N^\perp$.

We now give a sufficient condition for w^* in Corollary 1.4 (2) to be contained in N^\perp i.e. $\|w^*\|_N = 0$. This result (Proposition 1.8) follows (in the same way Proposition 1.2 was a consequence of Lemma 1.1) from a lemma which extends to Lemma 1.1. We need the following proposition of Dieudonné [3] for the proof of the lemma.

PROPOSITION 1.6. *Let E be a topological vector space and A, B two closed convex and nonempty subsets of E . Suppose A is locally compact and that $0^+A \cap 0^+B = \{0\}$. Then $B-A$ is closed in E .*

LEMMA 1.7. *Let C be a closed convex subset of a Banach space E and N a finite dimensional subspace of E . Suppose $0^+C \cap N = \{0\}$ and that $x^* \in N^\perp$, $x \in C$ and $\varepsilon > 0$ are such that*

$$\sup x^*(C) \leq \langle x, x^* \rangle + \varepsilon.$$

Then for any $k > 0$, there exist $w \in C$ and $w^ \in N^\perp$ such that*

$$\langle w, w^* \rangle = \sup w^*(C), \|x - w\|_{N^\perp} \leq \varepsilon/k \quad \text{and} \quad \|w^* - x^*\| \leq k.$$

Proof. By Proposition 1.6, the set $C + N$ is closed in E . Let $Q: E \rightarrow E/N$ be the quotient map. Since $Q^{-1}(Q(C)) = C + N$, it follows from the definition of the quotient topology that $Q(C)$ is closed. We identify $(E/N)^*$ with N^\perp and apply Lemma 1.1 to $Q(C)$ in E/N . Then for any $k > 0$ there exist $z + N \in C + N$ and $w^* \in N^\perp$ such that

$$\langle z + N, w^* \rangle = \sup w^*(C + N), \|x - z + N\| \leq \varepsilon/k \quad \text{and} \quad \|z^* - w^*\| \leq k.$$

Since $z + N \in C + N$, there exist $w \in C$ and $n \in N$ for which $z = w + n$. Because $w^* \in N^\perp$, we have

$$\langle w, w^* \rangle = \langle z + N, w^* \rangle = \sup w^*(C + N) = \sup w^*(C).$$

Also, we have

$$\|x - w\|_{N^\perp} = \|x - w + N\| = \|x - z + N\| \leq \varepsilon/k,$$

and the proof is complete.

PROPOSITION 1.8. *Let f be a l.s.c. proper convex function on a Banach space E and let N be a finite dimensional subspace of E . Suppose that $z^* \in S(f, \varepsilon, N)$, that $0^+ \text{epi } f \cap (N \times \{0\}) = \{0\}$, that $(0, 0) \in \text{epi } f$ and that $z^* \in N^\perp$. Then for each $k \in (0, 1)$ there exist $w \in E$ and $w^* \in N^\perp$ such that*

$$w^* \in \partial f(w), \|w\|_{N^\perp} \leq \varepsilon/k, |f(w)| \leq \varepsilon/k$$

and

$$\|z^* - w^*\| \leq \frac{k(1 + \|z^*\|)}{1 - k}.$$

Proof. By hypothesis, $\sup (z^*, -1)(\text{epi } f) \leq \varepsilon$ and $(z^*, -1) \in (N \times \{0\})^\perp = N^\perp \times R$ and Lemma 1.7 applies with $x = (0, 0)$. Thus for any $k > 0$ there exist $(w, f(w)) \in \text{epi } f$ and $G \in N^\perp \times R$ such that

$$G(w, f(w)) = \sup G(C), \|(w, f(w))\|_{N^\perp \times R} \leq \varepsilon/k$$

and

$$\|G - (z^*, -1)\| \leq k.$$

Thus $|G(0, -1) - 1| \leq k$ and since $k < 1$ we have $0 < 1 - k < G(0, -1)$. Hence there is an element $w^* \in E^*$ such that for each $y \in E$

$$\langle y, w^* \rangle = \frac{G(y, 0)}{G(0, -1)}.$$

Since $G \in N^\perp \times R$, we have $w^* \in N^\perp$; the verifications that $w^* \in \partial f(w)$ and that

$$\|z^* - w^*\| \leq \frac{k(1 + \|z^*\|)}{1 - k}$$

are the same as in Proposition 1.3.

It is a trivial consequence of Lemma 1.1 that if C is a closed convex subset of a Banach space E , then the support functionals of C are norm dense in the set of linear functionals bounded above on C . The following corollary to Proposition 1.8 shows that the support functionals of C which are bounded above on C and positive at some point of C have a norm dense intersection with any finite codimensional linear variety M satisfying $0^+C \cap 0^+M^0 = \{0\}$.

COROLLARY 1.9. *Let C be a closed convex and nonempty subset of a Banach space E and let N be a finite dimensional subspace of E with $0^+C \cap N = \{0\}$. Let $x^* \in E^*$ be such that $0 < S_C(x^*) < \infty$. Then for each $\varepsilon > 0$ there exist $z^* \in E^*$ and $z \in C$ satisfying*

$$\langle z, z^* \rangle = \sup z^*(C), \quad z^* \in N^\perp + x^*, \quad \|z^* - x^*\| \leq \varepsilon$$

and

$$|S_C(x^*) - S_C(z^*)| \leq \varepsilon.$$

Proof. Choose $\delta \in (0, S_C(x^*))$ such that $2\delta^{1/2}/(1 - 2\delta^{1/2}) < \varepsilon$ and $y \in C$ so that

$$(1) \quad S_C(x^*) \leq \langle y, x^* \rangle + \delta.$$

Define the function $h: E \rightarrow R \cup \{\infty\}$ by $h(x) = \psi_C(x) - \langle x, x^* \rangle + \langle y, x^* \rangle$. Clearly (1) implies $0 \in S(h, \delta, N)$. We check that

$$0^+ \text{epi } h \cap ((N \times R\{y\}) \times \{0\}) = \{0\}.$$

Suppose $(n, 0) \in 0^+ \text{epi } h$ for $n \in N$; then since $h(y) = 0$, we have for any $\lambda \geq 0$ that $(y + \lambda n, 0) \in \text{epi } h$ i.e., $h(y + \lambda n) \leq 0$. Hence $\psi_C(y + \lambda n) \leq \langle \lambda n, x^* \rangle < \infty$ and therefore $y + \lambda n \in C$. Thus $n \in 0^+C \cap N = \{0\}$. Suppose $(y, 0) \in 0^+ \text{epi } h$; then we have $((1 + \lambda)y, 0) \in \text{epi } h$ for each $\lambda \geq 0$ and hence $y + \lambda y \in C$ for each $\lambda \geq 0$; but since $\langle y, x^* \rangle > 0$ this contradicts $S_C(x^*) < \infty$. Finally suppose $(-y, 0) \in 0^+ \text{epi } h$; then $((1 - \lambda)y, 0) \in \text{epi } h$ for each $\lambda \geq 0$, so $\langle \lambda y, x^* \rangle \leq 0$ for each $\lambda \geq 0$ which contradicts $\langle y, x^* \rangle > 0$.

We can apply Proposition 1.8 with $k = 2\delta^{1/2}$ to obtain $w^* \in (NxR\{y\})^\perp$ and $z \in C$ satisfying $w^* \in \partial h(z)$ and $\|w^*\| \leq 2\delta^{1/2}/(1 - 2\delta^{1/2}) < \varepsilon$ and $|h(z)| \leq \delta^{1/2}/2$ and $\|z\|_{(N \times R\{y\})^\perp} \leq \delta/2$. Let $z^* = w^* + x^*$; then $\|z^* - x^*\| \leq \varepsilon$ and it is easy to check that $w^* \in \partial h(z)$ implies $\langle z, z^* \rangle = S_C(z^*)$. Since $h(z) = \langle y - z, x^* \rangle$ we have

$$\begin{aligned}
0 &\leq S_c(z^*) - \langle y, z^* \rangle \\
&= S_c(z^*) - \langle y, x^* \rangle \\
&= \langle z - y, z^* \rangle \\
&= \langle z, w^* \rangle + \langle z - y, x^* \rangle \\
&\leq \delta^{1/2} < \varepsilon,
\end{aligned}$$

and combining this inequality with

$$-\delta \leq \langle y, z^* \rangle - S_c(x^*) \leq 0$$

we obtain

$$|S_c(z^*) - S_c(x^*)| \leq \varepsilon$$

which completes the proof.

2. In this section we obtain dual results for most of those in §1. The key is the following lemma which is a dual version of Lemma 1.1. It was essentially proved in [8, Theorem 1] and can be easily obtained from the Brondsted-Rockafellar lemma.

LEMMA 2.1. *Let C be a weak*-closed convex subset of the dual E^* of the Banach space E . Suppose $x^* \in C$ and $x \in E$ and $\varepsilon > 0$ satisfy*

$$\sup C(x) \leq \langle x, x^* \rangle + \varepsilon.$$

Then for any $k > 0$ there exist $w^ \in C$ and $w \in E$ satisfying*

$$\langle w, w^* \rangle = \sup C(w), \|x - w\| \leq k \quad \text{and} \quad \|x^* - w^*\| \leq \varepsilon/k.$$

This lemma can be used to prove the following dual version of Proposition 1.3, by much the same method.

PROPOSITION 2.2. *Let f be a l.s.c. proper convex function on a Banach space E and N a subspace of E . Suppose $f^*(0) = 0$ and $z \in S(f^*, \varepsilon, N^\perp)$ where $\varepsilon > 0$. Then for each $k \in (0, 1)$ there exist $w \in E$ and $w^* \in E^*$ satisfying:*

$$\begin{aligned}
w &\in \partial f^*(w^*) \\
|\langle w, w^* \rangle| &\leq (\varepsilon/k) \left(\frac{1+k}{1-k} \right) \\
\|w\|_{N^\perp} &\leq \frac{k+\varepsilon}{1-k} \\
\|w^*\| &\leq \varepsilon/k \\
\|z - w\| &\leq \frac{k(1+\|z\|)}{1-k}.
\end{aligned}$$

Proof. Since $z \in S(f^*, \varepsilon, N^\perp)$ and $(0, 0) \in \text{epi } f^*$, it follows that $0 \leq \sup(z, -1)(\text{epi } f^*) \leq -\|z\|_{N^\perp} + \varepsilon$ and hence that

$$\sup(z, -1)(\text{epi } f^*) \leq \varepsilon \quad \text{and} \quad \|z\|_{N^\perp} \leq \varepsilon.$$

By Lemma 2.1, for any $k > 0$ there exist $G \in E \times R$ and $(w^*, f^*(w^*)) \in \text{epi } f^*$ satisfying

$$\sup G(\text{epi } f^*) = G(w^*, f^*(w^*)), \| (w^*, f^*(w^*)) \| \leq \varepsilon/k$$

and

$$\|(z, -1) - G\| \leq k.$$

Thus $|G(0, -1) - 1| \leq k$ and since $k < 1$, we have $0 < 1 - k < G(0, -1)$. Hence there is an element $w \in E$ such that for each $y^* \in E^*$

$$\langle w, y^* \rangle = \frac{G(y^*, 0)}{G(0, -1)}.$$

The rest of the proof is obtained by interchanging the roles of w , and w^* in the proof of Proposition 1.3.

We have seen that Theorem 1.5 requires some restriction in the subspace N . This is not the case with the following dual version which uses the weak* compactness of the unit ball of the weak* closed subspace N^\perp .

THEOREM 2.3. *Suppose f is a l.s.c. proper convex function on a Banach space E and that N is a subspace of E . Suppose also that $y \in E$ and $y^* \in \text{dom } f^*$ satisfy*

$$f^*(y^* + n^*) \geq f^*(y^*) + \langle y, n^* \rangle \quad \text{for each } n^* \in N^\perp.$$

Then for each $\varepsilon > 0$ there exist $z \in E$ and $z^ \in E^*$ satisfying*

$$z \in \partial f^*(z^*), \|z - y\|_{N^\perp} \leq \varepsilon, \|y^* - z^*\| \leq \varepsilon$$

and

$$|\langle z, y^* - z^* \rangle| \leq \varepsilon.$$

Proof. Let $h = f^*(y^* + \cdot) - f^*(y^*) - \langle y, \cdot \rangle$; then h is w^* -l.s.c. proper and convex. Also $h(0) = 0$ and $h \geq 0$ on N^\perp . Hence the weak* closed convex set $\text{epi } h$ is disjoint from the weak* compact convex set $B_{N^\perp} \times \{-\delta\}$ for every $\delta > 0$. By the separation theorem, there exists $G \in E \times R$ satisfying

$$\sup G(\text{epi } h) < \inf G(B_{N^\perp} \times \{-\varepsilon\}).$$

Since $(0, 0) \in \text{epi } h$, we have $G(0, -1) > 0$. Thus for any $\delta > 0$, the set $E \cap S(h, \delta, N^\perp)$ is nonempty. Choose $\delta > 0$ so that

$$\frac{\delta^{1/2} + \delta}{1 - \delta^{1/2}} < \frac{\varepsilon}{1 + |y|},$$

and apply Proposition 2.2 with δ in place of ε and $k = \delta^{1/2}$, to obtain $w \in E$ and $w^* \in E^*$ satisfying

$$w \in \partial f^*(w^*), |\langle w, w^* \rangle| \leq \delta, \|w\|_{N^\perp} \leq \delta \quad \text{and} \quad \|w^*\| \leq \delta.$$

Let $z^* = w^* + y^*$ and $z = w + y$; it is easy to check that z^* and z satisfy the conclusions of the theorem.

If one considers E as a subspace of E^{**} , then $gr \partial f$ is a subset of $E^{**} \times E^*$, as is $gr(\partial f^*)^{-1}$ where $gr(\partial f^*)^{-1} = \{(x^{**}, x^*) | x^{**} \in \partial f^*(x^*)\}$. Since $x^* \in \partial f(x)$ if and only if $x \in \partial f^*(x^*)$ it is evident that $gr \partial f \subset gr(\partial f^*)^{-1}$ and it is natural to ask if there is any other relationship between these two sets. Rockafellar [11] (cf. Gossez [4]) has answered this question: $\overline{gr \partial f} = gr(\partial f^*)^{-1}$, where the closure is taken with respect to the product of the \mathfrak{A} topology on E^{**} and the norm topology on E^* . The \mathfrak{A} topology on E^{**} is the weakest topology on E^{**} containing the weak ** topology and for which the norm on E^{**} is a continuous function. In general, it is not a vector topology.

We will give a short proof of this result of Rockafellar using Theorem 2.3, after first proving an easy lemma.

LEMMA 2.4. *Let f be a l.s.c. proper convex function on the Banach space E . Suppose $f^*(0) = 0$ and $x^{**} \in \partial f^*(0)$ and $\delta > 0$. Then $x^{**} \in \partial(f + \psi_B)^*(0)$, where $B = B(0; \|x^{**}\| + \delta)$.*

Proof. Let $K = \text{dom } f$ and let K_1 be the weak ** closure of K in E^{**} . By [6, p. 62], we have $(\psi_K + \|\cdot\|)^{**} = \psi_{K_1} + \|\cdot\|$; hence $\inf \{\|x\| | x \in K\} = \inf \{\|z^{**}\| | z^{**} \in K_1\}$ and it follows that $K \cap B \neq \emptyset$. By [6, p. 62], $(f + \psi_B)^{**} = f^{**} + \psi_B^{**}$; hence

$$\begin{aligned} 0 &= \inf f^{**} && (\text{since } f^*(0) = 0) \\ &= f^{**}(x^{**}) && (\text{since } x^{**} \in \partial f^*(0)) \\ &= (f + \psi_B)^{**}(x^{**}) \\ &= \inf (f + \psi_B)^{**}; \quad \text{so } 0 \in (f + \psi_B)^{**}(x^{**}). \end{aligned}$$

THEOREM 2.5. [Rockafellar]. *Let f be a l.s.c. proper convex function on the Banach space E . Consider $(\partial f^*)^{-1}: E^{**} \rightarrow E^*$. Then $gr(\partial f^*)^{-1} = \overline{gr f}$ where the closure is taken with respect to the product of the \mathfrak{A} topology on E^{**} and the norm topology on E^**

Proof. The bilinear function $\langle \cdot, \cdot \rangle: E^{**} \times E^* \rightarrow R$ is continuous for the $\mathfrak{A} \times$ norm topology, hence $\overline{gr \partial f}$ is monotone. Since we already

know $gr(\partial f^*)^{-1}$ is maximal monotone, it suffices to show that

$$gr(\partial f^*)^{-1} \subset \overline{gr \partial f}.$$

Let $(x^{**}, x^*) \in gr(\partial f^*)^{-1}$ and suppose we are given $\delta \in (0, 1)$ and $\{x_n^*\}_{n=1}^k$ in the unit sphere of E^* . By considering $g^*(\cdot) = f((\cdot) + x^*) - f^*(x^*)$, if necessary, we can assume $x^* = 0$ and $f^*(0) = 0$. Thus $0 \in \partial f^{**}(x^{**})$ and by Lemma 2.4 we have $0 \in \partial(f + \psi_B)^{**}(x^{**})$, where

$$B = B(0; \|x^{**}\| + \delta).$$

Choose $y \in E$ such that $\langle y, x_n^* \rangle = \langle x^{**}, x_n^* \rangle$ for $n = 1, \dots, k$ and apply Theorem 2.3 to $f + \psi_B$ with $y^* = 0$ and $N^\perp = \text{span}\{x_n^*\}_{n=1}^k$ and $\varepsilon = \varepsilon'/(1 + \|x^{**}\|)$ where $0 < \varepsilon' < \delta^2$. We then obtain $z^* \in E^*$ and $z \in E$ satisfying $z^* \in \partial(f + \psi_B)(z)$, $\|z - y\|_{N^\perp} \leq \varepsilon$, $|\langle z, z^* \rangle| \leq \varepsilon$ and $\|z^*\| \leq \varepsilon$. Thus, $z^* = u^* + t^*$ where $u^* \in \partial f(z)$ and $t^* \in \partial \psi_B(z)$. Clearly $z \in x^{**} + \delta(\{x_n^*\}_{n=1}^k)^\circ$ and $\|z\| \leq \|x^{**}\| + \delta$ and $\|u^*\| \leq \delta + \|t^*\|$. We will show $\|t^*\| \leq \delta$ and this will conclude the proof. Suppose $\|t^*\| > \delta$, then because $u^* \in \partial f(z)$ and $0 \in \partial f^{**}(x^{**})$ we have

$$0 \leq \langle x^{**} - z, -u^* \rangle.$$

Hence

$$\begin{aligned} \|z\| \|t^*\| &= \langle z, t^* \rangle \leq \langle x^{**}, t^* \rangle - \langle x^{**}, z^* \rangle + \langle z, z^* \rangle \\ &\leq \|x^{**}\| \|t^*\| + \|x^{**}\| \|z^*\| + \varepsilon \\ &\leq \|x^{**}\| \|t^*\| + (\|x^{**}\| + 1)\varepsilon \\ &\leq \|x^{**}\| \|t^*\| + \varepsilon' \\ &< \|x^{**}\| \|t^*\| + \delta^2, \end{aligned}$$

so $\|z\| < \|x^{**}\| + \delta^2/\|t^*\| < \|x^{**}\| + \delta$ and since $t^* \in \partial \psi_B(z)$, we have the contradiction $t^* = 0$.

Finally we prove a dual version of Proposition 1.8. We require a lemma which is a consequence of Proposition 1.6 and Lemma 2.1.

LEMMA 2.6. *Let C be a weak* closed convex subset of the dual of a Banach space E and N a closed finite codimensional subspace of E . Suppose $0^+C \cap N^\perp = \{0\}$, and that $x^* \in C$, $x \in N$ and $\varepsilon > 0$ satisfy*

$$\sup C(x) \leq \langle x, x^* \rangle + \varepsilon.$$

Then for any $k > 0$ there exist $z \in N$ and $z^ \in C$ satisfying*

$$\langle z, z^* \rangle = \sup C(z), \|x - z\| \leq k, \|x^* - z^*\|_N \leq \varepsilon/k.$$

Proof. By Proposition 1.6, the set $C + N^\perp$ is weak* closed in E^* . Let $Q: E^* \rightarrow E^*/N^\perp$ be the quotient map. Since $Q^{-1}(Q(C)) =$

$C + N^\perp$, it follows from the definition of the quotient topology that $Q(C)$ is weak* closed. We identify E^*/N^\perp with N^* and apply Lemma 2.1 to $Q(C)$ in E^*/N^\perp . Then for any $k > 0$ there exist $w^* + N^\perp \subset C + N^\perp$ and $z \in N$ satisfying $\langle z, w^* + N^\perp \rangle = \sup C(z)$, $\|z - x\| \leq k$ and $\|x^* - w^* + N^\perp\| \leq \varepsilon/k$. Let $n^* \in N^\perp$ be such that $w^* + n^* \in C$ and set $z^* = w^* + n^*$, so that $\langle z, z^* \rangle = \sup C(z)$. Finally, we have

$$\|x^* - z^*\|_N = \|x^* - z^* + N^\perp\| = \|x^* - w^* + N^\perp\| \leq \varepsilon/k.$$

PROPOSITION 2.7. *Let f be a l.s.c. proper convex function on a Banach space E . Let N be a closed subspace of E of finite codimension. Suppose that $0^+ \text{epi } f^* \cap (N^\perp \times \{0\}) = \{0\}$, that $(0, 0) \in \text{epi } f^*$ and that $z \in N \cap S(f^*, \varepsilon, N^\perp)$, where $\varepsilon > 0$. Then for any $k \in (0, 1)$ there exist $w \in N$ and $w^* \in E^*$ satisfying*

$$\begin{aligned} w &\in \partial f^*(w^*) \\ \|z - w\| &\leq \frac{k(1 + \|z\|)}{1 - k} \\ \|w^*\|_N &\leq \varepsilon/k \\ |f^*(w^*)| &\leq \varepsilon/k. \end{aligned}$$

Proof. By hypothesis, $\sup(z, -1)(\text{epi } f^*) \leq \varepsilon$ and $(z, -1) \in N \times R$. We can apply Lemma 2.6 with $x = (0, 0)$ and obtain, for any $k > 0$, $G \in N \times R$ and $(w^*, f^*(w^*)) \in \text{epi } f^*$ satisfying

$$G((w^*, f^*(w^*))) = \sup G(\text{epi } f^*), \|G - (z, -1)\| \leq k$$

and

$$\|(w^*, f^*(w^*))\|_{N \times R} \leq \varepsilon/k.$$

Thus $|G(0, -1) - 1| \leq k$ and since $k < 1$ we have $0 < 1 - k < G(0, -1)$. Hence there exists an element $w \in E$ such that

$$\langle w, y^* \rangle = \frac{G(y^*, 0)}{G(0, -1)} \quad \text{for each } y^* \in E^*.$$

Since $G \in N \times R$, we have $w \in N$.

The verifications that $w \in \partial f^*(w^*)$ and that

$$\|z - w\| \leq \frac{k(1 + \|z\|)}{1 - k}$$

are the same as in Proposition 2.2.

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