

# Pacific Journal of Mathematics

**DIMENSION THEORY IN ZERO-SET SPACES**

M. J. CANFELL

# DIMENSION THEORY IN ZERO-SET SPACES

M. J. CANFELL

The main purpose of this paper is to show that the zero-set spaces of Gordon provide a natural and very general setting in which to develop dimension theory. Defining covering dimension for zero-set spaces in the natural way, it is shown that the subspace theorem, the product theorem, and sum theorem hold. As a consequence it is possible to give a subspace theorem for arbitrary topological spaces.

1. A subspace theorem for arbitrary topological spaces. From the general theory in zero-set spaces it is possible to deduce a subspace theorem in arbitrary topological spaces. To express the result it is convenient to have on hand a definition of the dimension of a ring as defined in [1]. This notion also allows the simplification of certain proofs in dimension theory.

Let  $R$  be a commutative ring with identity. By a *basis* of  $R$  we mean a finite set of elements which generate  $R$ . The *order* of a basis is the largest integer  $n$  for which there exist  $n + 1$  members of the basis with nonzero product. A basis  $\{a_i\}$  of  $R$  is said to *refine* the basis  $\{b_j\}$  of  $R$  if each  $a_i$  is a multiple of some  $b_j$ . The *dimension* of  $R$ , denoted by  $d(R)$ , is the least cardinal  $m$  such that every basis of  $R$  has a refinement of order at most  $m$ .

Let  $C(X)$  denote the ring of continuous real-valued functions on a topological space  $X$ . It is shown in [1] that  $\dim X = d(C(X))$ . For subspaces  $A$  of  $X$ , the statement  $d(C(A)) \leq d(C(X))$  is equivalent to  $\dim A \leq \dim X$ , the assertion of the subspace theorem. These statements are not always true in arbitrary topological spaces [9, p. 264]. However we obtain a subspace theorem by replacing  $C(A)$  by another ring of functions associated with  $A$ . Let  $C_z(A)$  denote the set of all real-valued functions  $f$  defined on  $A$  such that for each real number  $r$ , the sets  $\{x \in A \mid f(x) \leq r\}$  and  $\{x \in A \mid f(x) \geq r\}$  are the intersections with  $A$  of zero-sets of  $X$ . Here a zero-set of  $X$  is the set of zeros of a continuous real-valued function on  $X$ . For general information about zero-sets the reader is referred to [9]. It follows from Theorem 3.5 of [10] that  $C_z(A)$  is a uniformly closed ring and is also a lattice.

The proof of the following subspace theorem will be discussed after Theorem 10.

**THEOREM 1.** *If  $A$  is a subspace of an arbitrary topological space  $X$ , then  $d(C_z(A)) \leq d(C(X))$ .*

It is also possible to formulate analogous versions of the product theorem and the sum theorem in arbitrary topological spaces. However, it will become apparent that the most natural way to formulate these results is in the context of zero-set spaces.

The ring  $C_z(A)$  introduced above seems to have been little studied in the literature and we digress to give some further information about it.

It is clear that  $C_z(A)$  is contained in  $C(A)$ . A condition for equality may be given using the concept of " $z$ -embedding" from [11]. A subspace  $A$  of a topological space  $X$  is said to be  $z$ -embedded in  $X$  if every zero-set of  $A$  is the intersection with  $A$  of a zero-set of  $X$ . In particular, if  $X$  is perfectly normal, every closed set of  $X$  is a zero-set of  $X$ , and it follows that every subspace of  $X$  is  $z$ -embedded in  $X$ .

**THEOREM 2.** *Let  $A$  be a subspace of a topological space  $X$ . Then  $C_z(A) = C(A)$  if and only if  $A$  is  $z$ -embedded in  $X$ .*

*Proof.* Suppose  $A$  is  $z$ -embedded in  $X$  and  $f \in C(A)$ . For each real number  $r$ , the sets  $\{x \in A \mid f(x) \leq r\}$  and  $\{x \in A \mid f(x) \geq r\}$  are zero-sets of  $A$  and by hypothesis are the intersections with  $A$  of zero-sets of  $X$ . Hence  $f \in C_z(A)$ . It follows that  $C_z(A) = C(A)$ .

Conversely, suppose  $C_z(A) = C(A)$  and let  $Z$  be a zero-set of  $A$ . Then  $Z = \{x \in A \mid f(x) = 0\}$  for some  $f \in C(A)$ . Since  $f \in C_z(A)$ , it follows that  $Z$  is the intersection with  $A$  of a zero-set of  $X$ . Hence  $A$  is  $z$ -embedded in  $X$ .

If  $f \in C(X)$ , the restriction of  $f$  to a subspace  $A$  clearly belongs to  $C_z(A)$ . In the case when  $A$  is a zero-set of  $X$ , we have a Urysohn extension theorem for members of  $C_z(A)$ .

**THEOREM 3.** *Let  $A$  be a zero-set of an arbitrary topological space  $X$ . Then each  $f \in C_z(A)$  has a continuous extension to  $X$ .*

*Proof.* Except for notation, the proof is the same as in Gillman and Jerison [9, pp. 18, 19].

**2. Zero-set spaces.** In the following,  $\mathbf{R}$  will denote the real numbers and  $\mathbf{N}$  the positive integers. For information on zero-set spaces, the reader is referred to Gordon [10]. In this section we review some of the main facts and give some further results.

A zero-set space is a pair  $(X, \mathcal{Z})$  where  $\mathcal{Z}$  is a family of subsets of  $X$  satisfying certain axioms. The sets in  $\mathcal{Z}$  are called zero-sets and their complements with respect to  $X$  are called cozero-sets. The first of the axioms given by Gordon concerning the separation of

distinct points of  $X$  by a set in  $\mathcal{Z}$  is not needed in dimension theory and we will omit this requirement. Further, we find it convenient to give the axioms for the cozero-sets rather than the zero-sets. Thus by a zero-set space we mean a pair  $(X, \mathcal{Z})$  where  $\mathcal{Z}$  is a family of subsets of  $X$  such that the family  $\mathcal{C}$  of all complements in  $X$  of members of  $\mathcal{Z}$  satisfies the following conditions.

- (1)  $\mathcal{C}$  is closed under countable unions; in particular,  $X \in \mathcal{C}$ .
- (2)  $\mathcal{C}$  is closed under finite intersections; in particular,  $\emptyset \in \mathcal{C}$ .
- (3) Whenever  $A$  and  $B$  are disjoint zero-sets of  $X$ , there exist disjoint cozero-sets  $C$  and  $D$  with  $A \subset C$  and  $B \subset D$ .
- (4) Each cozero-set of  $X$  is the countable union of zero-sets of  $X$ .

It is interesting to note that these conditions differ from the axioms for the open sets of a perfectly normal topological space only in the requirement that we have closure under countable unions rather than arbitrary unions. It is significant that a good theory of dimension is available for perfectly normal spaces [2], [3].

The family of all cozero-sets in an arbitrary topological space satisfies the conditions (1)–(4), [9, Chapter 1]. It is again significant that the most satisfactory theory of dimension in topological spaces involves the use of cozero-sets [9, p. 243], [8].

If  $X$  and  $Y$  are zero-set spaces, a mapping  $\alpha: X \rightarrow Y$  is called a zero-set mapping if the inverse image of every cozero-set is again a cozero-set. When  $Y = \mathbf{R}$ , the cozero-sets are taken to be the open sets of  $\mathbf{R}$ . If  $X$  is a zero-set space,  $S(X)$  will denote the set of real-valued zero-set mappings on  $X$ . It is shown in [9] that

- (i)  $S(X)$  is a uniformly closed ring and is also a lattice;
- (ii) For each cozero-set  $C$  of  $X$ , there is a zero-set function  $f \in S(X)$  such that  $C = \{x \in X \mid f(x) \neq 0\}$ .

Part (ii) is the justification for calling members of  $\mathcal{C}$  cozero-sets, and may be derived from a Urysohn's lemma argument in almost the same way that an open set in a perfectly normal topological space is shown to be the cozero-set of a continuous real-valued function as in [6, p. 148].

With subspaces defined in the natural way as in [10], we also have the Urysohn extension theorem holding for zero-set spaces. Again the proof is the same as in [9].

**THEOREM 4.** *Let  $X$  be a zero-set space and let  $A$  be a zero-set of  $X$ . Then each  $f \in S(A)$  has an extension to a zero-set function of  $S(X)$ .*

By a *basic cover* of a zero-set space, we mean a covering of  $X$  by cozero-sets. In defining locally finite families in zero-set spaces, we

use a condition which is somewhat stronger than the condition for local finiteness in topological spaces.

**DEFINITION.** A family  $\{U_\alpha | \alpha \in A\}$  of subsets of a zero-set space  $X$  is said to be *locally finite* if there exists a countable basic cover  $\{V_i | i \in \mathbb{N}\}$  of  $X$  such that each  $V_i$  meets at most finitely many of the  $U_\alpha$ .

We now give a result which is crucial in the development of dimension theory in zero-set spaces. This says in effect that every zero-set space is countably paracompact. Countably paracompact spaces were discussed by Dowker [4] who showed that each perfectly normal topological space is countably paracompact.

**THEOREM 5.** *Let  $\{U_i\}$  be a countable basic cover of a zero-set space. Then there exists a countable locally finite basic cover  $\{V_i\}$  of  $X$  such that  $V_i \subset U_i$  for each  $i$ .*

*Proof.* Choose zero-set functions  $f_j$  such that

$$\bigcup_{k \leq j} U_k = \{x | f_j(x) > 0\},$$

and let

$$B_{ji} = \{x | f_j(x) < 1/i\},$$

$$V_i = U_i \cap \left( \bigcap_{j < i} B_{ji} \right),$$

and

$$W_{jk} = \{x | f_j(x) > 1/k\}.$$

It is clear that  $V_i$  is a cozero-set and that  $V_i \subset U_i$ . For each  $x \in X$ , there is some first  $i$  for which  $x \in U_i$  but  $x \notin U_k$  for  $k < i$ . Suppose that  $x \notin V_i$ . Then  $x \notin B_{ji}$  for some  $j < i$  and therefore  $f_j(x) \geq 1/i > 0$ . Thus  $x \in U_k$  for some  $k \leq j < i$ . This contradiction shows that  $x \in V_i$  and so  $\{V_i\}$  is a basic cover of  $X$ .

Given  $x \in X$  there is some  $j$  for which  $f_j(x) > 0$ . Hence  $x \in W_{jk}$  for some  $k$  and therefore  $\{W_{jk}\}$  is a countable basic cover of  $X$ . If  $i > j$  and  $i > k$ , then  $y \in W_{jk}$  implies that  $f_j(y) > 1/k > 1/i$  so that  $y \notin B_{ji}$ . Thus  $y \notin V_i$  and  $W_{jk} \cap V_i = \emptyset$ . Therefore,  $W_{jk}$  meets only finitely many  $V_i$  and the proof is complete.

**REMARK.** It is possible to show as in [7, p. 221] that each countable basic cover has a star-finite countable basic refinement.

**3. Covering dimension for zero-set spaces.** The *order* of a basic cover is the largest integer  $n$  for which there exist  $n + 1$  members of the cover with nonempty intersection.

**DEFINITION.** Let  $X$  be a zero-set space. The dimension of  $X$ , denoted by  $\dim X$ , is the least cardinal  $m$  such that every finite basic cover of  $X$  has a finite basic refinement of order at most  $m$ .

If  $(X, \mathcal{T})$  is a topological space, then the family  $\mathcal{Z}$  of zero-sets of  $X$  form a zero-set structure on  $X$ . The definition of dimension for a topological space  $(X, \mathcal{T})$  as given in [9, p. 243] is the above definition for the case of the zero-set space  $(X, \mathcal{Z})$ .

The subspace theorem for zero-sets of a zero-set space is an easy consequence of the definition. If  $X$  is a zero-set space with  $\dim X \leq n$ , and if  $A$  is a zero-set of  $X$ , then  $\dim A \leq n$ .

We remark that if  $\{U_i\}$  is a basic cover and  $\{V_j\}$  is a countable basic refinement of order  $\leq n$ , then there exists a basic cover  $\{W_i\}$  of order  $\leq n$  such that  $W_i \subset U_i$  for each  $i$ . In fact we only have to set  $W_i = \bigcup \{V_j \mid V_j \subset U_i, V_j \not\subset U_k \text{ for } k < i\}$ . Some of the  $W_i$  may possibly be empty.

The following lemma is proved using an argument due to deVries and given in Nagata [13, p. 22].

**LEMMA 6.** *Let  $\{U_i \mid i \in N\}$  be a countable basic cover of a zero-set space  $X$ , and let  $\{Z_j \mid j \in N\}$  be a countable locally finite cover of  $X$  consisting of zero-sets  $Z_j$  such that  $\dim Z_j \leq n$  for each  $j$  and such that each  $Z_j$  meets at most finitely many  $U_i$ . Then there exists a basic cover  $\{V_i \mid i \in N\}$  of order  $\leq n$  such that  $V_i \subset U_i$  for each  $i \in N$ .*

*Proof.* We construct by induction a sequence of basic covers  $\{U_{j,i} \mid i \in N\}$  such that  $U_{0,i} = U_i$  and  $U_{j,i} \subset U_{k,i}$  for  $j > k$ , and such that each point of  $Z_j$  is contained in at most  $n + 1$  members of  $\{U_{j,i} \mid i \in N\}$ . Suppose that  $\{U_{k,i} \mid i \in N\}$  has been constructed. To get  $\{U_{k+1,i} \mid i \in N\}$  we restrict  $\{U_{k,i} \mid i \in N\}$  to  $Z_{k+1}$  and choose a basic refinement  $\{W_{k,i} \mid i \in N\}$  of it of order  $\leq n$  such that  $W_{k,i} \subset U_{k,i}$  for each  $i$ . If we put  $U_{k+1,i} = (U_{k,i} \setminus Z_{k+1}) \cup W_{k,i}$ , the induction step is completed.

Let  $V_i = \bigcap \{U_{k,i} \mid k \in N\}$ . To see that  $V_i$  is a cozero-set, we let  $\{A_s\}$  be a countable basic cover such that each  $A_s$  meets at most finitely many  $Z_k$ . Since  $U_{k+1,i}$  was obtained from  $U_{k,i}$  by removing part of  $Z_{k+1}$ , it follows that for some integer  $K$  we have  $U_{k,i} \cap A_s = U_{K,i} \cap A_s$  for all  $k > K$ . Since  $V_i$  is a countable union of sets of the form  $U_{K,i} \cap A_s$ , it is a cozero-set. Finally it is clear that each point of  $X$  is contained in at most  $n + 1$  of the  $V_i$ .

**LEMMA 7.** *Let  $\{V_k\}$  be a countable basic cover of a zero-set space*

*X. There exists a basic cover  $\{W_k\}$  and zero-sets  $Z_k$  such that  $W_k \subset Z_k \subset V_k$  for each  $k$ .*

*Proof.* The sets  $W_k$  and  $Z_k$  may be found inductively as in [9, p. 243].

For normal topological spaces, Dowker [5] has shown that one may use locally finite covers instead of finite covers in the definition of dimension. For zero-set spaces one may use countable basic covers instead of finite basic covers as shown in the next result.

**THEOREM 8.** *Let  $X$  be a zero-set space. Then  $\dim X \leq n$  if and only if every countable basic cover of  $X$  has a countable basic refinement of order  $\leq n$ .*

*Proof.* The sufficiency of the condition follows from the remarks preceding Lemma 6.

Suppose now that  $\dim X \leq n$ . In view of Theorem 5, it is sufficient to prove the result for locally finite covers. Let  $\{U_i\}$  be a countable locally finite basic cover of  $X$  and let  $\{A_k\}$  be a countable basic cover of  $X$  such that each  $A_k$  meets at most finitely many  $U_i$ . Let  $\{V_k\}$  be a locally finite basic cover of  $X$  such that  $V_k \subset A_k$  for each  $k$ . Choose zero-sets  $Z_k$  as in Lemma 7. Since  $\{V_k\}$  is locally finite, so is  $\{Z_k\}$  and since  $Z_k \subset A_k$ , each  $Z_k$  meets at most finitely many  $U_i$ . We have  $\dim Z_k \leq n$  for each  $k$ . An application of Lemma 6 now yields a basic refinement of  $\{U_i\}$  of order  $\leq n$ . This completes the proof.

Further characterizations of dimension in terms of mappings and separation of zero-sets may be obtained for zero-set spaces. These results, and their proofs, are similar to the results in topological spaces as in [12] and [8].

#### 4. The sum, subspace, and product theorems.

**THEOREM 9.** (The sum theorem.) *Let  $X$  be a zero-set space and let  $X = \bigcup_{i=1}^{\infty} Z_i$  where each  $Z_i$  is a zero-set. If  $\dim Z_i \leq n$  for each  $i$ , then  $\dim X \leq n$ .*

*Proof.* This result may be proved as in Hemmingsen [12, Theorem 4.2]. For the case when the family  $\{Z_i\}$  is locally finite, the result is immediate from Lemma 6. It is only the latter case which is needed in the proof of the subspace theorem.

**THEOREM 10.** (The subspace theorem.) *If  $X$  is a zero-set space and  $A$  is a subspace of  $X$ , then  $\dim A \leq \dim X$ .*

*Proof.* We have already observed that the result holds when  $A$  is a zero-set of  $X$ .

Suppose next that  $A$  is a cozero-set of  $X$ . Choose a zero-set function  $f$  such that  $0 \leq f \leq 1$  and  $A = \{x \mid f(x) > 0\}$ . Let  $Z_i = \{x \mid 1/(i+1) \leq f(x) \leq 1/i\}$ , ( $i = 1, 2, \dots$ ), and let  $U_i = \{x \mid f(x) > 1/i\}$ . Then  $\{U_i\}$  is a basic cover of  $A$  and each  $U_i$  meets finitely many  $Z_i$ . Thus  $\{Z_i\}$  is a locally finite collection in  $A$ . Also  $A = \bigcup_{i=1}^{\infty} Z_i$  and since  $\dim Z_i \leq n$  for each  $i$ , we have  $\dim A \leq n$  by Theorem 9.

Finally we consider an arbitrary subspace  $A$  of  $X$ . Let  $\{U_i\}$  be a finite basic cover of  $A$ . We can write  $U_i = V_i \cap A$  for cozero-sets  $V_i$  of  $X$ . The cozero-set  $V = U_i V_i$  has dimension  $\leq n$  (by the preceding paragraph) and so the basic cover  $\{V_i\}$  of  $V$  has a basic refinement  $\{W_j\}$  (in  $V$ ) of order  $\leq n$ . The trace of  $\{W_j\}$  on  $A$  is then a basic refinement of  $\{U_i\}$  of order  $\leq n$ . Thus  $\dim A \leq n$  and the proof is complete.

If  $X$  is a zero-set space, we may show, as in [1], that  $\dim X = d(S(X))$ . Moreover, if  $S^*(X)$  denotes the bounded functions in  $S(X)$ , the same argument may be used to show that  $\dim X = d(S^*(X))$ .

Theorem 1 now follows from Theorem 10. If  $\mathcal{T}$  is a topology on  $X$ , we let  $\mathcal{Z}$  denote the family of zero-sets in  $X$  of elements of  $C(X)$ . If  $A$  is a zero-set subspace of  $(X, \mathcal{Z})$ , then  $\dim A \leq \dim X$ . Hence  $d(S(A)) \leq d(S(X))$ . The result follows if we observe that  $C_{\mathcal{Z}}(A) = S(A)$  and  $C(X) = S(X)$ .

The product of zero-set spaces is discussed in [10, p. 152]. The product theorem for zero-set spaces will be derived from the subspace theorem and the product theorem for compact topological spaces.

Let  $X$  be a zero-set space for which  $S(X)$  separates the points of  $X$ . Gordon [10] has shown that such a space has a compactification  $\beta X$  with the property that  $S^*(X)$  is isomorphic to  $S(\beta X)$ . Moreover, if  $\beta X$  is given the weak topology induced by  $S(\beta X)$ , then  $C(\beta X)$ , the ring of continuous functions on  $\beta X$  for this topology, coincides with  $S(\beta X)$ . Thus  $d(C(\beta X)) = d(S(\beta X))$  and it follows that the dimension of  $\beta X$  is the same whether considered as a zero-set space or as a topological space.

**THEOREM 11.** *If  $X$  is a zero-set space, then  $\dim X = \dim \beta X$ .*

*Proof.* Since  $S^*(X)$  and  $S(\beta X)$  are isomorphic,  $\dim X = n \Leftrightarrow d(S^*(X)) = n \Leftrightarrow d(S(\beta X)) = n \Leftrightarrow \dim \beta X = n$ .

**THEOREM 12.** (The product theorem.) *If  $X$  and  $Y$  are zero-set spaces, then*



$$\dim(X \times Y) \leq \dim X + \dim Y.$$

*Proof.* Suppose first that  $S(X)$  and  $S(Y)$  separate the points of  $X$  and  $Y$  respectively. Since  $X \times Y \subset \beta X \times \beta Y$ , the subspace theorem yields  $\dim(X \times Y) \leq \dim(\beta X \times \beta Y)$ . By the product theorem for compact topological spaces [12], we have  $\dim(\beta X \times \beta Y) \leq \dim \beta X + \dim \beta Y$ . Since  $\dim X = \dim \beta X$  and  $\dim Y = \dim \beta Y$ , it follows that  $\dim(X \times Y) \leq \dim X + \dim Y$ .

Now suppose that  $X$  and  $Y$  are arbitrary zero-set spaces. We may define, as in the construction on p. 41 of [9], zero-set spaces  $X'$  and  $Y'$  such that  $S(X)$  is isomorphic to  $S(X')$  and  $S(Y)$  is isomorphic to  $S(Y')$ , and  $S(X')$  and  $S(Y')$  separate the points of  $X'$  and  $Y'$  respectively. It is clear from the construction that  $S(X \times Y)$  is isomorphic to  $S(X' \times Y')$ . It follows that  $\dim(X \times Y) = \dim(X' \times Y')$ . The result follows on applying the first part of the theorem to  $X'$  and  $Y'$ .

## REFERENCES

1. M. J. Canfell, *An algebraic characterization of dimension*, Proc. Amer. Math. Soc., **32** (1972), 619-620.
2. E. Čech, *Contribution à la théorie de la dimension*, Čas. Mat. Fys., **62** (1933), 277-291.
3. C. H. Dowker, *Inductive dimension of completely normal spaces*, Quart. J. Math. Oxford Ser., (2) **4** (1953), 267-281.
4. ———, *On countably paracompact spaces*, Canad. J. Math., **3** (1951), 219-224.
5. ———, *Mapping theorems for non-compact spaces*, Amer. J. Math., **69** (1947), 200-242.
6. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
7. R. Engelking, *Outline of General Topology*, North Holland, Amsterdam, 1968.
8. J. R. Gard and R. D. Johnson, *Four dimension equivalences*, Canad. J. Math., **20** (1968), 48-50.
9. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.
10. H. Gordon, *Rings of functions determined by zero-sets*, Pacific J. Math., **36** (1971), 133-157.
11. A. W. Hager, *On inverse-closed subalgebras of  $C(X)$* , Proc. London Math. Soc., (3) **19** (1969), 233-257.
12. E. Hemmingsen, *Some theorems in dimension theory for normal Hausdorff spaces*, Duke Math. J., **13** (1946), 495-504.
13. J. Nagata, *Modern Dimension Theory*, John Wiley and Sons Inc., New York, 1966.

Received June 2, 1972 and in revised form October 11, 1972.

UNIVERSITY OF NEW ENGLAND, AUSTRALIA

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

D. GILBARG AND J. MILGRAM

Stanford University  
Stanford, California 94305

J. DUGUNDJI\*

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington  
Seattle, Washington 98105

RICHARD ARENS

University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

\* C. DePrima will replace J. Dugundji until August 1974.

Copyright © 1973 by  
Pacific Journal of Mathematics  
All Rights Reserved

David Parham Bellamy, <i>Composants of Hausdorff indecomposable continua; a mapping approach</i> .....	303
Colin Bennett, <i>A Hausdorff-Young theorem for rearrangement-invariant spaces</i> .....	311
Roger Daniel Bleier and Paul F. Conrad, <i>The lattice of closed ideals and <math>a^*</math>-extensions of an abelian <math>l</math>-group</i> .....	329
Ronald Elroy Bruck, Jr., <i>Nonexpansive projections on subsets of Banach spaces</i> .....	341
Robert C. Busby, <i>Centralizers of twisted group algebras</i> .....	357
M. J. Canfell, <i>Dimension theory in zero-set spaces</i> .....	393
John Dauns, <i>One sided prime ideals</i> .....	401
Charles F. Dunkl, <i>Structure hypergroups for measure algebras</i> .....	413
Ronald Francis Gariepy, <i>Geometric properties of Sobolev mappings</i> .....	427
Ralph Allen Gellar and Lavon Barry Page, <i>A new look at some familiar spaces of intertwining operators</i> .....	435
Dennis Michael Girard, <i>The behavior of the norm of an automorphism of the unit disk</i> .....	443
George Rudolph Gordh, Jr., <i>Terminal subcontinua of hereditarily unicoherent continua</i> .....	457
Joe Alston Guthrie, <i>Mapping spaces and <math>cs</math>-networks</i> .....	465
Neil Hindman, <i>The product of <math>F</math>-spaces with <math>P</math>-spaces</i> .....	473
M. A. Labbé and John Wolfe, <i>Isomorphic classes of the spaces <math>C_\sigma(S)</math></i> .....	481
Ernest A. Michael, <i>On <math>k</math>-spaces, <math>k_R</math>-spaces and <math>k(X)</math></i> .....	487
Donald Steven Passman, <i>Primitive group rings</i> .....	499
C. P. L. Rhodes, <i>A note on primary decompositions of a pseudovaluation</i> .....	507
Muril Lynn Robertson, <i>A class of generalized functional differential equations</i> .....	515
Ruth Silverman, <i>Decomposition of plane convex sets. I</i> .....	521
Ernest Lester Stitzinger, <i>On saturated formations of solvable Lie algebras</i> .....	531
B. Andreas Troesch, <i>Sloshing frequencies in a half-space by Kelvin inversion</i> .....	539
L. E. Ward, <i>Fixed point sets</i> .....	553
Michael John Westwater, <i>Hilbert transforms, and a problem in scattering theory</i> .....	567
Misha Zafran, <i>On the spectra of multipliers</i> .....	609