Pacific Journal of Mathematics

ISOMORPHIC CLASSES OF THE SPACES $C_\sigma(S)$

M. A. Labbé and John Wolfe

Vol. 47, No. 2 February 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 47, No. 2, 1973

ISOMORPHIC CLASSES OF THE SPACES $C_\sigma(S)$

M. A. LABBÉ and JOHN WOLFE

Jerison introduced the Banach spaces $C_\sigma(S)$ of continuous real or complex-valued odd functions with respect to an involutory homeomorphism $\sigma: S \to S$ of the compact Hausdorff space S. It has been conjectured that any Banach space of the type $C_\sigma(S)$ is isomorphic to a Banach space of all continuous functions on some compact Hausdorff space. This conjecture is shown to be true if either (1) S is a Cartesian product of compact metric spaces or (2) S is a linearly ordered compact Hausdorff space and σ has at most one fixed point.

Introduction. Let S always denote a compact Hausdorff space. $C(S)$ well denote the Banach space of real or complex-valued continuous functions on S equipped with the supremum norm. A homeomorphism $\sigma: S \to S$ is involutory if $\sigma(\sigma(s)) = s$ for each $s \in S$. Jerison [2] introduced the Banach space $C_\sigma(S) = \{f \in C(S): f(\sigma(s)) = -f(s)\}$ of odd functions with respect to an involutory homeomorphism $\sigma: S \to S$. If X and Y are Banach spaces then X is *isomorphic* (*isometric*) to Y, and we will write $X \sim Y$ ($X \approx Y$), if there is a bounded (norm preserving) one-to-one linear operator from X onto Y.

A special case of a conjecture due to A. Pełczyński [8] is as follows: for any Banach space $C_\sigma(S)$ there is a compact Hausdorff space T with $C_\sigma(S) \sim C(T)$. In this paper we prove this conjecture when S is either a Cartesian product of compact metric spaces or a linearly ordered compact Hausdorff space (in the second case we assume σ has at most one fixed point). The results and techniques of this paper generalize, and provide shorter proofs of, some results of Samuel [11].

1. Linearly ordered spaces. A topological space A is a *linearly ordered topological space* if the topology on A is the order topology ([4], page 57) arising from some linear ordering on the set A. Examples of linearly ordered spaces are the closed interval [0,1], every space of ordinal numbers, every totally disconnected compact metric space ([5], Corollary 2a), and every compact subset of a linearly ordered space.

THEOREM 1. *Let S be an infinite linearly ordered compact Hausdorff space. If σ is an involutory homeomorphism on S with at most one fixed point, then $C_\sigma(S) \sim C(T)$ for some compact Hausdorff space T.*

Proof. The function $\Psi: S \to S$ defined by $\Psi(s) = \min\{s, \sigma(s)\}$ is continuous on S. Set $T = \Psi(S)$; the compact set T contains exactly one point from each of the pairs $\{s, \sigma(s)\}$ and thus $T \cup \sigma(T) = S$ and $T \cap \sigma(T)$ contains at most the fixed point of σ. If $T \cap \sigma(T) = \emptyset$, then $C_\sigma(S)$ is isometric to $C(T)$ via the restriction map. If $T \cap \sigma(T) = \{t_0\}$, where t_0 is the fixed point of σ, then restriction of the functions in $C_\sigma(S)$ to T is an isometry of $C_\sigma(S)$ onto the closed hyperplane $C(T, t_0) = \{f \in C(T): f(t_0) = 0\}$ of $C(T)$. By [1], $C(T, t_0) \sim C(T)$ if T contains a convergent sequence with distinct terms. Since T is infinite, it contains a strictly monotone sequence (t_n). This sequence converges either to its supremum or to its infimum and thus $C(T, t_0) \sim C(T)$.

Remark. The first part of the proof shows that if $\sigma: S \to S$ is an arbitrary involutory homeomorphism on a linearly ordered compact Hausdorff space S, T is as in the proof, and $T_0 = \{s \in S: \sigma(s) = s\}$, then $C_\sigma(S) \approx C(T, T_0) = \{f \in C(S): f(T_0) \subset \{0\}\}$.

If S is a countable compact metric space, then S is linearly ordered since it is homeomorphic to a closed subset of the Cantor set ([5], page 286). Thus the following result due to Samuel [11] is an easy consequence.

Corollary 2. *Suppose S is a countably infinite compact metric space and $\sigma: S \to S$ is an involutory homeomorphism on S with at most one fixed point. Then $C_\sigma(S) \sim C(S)$.*

Proof. If T is an infinite compact metric space, then $C(T) \sim C(T) \oplus C(T)$ ([10], page 514) where $\oplus$ denotes the Cartesian product normed by taking the maximum of the norms of the two coordinates. Now, if T is as in Theorem 1 so that $S = T \cup \sigma T$ and $T \cap \sigma T$ has at most one point, it follows that $C(S) \sim C(T) \oplus C(\sigma(T))$: that is immediate if $T \cap \sigma(T) = \emptyset$; if $T \cap \sigma(T) = \{t_0\}$, then we have the string of isomorphisms $C(S) \sim C(S, t_0) \approx C(T, t_0) \oplus C(\sigma(T), t_0) \sim C(T) \oplus C(\sigma(T))$. Thus $C_\sigma(S) \sim C(T) \sim C(T) \oplus C(T) \sim C(T) \oplus C(\sigma(T)) \sim C(S)$ if S is countably infinite compact metric and σ has at most one fixed point.

Remark. In general, even for an involutory homeomorphism $\sigma: S \to S$ having no fixed points on an ordinal space S, it is not true that $C_\sigma(S) \sim C(S)$. We are indebted to J. J. Schäffer for the following example. Let ω_1 be the first uncountable ordinal number and let $S = \{\alpha: \alpha$ an ordinal and $1 \leqq \alpha \leqq \omega_1 \cdot 2\}$. Let $F_1 = \{\alpha \in S: \alpha \leqq \omega_1\}$ and $F_2 = \{\alpha \in S: \alpha > \omega_1\}$. Then $\tau: \alpha \to \omega_1 + \alpha: F_1 \to F_2$ is a homeomorphism, and we define the involutory homeomorphism $\sigma: S \to S$ by

$\sigma(\alpha) = \tau(\alpha)$ if $\alpha \in F_1$, $\sigma(\alpha) = \tau^{-1}(\alpha)$ if $\alpha \in F_2$. Then $C_\sigma(S)$ is isomorphic to $C(F_1)$. However, $C(F_1)$ is not isomorphic to $C(S)$ ([12], Theorem 2).

2. Products of compact metric spaces. We begin this section with some terminology and preliminary facts from [9]. A subspace Z of a Banach space X is *complemented* if there is a bounded linear projection $P: X \to X$ with range Z, i.e., $P^2 = P$ and $P(X) = Z$. For Banach spaces Y and X, Y is *a factor of* X if there is a complemented subspace Z of X with $Y \sim Z$. If $\sigma: S \to S$ is an involutory homeomorphism, then the operator $P: C(S) \to C(S)$ defined by $(Pf)(s) = (1/2)[f(s) - f(\sigma(s))]$ projects $C(S)$ onto the subspace of odd functions $C_\sigma(S)$. Thus $C_\sigma(S)$ is a factor of $C(S)$.

D will denote the two point discrete space $\{0, 1\}$ and, for each cardinal number m, D^m will denote the generalized Cantor set which is the Cartesian product of m copies of D. We will need the following isomorphism criterion due to A. Pełczyński ([9], Proposition 8.3): if X is a Banach space and X is a factor of $C(D^m)$ and $C(D^m)$ is a factor of X, then $X \sim C(D^m)$.

Following [9], we say that a space S is an *almost Milutin space* if, for some cardinal number m, there is a continuous onto map $\theta: D^m \to S$ such that the subspace $X = \{f \circ \theta : f \in C(S)\}$ of $C(D^m)$ is complemented. If T is a closed subset of the space S, an *extension operator* is a bounded linear operator $E: C(T) \to C(S)$ such that, for each $f \in C(T)$, $Ef \mid T = f$ where "$\mid$" denotes the restriction. A compact Hausdorff space T is an *almost Dugundji space* if, for every embedding $i: T \to S$ of T into a compact Hausdorff space S, there is an extension operator $E: C(i(T)) \to C(S)$. Every Cartesian product of compact metric spaces (in particular, every space D^m) is both an almost Milutin and an almost Dugundji space ([9], Theorems 5.6 and 6.6). The *weight* of a space S is the smallest cardinal number m such that there is a base for the topology of S consisting of m open sets. If S is either an almost Milutin or an almost Dugundji space, then $C(S)$ is a factor of $C(D^m)$, where m is the weight of S (see the proof of [9], Proposition 8.4).

PROPOSITION 3. *Let S be either an almost Milutin space or an almost Dugundji space and let $\sigma: S \to S$ be an involutory homeomorphism on S. Suppose there is a closed subset F of S with $\sigma(F) \cap F = \varnothing$ such that F is homeomorphic to D^m, where m is the weight of S. Then $C_\sigma(S) \sim C(S)$.*

Proof. Since $C_\sigma(S)$ is a factor of $C(S)$ and $C(S)$ is a factor of $C(D^m)$, $C_\sigma(S)$ is a factor of $C(D^m)$. Thus, by Pełczyński's criterion, it suffices to show that $C(D^m)$ is a factor of $C_\sigma(S)$. Since F and $\sigma(F)$

are disjoint and each is homeomorphic to D^m, $F \cup \sigma(F)$ is homeomorphic to the almost Dugundji space D^{m+1}. Hence there exists an extension operator $E: C(F \cup \sigma(F)) \to C(S)$. Let σ' be the restriction of σ to the invariant set $F \cup \sigma(F)$ and let $P: C(S) \to C_\sigma(S)$ be the above-defined projection onto the odd functions. Then $C_{\sigma'}(F \cup \sigma(F))$ is isomorphic to the range of the projection Q defined on $C_\sigma(S)$ by $Qf = PE(f \mid (F \cup \sigma(F)))$. Since $C_{\sigma'}(F \cup \sigma(F))$ is trivially isometric to $C(F)$, which is isometric to $C(D^m)$, it follows that $C(D^m)$ is a factor of $C_\sigma(S)$.

LEMMA 4. *If S is an infinite product of nontrivial compact metric spaces and $\sigma: S \to S$ is an involutory homeomorphism on S that is not the identity, then $C_\sigma(S) \sim C(S)$.*

Proof. Let $S = \prod_{i \in I} S_i$, where each S_i has at least two points. A basis for the topology of S is given by the open sets U of the form $U = (\prod_{i \in I \setminus A} S_i) \times (\prod_{i \in A} U_i)$ where A is a finite subset of I and U_i is an open set in S_i for $i \in A$. If I is infinite, then the weight m of S is the cardinality of I. So it suffices, by Proposition 3, to construct a closed set F in S which is homeomorphic to D^m with $\sigma(F) \cap F = \varnothing$. There exists $s \in S$ with $\sigma(s) \neq s$; choose a basic neighborhood U of s with $\sigma(U) \cap U = \varnothing$. Then $U = (\prod_{i \in I \setminus A} S_i) \times (\prod_{i \in A} U_i)$ for some finite set A in I. For each i, let $\{t_i^1, t_i^2\}$ be any pair of distinct points in S_i if $i \in I \setminus A$, and just any pair of points in U_i if $i \in A$. Let $F = \prod_{i \in I} \{t_i^1, t_i^2\}$. Then F is homeomorphic to D^m and $\sigma(F) \cap F = \varnothing$.

LEMMA 5. *If S is an uncountable compact metric space and σ is an involutory homeomorphism on S such that $\{s: \sigma(s) = s\}$ is countable, then $C_\sigma(S) \sim C(S)$.*

Proof. Let P be the set of condensation points of S, i.e., $s \in P$ iff every neighborhood of s is uncountable. By the Cantor-Bendixson Theorem ([5], page 253), the complement of P is countable. Thus P is uncountable and there is a point $s \in P$ with $\sigma(s) \neq s$. Let F_0 be a closed neighborhood of s with $\sigma(F_0) \cap F_0 = \varnothing$. Since F_0 is an uncountable compact metric space, it must contain a closed subset F homeomorphic to $D^{\aleph_0}$ ([5], page 445). Clearly $\sigma(F) \cap F = \varnothing$. Since the weight of S is $\aleph_0$, the conclusion follows from Proposition 3.

THEOREM 6. *If S is a product of compact metric spaces and σ is an involutory homeomorphism on S that is not the identity, then $C_\sigma(S) \sim C(T)$ for some compact Hausdorff space T.*

Proof. If S is an infinite product of nontrivial compact metric spaces, then $C_\sigma(S) \sim C(S)$ by Lemma 4. If S is a finite product of compact metric spaces, then S is compact metric. Let T be the quotient space obtained from S by identifying the fixed points of σ. Let σ' denote the involutory homeomorphism on T which is induced by σ; it has at most one fixed point. Then $C_\sigma(S) \approx C_{\sigma'}(T)$, and $C_{\sigma'}(T) \sim C(T)$ by Lemma 5 if T is uncountable; by Corollary 2 if T is countably infinite. The conclusion is obvious if T is finite.

We conclude with an application to the problem of the isomorphic classification of complemented subspaces of the Banach spaces of type $C(S)$. This result is due to Samuel [11].

COROLLARY 7. *Let X be a subspace of $C(S)$, where S is a compact metric space. If X is the range of a norm-1 projection on $C(S)$, then $X \sim C(T)$ for some compact metric space T.*

Proof. By [7] or [3] (see also [6]), we have $X \approx C_\sigma(K)$ where σ is an involutory homeomorphism on a certain subspace K of a Hausdorff quotient space of S. Since a Hausdorff quotient of a compact metric space is metric, $C_\sigma(K) \sim C(T)$ for some compact metric space T by the preceding theorem.

REFERENCES

1. D. W. Dean, *Projections in certain continuous function spaces $C(H)$ and subspaces of $C(H)$ isomorphic with $C(H)$*, Canad. J. Math., **14** (1962), 385-401.

2. M. Jerison, *Certain spaces of continuous functions*, Trans. Amer. Math. Soc., **70** (1951), 103-113.

3. M. Jonac and C. Samuel, *Sur les sous-espaces complémentés de $C(S)$*, Bull. Sci. Math., 2^e série **94** (1970), 159-163.

4. J. L. Kelley, *General Topology*, Princeton, New Jersey, 1955.

5. K. Kuratowski, *Topology*, vol. I., New York, 1966.

6. K. J. Lindberg, *Contractive projections in Orlicz sequence spaces and continuous function spaces*, Thesis, University of California at Berkeley, 1971.

7. J. Lindenstrauss and D. E. Wulbert, *On the classifications of the Banach spaces whose duals are L_1 spaces*, J. Functional Analysis, **4** (1969), 332-349.

8. A. Pełczyński, *Projections in certain Banach spaces*, Studia Math., **19** (1960), 209-228.

9. ———, *Linear extensions, linear averagings and their applications to linear classification of spaces of continuous functions*, Rozprawy Matematyczne, **58** (1968).

10. ———, *On $C(S)$—subspaces of separable Banach spaces*, Studia Math., **31** (1968), 513-522.

11. C. Samuel, *Sur certains espaces $C_\sigma(S)$ et sur les sous-espaces complémentés de $C(S)$*, Bull. Sci. Math., 2^e série **95** (1971), 65-82.

12. Z. Semadeni, *Banach spaces non-isomorphic to their Cartesian squares. II*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., **8** (1960), 81-84.

Received March 6, 1972

UNIVERSITY OF PITTSBURGH
AND
CARNEGIE-MELLON UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. DePrima will replace J. Dugundji until August 1974.

Pacific Journal of Mathematics

Vol. 47, No. 2 February, 1973