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# DOMAINS OF NEGATIVITY AND APPLICATION TO GENERALIZED CONVEXITY ON A REAL TOPOLOGICAL VECTOR SPACE

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# DOMAINS OF NEGATIVITY AND APPLICATION TO GENERALIZED CONVEXITY ON A REAL TOPOLOGICAL VECTOR SPACE

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The purpose of this paper is to derive conditions for the existence of domains of negativity, and then to determine maximal domains of convexity, quasi-convexity, and pseudoconvexity for a quadratic function defined on a real topological vector space.

1. Introduction. Martos, in [14] and [15], and Cottle and the author, in [3], [4], [6], and [7], study quasi-convex and pseudo-convex quadratic functions defined on  $E^n$ , the *n*-dimensional Euclidean space. Furthermore, in [6] and [7], the author uses the concept of domains of negativity that was introduced, mutatis mutandis, by Koecher in [11]. The purpose of this paper is to derive conditions for the existence of domains of negativity, and then to generalize the results found in [6].

In §2, we briefly review definitions needed in the rest of this paper. We also state relations between the classes of convex, quasiconvex, and pseudo-convex quadratic functions on a convex set. Conditions for the existence of domains of negativity and properties of these are given in §3. In §4, convex quadratic functions are studied. Then, domains of quasi-convexity and pseudo-convexity for quadratic forms are specified in §5, and, in §6, we extend this analysis to quadratic functions.

*Note.* Another approach to this theory have been used by Siegfried Schaible in "Quasi-convex Optimization in General Real Linear Spaces", Zeitschrift für Operations Research, 1972.

2. DEFINITIONS. Let  $E^1$  denote the field of real numbers with the natural topology and let X be a vector space over  $E^1$ . We assume that X admits a *norm*, i.e., there exists a mapping  $x \to |x|$  from X into  $E^1_+ = \{\alpha \in E^1 | \alpha \ge 0\}$  with the following properties:

(i) |x| = 0 if and only if x = 0,

(ii)  $|\lambda x| = |\lambda| |x|$  for all  $\lambda \in E^1$  and all  $x \in X$ ,

(iii)  $|x + y| \leq |x| + |y|$  for all x and y in X.

A topology on X is determined by this norm, and X, so endowed, is called a topological vector space over  $E^{1}$ .

Let X and Y be two real vector spaces. The mapping  $A: X \rightarrow Y$  is a *linear transformation* if and only if for all vectors x and y in

X and for all real numbers  $\alpha$  and  $\beta$ 

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y).$$

If  $Y = E^{1}$ , then A is said to be a *linear form* from X into  $E^{1}$ .

The mapping  $L: X \times X \rightarrow E^1$  is a bilinear form on X if and only if (i) L(x, y) = L(y, x) for all x and y in X,

(ii) L(x, y) is linear and continuous in y for each fixed x.

With each bilinear form L is associated a unique quadratic form  $Q: X \rightarrow E^1$  defined by

Q(x) = L(x, x) for all  $x \in X$ .

A quadratic function on a real vector space X is a mapping  $R: X \rightarrow E^1$  defined by

$$R(x) = 1/2Q(x) + P(x)$$
 for all  $x \in X$ ,

where 
$$Q$$
 is a quadratic form and  $P$  is a linear form, both defined on  $X$ .  
The *radical* of a bilinear form  $L$  is the set

$$X(L) = \{x \in X | L(x, y) = 0 \text{ for all } y \in X\}$$
.

L is nondegenerate on X if X(L) = 0. Otherwise, L is degenerate.

If  $X_1$  and  $X_2$  are subsets of X, then the complement of  $X_2$  relative to  $X_1$  is the set

$$X_{\scriptscriptstyle 1}ackslash X_{\scriptscriptstyle 2}=\{x\in X_{\scriptscriptstyle 1}\,|\,x
otin X_{\scriptscriptstyle 2}\}$$
 .

Also, the sum of  $X_1$  and  $X_2$  is the set

$$X_1 + X_2 = \{x \in X | x = u + v, u \in X_1, \text{ and } v \in X_2\}$$
.

If  $E_1$  and  $E_2$  are subspaces of X, then  $X = E_1 \bigoplus E_2$ , the direct sum of  $E_1$  and  $E_2$ , if and only if for each  $x \in X$  there exists a unique pair  $u \in E_1$  and  $v \in E_2$  such that x = u + v.

In [11], Koecher introduces the notion of domains of positivity in a real topological vector space, and mutatis mutandis, we define a *domain of negativity* in X determined by L as a subset Y of X having the following properties:

(i) Y is open and nonempty,

(ii) L(x, y) < 0 for all x and  $y \in Y$ ,

(iii) for all  $x \notin Y$  there exits a vector  $y \in \overline{Y} \setminus X(L)$  such that  $L(x, y) \ge 0$ . (Note that  $\overline{Y}$  is the closure of Y.)

A subset S of X is said to be *convex* if and only if for all x, y in S and for all  $\theta \in [0, 1]$ 

$$x(\theta) = (1 - \theta)x + \theta y \in S.$$

Furthermore, S is solid if and only if it has a nonempty interior,  $S^{\circ}$ .

The quadratic function R(x) = 1/2Q(x) + P(x) is convex on a convex set S in X if and only if for all x and y in S and for all  $\theta \in [0, 1]$ ,

(1) 
$$R((1-\theta)x + \theta y) \leq (1-\theta)R(x) + \theta R(y) .$$

The quadratic function R(x) = 1/2Q(x) + P(x) is quasi-convex on a set S in X if and only if for all x and y in S

(2) 
$$R(y) \leq R(x)$$
 implies  $L(x, y - x) + P(y - x) \leq 0$ .

The quadratic function R(x) = 1/2Q(x) + P(x) is pseudo-convex on a set S in X if and only if for all x and y in S

(3) 
$$L(x, y - x) + P(y - x) \ge 0$$
 implies  $R(y) \ge R(x)$ .

Observe that if we take P(x) = 0 for all  $x \in X$ , then (1), (2), and (3) are the conditions for the quadratic form Q to convex, quasi-convex, and pseudo-convex, respectively.

If S is a convex set, then denote by C(S), QC(S), and PC(S) the classes of all quadratic functions R that are convex on S, quasi-convex on S, and pseudo-convex on S, respectively.

Notice that Mangasarian's results in Chapters 6 and 9 of [13] hold for a quadratic function R(x) = 1/2Q(x) + P(x) defined on an arbitrary real topological vector space if we replace the expression  $(\nabla R(x), y - x)$  by L(x, y - x) + P(y - x). (Recall that in  $E^n$  the gradient of Revaluated at  $x, \nabla R(x)$ , is the column vector of the partial derivatives of R at x.) Thus, from [13, Theorem 9.1.4], we have this equivalent definition: a quadratic function R(x) is quasi-convex on a convex set S in X if and only if for all  $x, y \in S$  and for all  $\theta \in [0, 1]$ 

(4) 
$$R((1-\theta)x + \theta y) \leq Max \{R(x), R(y)\}.$$

Furthermore the results in [13], [Chapters 6 and 9] imply that if S is a convex set in X, then

$$(5) C(S) \subset PC(S) \subset QC(S) .$$

In [3], Cottle and the author have shown the following.

(6) PROPOSITION. If the real valued function h is quasi-convex on a nonempty convex set S in  $E^*$  and continuous on  $\overline{S}$ , then h is quasi-convex on  $\overline{S}$ , the closure of S.

Since this result holds for a quadratic function R defined on an arbitrary real topological vector space, if S is convex, then

$$QC(S) \subset QC(\bar{S}) .$$

It follows from (5) and (7) that for a convex set  $S \subset X$ 

(8) 
$$C(S) \subset PC(S) \subset QC(S) \subset QC(\bar{S})$$
.

Observe the similarity with Ponstein's results for  $X = E^n$ . See [16].

3. Domains of negativity. In this section we give necessary and sufficient conditions for a bilinear form to determine a pair of domains of negativity in a real topological vector space. The importance of domains of negativity in the study of quasi-convexity and pseudo-convexity will become apparent in  $\S$  and 6.

First we introduce the following notation. For each  $x \in X$  we denote by E(x) the subspace generated by x, i.e.,

$$E(x) = \{z \in X \mid z = \alpha x, \alpha \in E^{\scriptscriptstyle 1}\}$$
.

Given a certain bilinear form L and an arbitrary subspace E of X, we denote

$$E_L = \{z \in X \mid L(x, z) = 0 \text{ for all } x \in E\}.$$

Referring to [10, p. 6], the following is true.

(9) PROPOSITION. If  $x \in X$  and  $Q(x) \neq 0$ , then  $X = E(x) \bigoplus E_L(x)$ .

Relative to a bilinear form L, we say that a nonzero vector  $z \in X$  is

positive-valued if and only if Q(z) > 0, negative-valued if and only if Q(z) < 0, zero-valued if and only if Q(x) = 0.

Suppose that L is a nondegenerate bilinear form, i.e., X(L) = 0. Furthermore, suppose there exists a vector  $x \in X$  such that Q(x) = -1and  $E_L(x)$  is an *inner product space* where L(u, v) is the inner product, i.e.,

$$L(u, v) = L(v, u)$$
 for all  $u, v \in E_L(x)$   
 $Q(u) \ge 0$  for all  $u \in E_L(x)$   
 $Q(u) = 0$  implies  $u = 0$ .

For details see Schaefer [17, p. 44] or Greub [9, p. 160]. From (9),

$$X = E(x) \bigoplus E_{L}(x)$$
.

Using the same type of argument as in [9, p. 268], the following can be shown.

(10) PROPOSITION. If z is a negative-valued vector or if z is a nonzero but zero-valued vector, then  $L(x, z) \neq 0$ .

Define the sets

 $Y^+ = \{z \in X \,|\, Q(z) < 0 \, \, ext{and} \, \, L(x,z) < 0\} \;, \ Y^- = \{z \in X \,|\, Q(z) < 0 \, \, ext{and} \, \, L(x,z) > 0\} \;,$ 

Notice that  $Y^+$  and  $Y^-$  are nonempty since  $x \in Y^+$  and  $-x \in Y^-$ . It is easy to verify that

$$ar{Y}^+ = \{ z \in X \, | \, Q(z) \leqslant 0 \, ext{ and } L(x, \, z) < 0 \} \cup \{ 0 \} \ ar{Y}^- = \{ z \in X \, | \, Q(z) \leqslant 0 \, ext{ and } L(x, \, z) > 0 \} \cup (0) \; ,$$

and that  $Y^+ \cup \{0\}$ ,  $\overline{Y}^- \cup \{0\}$ ,  $\overline{Y}^+$ , and  $\overline{Y}^-$  are solid convex cones. Furthermore, a modified version of arguments [6, (3.22) and (3.32)] shows that  $Y^+$  and  $Y^-$  are domains of negativity.

The definitions of  $Y^+$  and  $Y^-$  and (10) imply the following result.

(11) THEOREM. Given the pair of domains of negativity  $Y^+$  and  $Y^-$  in X determined by L, then

- (a)  $z \in X^- = Y^+ \cup Y^-$  if and only if Q(z) < 0,
- (b)  $z \in X^{\circ} = (\overline{Y}^{+} \setminus Y^{+}) \cup (\overline{Y}^{-} \setminus Y^{-})$  if and only if Q(z) = 0,
- (c)  $z \in X^+ = X \setminus (\overline{Y}^+ \cup \overline{Y}^-)$  if and only if Q(z) > 0.

Since  $Y^+$  and  $Y^-$  are maximal ([11, p. 5]), then it follows from (11) that the pair  $Y^+$  and  $Y^-$  in X determined by L is unique.

In summary, if the vector  $x \in X$  is such that Q(x) = -1 and  $E_L(x)$  is an inner product space, then there exists a pair of domains of negativity in X determined by L. This sufficient condition can be expressed into another form. To see this, we need the following result.

(12) PROPOSITION. If there exists a vector  $x \in X$  such that Q(x) = -1and  $E_L(x)$  is an inner product space, then for all  $z \in X$  such that Q(z) < 0 the subspace  $E_L(z)$  is an inner product space.

*Proof.* For contradiction, suppose that Q(z) < 0 for some  $z \in X$  and  $E_L(z)$  is not an inner product space. Hence, there exists a nonzero vector  $y \in E_L(z)$  such that  $Q(y) \leq 0$ . On the other hand, by definition of x there exists a pair  $Y^+$  and  $Y^-$  of domains of negativity in X determined by L.

Suppose  $z \in Y^+$ . If Q(y) < 0, then via (11), either the pair y and z belongs to  $Y^+$  or the pair -y and z belongs to  $Y^+$ . Since L(y, z) =

L(-y, z) = 0, in either case we have a contradiction to the definition of domains of negativity.

If Q(y) = 0, then, via (11), either  $y \in \overline{Y}^+ \setminus Y^+$  or  $-y \in \overline{Y}^+ \setminus Y^+$ . Since  $y \neq 0$ , either the pair z and y or the pair z and -y contradicts the property that if  $u \in Y^+$  and  $v \in \overline{Y}^+ \setminus X(L)$ , then L(u, v) < 0 ([11, Theorem 1 a.]). The proof is complete.

Relying on (12), if the set  $\{x \in X | Q(x) < 0\}$  is nonempty and for each x in this set the subspace  $E_L(x)$  is an inner product space, then there exists a pair of domains of negativity. Other trivial sufficient conditions for the existence of such a pair are Q(x) < 0 and  $E_L(x)$  empty (i.e., dim X = 1). Now we turn to the necessity of these conditions.

(13) THEOREM. If there exists a pair  $Y^+$  and  $Y^-$  of domains of negativity in X determined by L, then the set  $\{x \in X | Q(x) < 0\}$  is nonempty and for all  $x \in X$  such that Q(x) < 0 the subspace  $E_L(x)$  is an inner product space or is empty.

*Proof.* Since  $Y^+$  is nonempty, it follows that  $\{x \in X \mid Q(x) < 0\}$  is nonempty. The second condition is shown by a similar argument as in (12), and this completes the proof.

We are left with the problem of studying conditions for the existence of domains of negativity when the bilinear form L is degenerate in X, i.e., when  $X(L) \neq 0$ . Referring to Schaefer [17, p. 20], the vector space X can always be expressed as

$$X = (X/X(L)) \bigoplus X(L)$$

where X/X(L) is called the *quotient space* of X over X(L). It is well-known that the bilinear form L is nondegenerate on X/X(L).

If there exists a pair  $Y_L^+$  and  $Y_L^-$  of domains of negativity in X/X(L) determined by L, then denote

$$egin{array}{lll} Y^+ &=& Y^+_L \bigoplus X(L) \ Y^- &=& Y^-_L \bigoplus X(L) \end{array}$$
 .

First, since  $Y_L^+$  and  $Y_L^-$  are nonempty and open, so are  $Y^+$  and  $Y^-$ . The other conditions for  $Y^+$  and  $Y^-$  to be domains of negativity in X follow from the fact that if  $x, y \in X$ , then

$$egin{array}{lll} x = u + t \ , & u \in X/X(L) \ ext{and} \ t \in X(L) \ , \ y = v + \ z, & v \in X/X(L) \ ext{and} \ z \in X(L) \ , \end{array}$$

and

$$L(x, y) = L(u, v) + L(t, z) = L(u, v)$$
.

Hence a pair  $Y^+$  and  $Y^-$  of domains of negativity in X determined by L exists if and only if such a pair exists when L is restricted to X/X(L).

4. Domains of convexity for a quadratic function. In this section, we want to determine the convex sets in X over which a quadratic function is convex. In [2], Cottle has studied this problem for quadratic functions defined on  $E^n$ , and, as we shall see, these results hold on an arbitrary real topological vector space.

Using definition (1), this result follows immediately.

(14) PROPOSITION. The quadratic function R is convex on a convex set S in X if and only if the quadratic form Q is convex on S.

The same kind of argument, as when the quadratic form is defined on  $E^n$ , can be used to show the following result.

(15) PROPOSITION. The quadratic form Q is convex on a convex set S in X if and only if for all x and y in S

$$Q(x-y) \ge 0$$
.

Notice this generalization of Cottle's result [2, (2)].

Recall that a set K in X is said to be a linear manifold if it is of the form

$$K = E + x$$

where  $x \in X$  and E is a vector subspace of X. ([1]).

With each convex set S in X is associated a carrying plane K(S) defined as the linear manifold of least dimension which contains S. The same argument as in [2] shows the following property.

(16) PROPOSITION. If the quadratic form Q is convex on a convex set S in X, then Q is convex on K(S).

It follows that if the quadratic form Q is convex on a solid convex set S in X, then Q is convex on X.

5. Domains of quasi-convexity and pseudo-convexity for quadratic forms. The results found in Chapter 3 of [6] hold even for quadratic forms defined on a real topological vector space. Since only slight modifications of these arguments are needed for the generalization, we will restrict ourselves to the statements of the results. Suppose that Y is a domain of negativity in X determined by L.

(17) THEOREM. The quadratic form Q is quasi-convex on  $\overline{Y}$  and pseudo-convex on  $\overline{Y} \setminus X(L)$ .

(18) THEOREM. If the quadratic form Q is quasi-convex, but not convex, on a solid convex set S, then there exists a unique pair of domains of negativity,  $Y^+$  and  $Y^-$ , in X determined by L, and  $S \subset \overline{Y^+}$  or  $S \subset \overline{Y^-}$ .

(19) THEOREM. If the quadratic form Q is pseudo-convex, but not convex, on a solid convex set S, then there exists a unique pair of domains of negativity,  $Y^+$  and  $Y^-$ , in X determined by L, and  $S \subset \overline{Y}^+ \setminus X(L)$  or  $S \subset \overline{Y}^- \setminus X(L)$ .

Therefore, if  $Y^+$  and  $Y^-$  is a pair of domains of negativity in X determined by L, then  $\overline{Y}^+$  and  $\overline{Y}^-$  are maximal domains of quasiconvexity, and  $\overline{Y}^+ \setminus X(L)$  and  $\overline{Y}^- \setminus X(L)$  are maximal domains of pseudoconvexity for a quadratic form Q.

6. Domains of quasi-convexity and pseudo-convexity for quadratic functions.

We wish to extend the analysis of Section 5 to quadratic functions. With each quadratic function R(x) = 1/2Q(x) + P(x), associate the set

$$M = \{a \in X | L(a, x) + P(x) = 0 \text{ for all } x \in X\}$$
.

A direct generalization of results in Chapter 4 of [6] gives this sufficient condition.

(20) THEOREM. If  $Y \subset X$  is a domain of negativity determined by L and M is nonempty, then the quadratic function R(x) is quasi-convex on  $\overline{Y} + M$  and pseudo-convex on  $\overline{Y} \setminus X(L) + M$ .

Before we proceed to determine necessary conditions for the quasi-convexity of a quadratic function on a solid convex set, we have to specify under what conditions the set M is nonempty.

It is obvious that the real topological vector space X can be expressed as

$$X = E^+ \oplus E^- \oplus E^0$$

where  $E^+$ ,  $E^-$  and  $E^0$  are subspaces of X such that

$$egin{array}{ll} Q(x)>0 ext{ for all } x\in E^+ackslash 0 \ , \ Q(x)<0 ext{ for all } x\in E^-ackslash 0 \ . \ Q(x)=0 ext{ for all } x\in E^{\, 0} \ , \end{array}$$

This decomposition may not be unique, but for the rest of this section we make the following *assumption*:

(21) There exists at least one decomposition

$$X = E^+ \oplus E^- \oplus E^\circ$$

where  $E^+$  and  $E^-$  are *complete* (i.e., each Cauchy sequence in  $E^+$  or  $E^-$  is convergent).

Under this assumption the following is true:

(22) PROPOSITION. If R(x) = 1/2Q(x) + P(x), then either the set  $M = \{a \in X | L(a, x) + P(x) = 0 \text{ for all } x \in X\}$  is nonempty or there exists a vector  $t \in X$  such that  $P(t) \neq 0$  and L(x, t) = 0 for all  $x \in X$ .

*Proof.* First we show that both conditions cannot hold simultaneously. Indeed, suppose there is an  $a \in M$ ; i.e., L(a, x) + P(x) = 0 for all  $x \in X$ . On the other hand, if t is such that L(x, t) = 0 for all  $x \in X$  and  $P(t) \neq 0$ , then x = a gives a contradiction.

Next, suppose that if L(x, t) = 0 for all  $x \in X$ , then P(t) = 0. Hence  $X = E^+ \bigoplus E^- \bigoplus E^0$  implies that for all  $x \in X$ 

$$L(a, x) + P(x) = (L(a^+, x^+) + P(x^+)) + (L(a^-, x^-) + P(x^-))$$

where  $a^+, x^+ \in E^+$  and  $a^-, x^- \in E^-$ . Relying on [17, p. 44] it follows that there exist at least one  $a^+ \in E^+$  and one  $a^- \in E^-$  such that for all  $x^+ \in E^+$ 

$$L(a^{+}, x^{+}) + P(x^{+}) = 0$$

and for all  $x^- \in E^-$ 

$$L(a^{-}, x^{-}) + P(x^{-}) = 0$$
.

This shows that M is nonempty and the proof is complete.

Notice this proposition generalizes to an arbitrary real topological vector space X, satisfying assumption (21), a well-known result proved in Gale's book [8, Theorem 2.5] for the case  $X = E^{n}$ .

This proposition and similar arguments as in [6, (4.4), (4.13), and (4.15)] are combined to show these results.

(23) THEOREM. If the quadratic function R(x) = 1/2Q(x) + P(x) is quasi-convex, but not convex, on a solid convex set S, then

(i) M is not empty,

- (ii) there exists a unique pair of domains of negativity,  $Y^+$  and  $Y^-$ , in X determined by L,
- (iii)  $S \subset \overline{Y}^+ + M \text{ or } S \subset \overline{Y}^- + M.$

(24) THEOREM. If the quadratic function R(x) = 1/2Q(x) + P(x) is pseudo-convex, but not convex, on a solid convex set S in X, then

- (i) M is not empty,
- (ii) there exists a unique pair of domains of negativity,  $Y^+$  and  $Y^-$ , in X determined by L,
- (iii)  $S \subset (\overline{Y}^+ \setminus X(L) + M)$  or  $S \subset (\overline{Y}^- \setminus X(L) + M)$ .

Therefore, if M is nonempty and  $Y^+$  and  $Y^-$  are a pair of domains of negativity in X determined by L, then  $\overline{Y}^+ + M$  and  $\overline{Y}^- + M$  are maximal domains of quasi-convexity, and  $\overline{Y}^+ \setminus X(L) + M$  and  $\overline{Y} - X(L) + M$ M are maximal domains of pseudo-convexity for a quadratic function R.

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