Pacific Journal of Mathematics

COMMON FIXED POINTS OF TWO MAPPINGS

CHI SONG WONG

Vol. 48, No. 1

March 1973

COMMON FIXED POINTS OF TWO MAPPINGS

Chi Song Wong

Let S, T be functions on a nonempty complete metric space (X, d). The main result of this paper is the following. S or T has a fixed point if there exist decreasing functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of $(0, \infty)$ into [0, 1) such that (a) $\sum_{i=1}^5 \alpha_i < 1$; (b) $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$, (c) $\lim_{t\downarrow 0} (\alpha_1 + \alpha_2) < 1$ and $\lim_{t\downarrow 0} (\alpha_3 + \alpha_4) < 1$ and (d) for any distinct x, y in X,

$$\begin{aligned} d(S(x), \ T(y)) &\leq a_1 d(x, \ S(x)) + a_2 d(y, \ T(y)) + a_3 d(x, \ T(y)) \\ &+ a_4 d(y, \ S(x)) + a_5 d(x, \ y) , \end{aligned}$$

where $a_i = \alpha_i(d(x, y))$. A number of related results are obtained.

1. Introduction. Let (X, d) be a nonempty complete metric space and let S, T be mappings of X into itself which are not necessarily continuous nor commuting. Suppose that there are nonnegative real numbers a_1 , a_2 , a_3 , a_4 , a_5 such that

(a)
$$a_1 + a_2 + a_3 + a_4 + a_5 < 1$$
,

(b)
$$a_1 = a_2$$
 or $a_3 = a_4$,

and for any x, y in X,

$$\begin{array}{ll} ({\rm c}\,) & d(S(x),\,T(y)) \leq a_1 d(x,\,S(x)) \,+\, a_2 d(y,\,T(y)) \,+\, a_3 d(x,\,T(y)) \\ & +\, a_4 d(y,\,S(x)) \,+\, a_5 d(x,\,y) \,\,. \end{array}$$

It is proved in this paper that each of S, T has a unique fixed point and these two fixed points coincide. Among others, a generalization is obtained by replacing a_1 , a_2 , a_3 , a_4 , a_5 with nonnegative real-valued functions on $(0, \infty)$. This result generalizes the Banach contraction mapping theorem and some results of G. Hardy and T. Rogers [5], R. Kannan [7], E. Rakotch [8], S. Reich [9], P. Srivastava, and V. K. Gupta [10]. It also gives a different proof for these special cases. Note that even if X = [0, 1] and if T_1 , T_2 are commuting continuous functions of X into itself, T_1 , T_2 need not have a common fixed point [1], [2], and [6].

2. Basic results.

THEOREM 1. Let S, T be mappings of a complete metric space (X, d) into itself. Suppose that there exist nonnegative real numbers a_1, a_2, a_3, a_4, a_5 which satisfy (a), (b), and (c). Then each of S, T

has a unique fixed point and these two fixed points coincide.

Proof. Let $x_0 \in X$. Define

$$x_{2n+1} = S(x_{2n}), x_{2n+2} = T(x_{2n+1}), \qquad n = 0, 1, 2, \cdots$$

From (c),

$$egin{aligned} d(x_1,\,x_2) &= d(S(x_0),\,\,T(x_1)) \ &\leq (a_1\,+\,a_5) d(x_0,\,x_1)\,+\,a_2 d(x_1,\,x_2)\,+\,a_3 d(x_0,\,x_2) \ &\leq (a_1\,+\,a_5) d(x_0,\,x_1)\,+\,a_2 d(x_1,\,x_2)\,+\,a_3 (d(x_0,\,x_1)\,+\,d(x_1,\,x_2)) \ . \end{aligned}$$

 \mathbf{So}

$$(1)$$
 $d(x_1, x_2) \leq rac{a_1 + a_3 + a_5}{1 - a_2 - a_3} d(x_0, x_1)$.

Similarly,

$$(2) d(x_2, x_3) \leq \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} d(x_1, x_2) .$$

Let

$$r=rac{a_1+a_3+a_5}{1-a_2-a_3}$$
 , $s=rac{a_2+a_4+a_5}{1-a_1-a_4}$

Repeating the above argument, we obtain, for each $n = 0, 1, 2, \dots$,

$$(3)$$
 $d(x_{2n+1}, x_{2n+2}) \leq rd(x_{2n+1}, x_{2n})$,

$$(4) d(x_{2n+3}, x_{2n+2}) \leq sd(x_{2n+2}, x_{2n+1})$$

By (3), (4), and induction, we have, for each $n = 0, 1, 2, \dots$,

$$(5)$$
 $d(x_{2n+1}, x_{2n+2}) \leq r(rs)^n d(x_0, x_1)$,

$$(6) d(x_{2n+2}, x_{2n+3}) \leq (rs)^{n+1} d(x_0, x_1) .$$

Since rs < 1 and

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq (1 + r) \sum_{n=0}^{\infty} (rs)^n d(x_0, x_1)$$
,

 $\{x_n\}$ is Cauchy. By completeness of (X, d), $\{x_n\}$ converges to some point x in X. We shall now prove that x is a fixed point of S and T. Let n be given. Then

(7)
$$d(x, S(x)) \leq d(x, x_{2n+2}) + d(S(x), x_{2n+2}) \\ = d(x, x_{2n+2}) + d(S(x), T(x_{2n+1})) .$$

By (c),

$$(8) \qquad d(S(x), T(x_{2n+1})) \leq a_1 d(x, S(x)) + a_2 d(x_{2n+1}, x_{2n+2}) + a_3 d(x, x_{2n+2}) \\ + a_4 d(x_{2n+1}, S(x)) + a_5 d(x, x_{2n+1}) .$$

Combining (7) and (8) and letting n tend to infinity, we obtain

$$d(x, S(x)) \leq (a_1 + a_4)d(x, S(x)) .$$

Since $a_1 + a_4 < 1$, S(x) = x. Similarly T(x) = x. Let y be a fixed point of T. Then from d(x, y) = d(S(x), T(y)) and (c), we obtain

$$d(x, y) \leq (a_3 + a_4 + a_5)d(x, y)$$
.

Since $a_3 + a_4 + a_5 < 1$, d(x, y) = 0. So T has a unique fixed point. Similarly, S has a unque fixed point.

When $a_3 = a_4 = a_5 = 0$, S = T and T is continuous (or even $x \rightarrow d(x, T(x))$ is lower semicontinuous) on X, Theorem 1 can be obtained by an earlier result of the author [11, Theorem 1].

From the proof of Theorem 1, we know that S, T still have a common fixed point if conditions (a), (b) are replaced by the following conditions:

$$(9) \qquad (a_1 + a_3 + a_5)(a_2 + a_4 + a_5) < (1 - a_2 - a_3)(1 - a_1 - a_4) ,$$

(10)
$$a_1 + a_4 < 1$$

If in addition,

$$(11) a_3 + a_4 + a_5 < 1$$

then the common fixed point of S, T is the unique fixed point of S(and T). Note that conditions (a) and (b) imply (9), but (a) alone does not. Indeed, for any a_1 , a_2 , a_5 in $[0, \infty)$ with $a_1 \neq a_2$ and $a_1 + a_2 + a_5 < 1$, we can always find a_3 , a_4 in $[0, \infty)$ such that (a) holds but (9) does not. This can be seen by considering the affine function f:

$$f(x, y) = (1 - a_2 - x)(1 - a_1 - y) - (a_1 + x + a_5)(a_2 + y + a_5)$$

defined on the compact convex set

$$K = \{(x, y) \in [0, 1] \times [0, 1]: a_1 + a_2 + x + y + a_5 \leq 1\}$$
.

f takes its minimum value at one of the extreme points of K. With some computation, we conclude that

$$\min f(K) = - |a_1 - a_2| (1 - a_1 - a_2 - a_5) .$$

Since $a_1 + a_2 + a_5 > 1$, min f(K) < 0 if and only if $a_1 \neq a_2$. Thus if $a_1 \neq a_2$, then by continuity of f, there exists a point (a_3, a_4) in

$$K \setminus \{(x, y) \in K: a_1 + a_2 + x + y + a_5 = 1\}$$

such that $f(a_3, a_4) < 0$.

COROLLARY 1. R. Kannan [7, Theorem 1]. Let S be a mapping of a complete metric space (X, d) into itself. Suppose that there exists a number r in [0, 1/2) such that

$$d(S(x), S(y)) \leq r(d(x + S(x)) + d(y, S(y)))$$

for all x, y in X. Then S has a unique fixed point.

COROLLARY 2. P. Srivastava and V. K. Gupta [10, Theorem 1]. Let S, T be mappings of a complete metric space (X, d) into itself. Suppose that there exists nonnegative real numbers a_1, a_2 such that

(a)
$$a_1 + a_2 < 1$$

and

(b)
$$d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y))$$

for all x, y in X.

Srivastava and Gupta stated the above result in a more general form with S, T replaced by S^p , T^q for some positive integers p, q. Since the unique fixed point of S^p (similarly T^q) is the unique fixed point of S, this result is equivalent to Corollary 2.

For Corollaries 1 and 2, we have the following related result.

PROPOSITION. Let S, T be self-maps of a nonempty complete metric space (X, d). Suppose that there exist nonnegative real numbers a_1 , a_2 such that $a_1 + a_2 < 1$ and

$$(*)$$
 $d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y)), x, y \in X.$

Then either (*) is true when all of its S are replaced by T or (*) is true when all of its T are replaced by S.

The following example proves that our result is actually more general than that of Srivastava and Gupta.

EXAMPLE. Let $X = \{1, 2, 3\}$. Let d be the metric for X determined by

$$d(1, 2) = 1, \ d(2, 3) = \frac{4}{7}, \ d(1, 3) = \frac{5}{7}.$$

Let S, T be the function on X such that

$$S(1) = S(2) = S(3) = 1;$$

 $T(1) = T(3) = 1, \quad T(2) = 3$

Let $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 5/7$, $a_5 = 0$. Then the conditions of Theorem 1 are satisfied. However, no nonnegative real numbers a_1 , a_2 , a_3 , a_5 can be chosen such that $a_1 + a_2 + a_3 + a_5 < 1$ and for $x, y \in X$,

$$d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_5 d(x, y) .$$

For if there exist such a_1 , a_2 , a_3 , a_5 , then

$$d(S(3), T(2)) \leq a_1 d(3, S(3)) + a_2 d(2, T(2)) + a_3 d(3, T(2)) + a_5 d(3, 2)$$
.

So

$$rac{5}{7} \leq rac{5a_1}{7} + rac{4a_2}{7} + rac{4a_5}{7} \leq rac{5}{7} \left(a_1 + a_2 + a_3
ight) < rac{5}{7}$$
 ,

a contradiction.

COROLLARY 3. G. Hardy and T. Rogers [5, Theorem 1]. Let S be a mapping of a nonempty complete metric space (X, d) into itself. Suppose that there exist nonnegative real numbers a_1, a_2, a_3, a_4, a_5 such that

(a)
$$a_1 + a_2 + a_3 + a_4 + a_5 < 1$$

and

(b)
$$d(S(x), S(y)) \leq a_1 d(x, S(x) + a_2 d(y, S(y)) + a_3 d(x, S(y)) + a_4 d(y, S(x)) + a_5 d(x, y)$$

for all x, y in X.

Then S has a unique fixed point.

Note that in the above case, we may without loss of generality assume that $a_1 = a_2$, $a_3 = a_4$ (replace a_1 , a_2 , a_3 , a_4 , a_5 respectively by

$$rac{a_1+a_2}{2}$$
 , $rac{a_1+a_2}{2}$, $rac{a_3+a_4}{2}$, $rac{a_3+a_4}{2}$, $rac{a_3+a_4}{2}$, a_3

if necessary). So the above result follows from Theorem 1. The above example shows that there is no such symmetry $(a_1 = a_2, a_3 = a_4)$ for the general case. Indeed, we cannot even assume $a_3 = a_4$. For if $a_3 = a_4$, then for the above example, we have

$$egin{array}{ll} rac{5}{7} &= d(S(3),\ T(3)) \leq rac{5}{7}\,a_{_1} + rac{4}{7}\,a_{_2} + a_{_4} + rac{4}{7}\,a_{_5}\,. \ &= rac{5}{7}\,a_{_1} + rac{4}{7}\,a_{_2} + rac{1}{2}\,a_{_3} + rac{1}{2}\,a_{_4} + rac{4}{7}\,a_{_5} \ &< rac{5}{7}\,(a_{_1} + a_{_2} + a_{_3} + a_{_4} + a_{_5}) < rac{5}{7}\,, \end{array}$$

a contradiction.

2. Extensions and some ralated results. The following result generalizes Theorem 1. Its proof is different from the one we gave for Theorem 1.

THEOREM 2. Let S, T be functions on a nonempty complete metric space (X, d). Suppose that there exist decreasing functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of $(0, \infty)$ into [0, 1) such that

(a) $\sum_{i=1}^{5} \alpha_i < 1;$ (b) $\alpha_1 = \alpha_2 \text{ or } \alpha_3 = \alpha_4;$

 $(c) \quad \lim_{t\downarrow 0} (\alpha_2 + \alpha_3) < 1 \text{ and } \lim_{t\downarrow 0} (\alpha_1 + \alpha_4) < 1;$

(d) for any distinct x, y in X,

$$egin{aligned} d(S(x),\ T(y)) &\leq a_1 d(x,\ S(x)) \,+\, a_2 d(y,\ T(y)) \,+\, a_3 d(x,\ T(y)) \ &+\, a_4 d(y,\ S(x)) \,+\, a_5 d(x,\ y) \;, \end{aligned}$$

where $a_i = \alpha_i(d(x, y))$.

Then at least one of S, T has a fixed point. If both S and T have fixed points, then each of S, T has a unique fixed point and these two fixed points coincide.

Proof. Let $x_0 \in X$. Define for each $n = 0, 1, 2, \dots$,

$$x_{2n+1} = S(x_{2n})$$
, $x_{2n+2} = T(x_{2n+1})$, $b_n = d(x_n, x_{n+1})$.

We may assume that $b_n > 0$ for each n, for otherwise some x_n is a fixed point of S or T. Let

$$r(t) = rac{lpha_{1}(t) + lpha_{3}(t) + lpha_{5}(t)}{1 - lpha_{2}(t) - lpha_{3}(t)} \;, \qquad t > 0 \;,$$

$$s(t) = rac{lpha_2(t) + lpha_4(t) + lpha_5(t)}{1 - lpha_1(t) - lpha_4(t)} , \qquad t > 0 .$$

Then r, s are decreasing. From (a) and (c), the limits

$$r_{\scriptscriptstyle 0} = \lim_{t \downarrow 0} r(t)$$
, $s_{\scriptscriptstyle 0} = \lim_{t \downarrow 0} s(t)$

are nonnegative real numbers. Let

$$f(t) = r(t)s(t)$$
 , $t > 0$.

Then f is decreasing and f(t) < 1 for each t > 0. As in the proof of Theorem 1, we have for each $n = 0, 1, 2, \dots$,

(12)
$$b_{2n+1} \leq r(b_{2n})b_{2n}$$
,

(13)
$$b_{2n+2} \leq s(b_{2n+1})b_{2n+1}$$
.

Let n be given. Then

(14) $b_{2n+3} \leq r(b_{2n+2})s(b_{2n+1})b_{2n+1}$,

(15)
$$b_{2n+2} \leq s(b_{2n+1}) r(b_{2n}) b_{2n}$$
.

Since r, s are decreasing,

(16)
$$b_{2n+3} \leq f(\min\{b_{2n+2}, b_{2n+1}\})b_{2n+1}$$

(17)
$$b_{2n+2} \leq f(\min\{b_{2n+1}, b_{2n}\})b_{2n}$$
.

Since f(t) < 1 for each t > 0, $\{b_{2n+1}\}$, $\{b_{2n}\}$ are decreasing sequences. So $\{b_{2n+1}\}$, $\{b_{2n}\}$ converge respectively to some points c_1 , c_2 . We shall prove that $c_1 = 0$, $c_2 = 0$. From (12) and (13),

$$c_1 \leqq r_{\scriptscriptstyle 0} c_2$$
 , $c_2 \leqq s_{\scriptscriptstyle 0} c_1$.

So either both c_1 , c_2 are zero or both c_1 , c_2 are not zero. Suppose to the contrary that $c_1 \neq 0$, $c_2 \neq 0$. Then from (16) and (17),

(18)
$$b_{n+2} \leq f(\min\{c_1, c_2\})b_n, \qquad n = 0, 1, 2, \cdots$$

By induction,

(19)
$$b_{2n} \leq (f(\min\{c_1, c_2\}))^n b_0 \qquad n = 0, 1, 2, \cdots$$

So $c_2 = 0$, a contradiction. Therefore, $c_1 = c_2 = 0$. This proves that $\{b_n\}$ converges to 0.

Now we shall prove that $\{x_n\}$ is Cauchy. Suppose not. Then there exist $\varepsilon \in (0, \infty)$ and sequences $\{p(n)\}, \{q(n)\}$ such that for each $n \ge 0$,

$$(20) p(n) > q(n) > n ,$$

(21)
$$d(x_{p(n)}, x_{q(n)}) \geq \varepsilon,$$

and (by the well-ordering principle),

$$(22) d(x_{p(n)-1}, x_{q(n)}) < \varepsilon.$$

Let $n \ge 0$ be given, $c_n = d(x_{p(n)}, x_{q(n)})$. Then

(23)

 $\varepsilon \leq c_{m}$

$$\leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < b_{p(n)-1} + \varepsilon$$
 .

From $c_1 = c_2 = 0$, we conclude that $\{c_n\}$ converges to ε from the right. Let

$$I_{1} = \{n: p(n), q(n) \text{ are odd} \},\$$

$$I_{2} = \{n: p(n) \text{ is odd}, q(n) \text{ is even} \}.$$

$$I_{3} = \{n: p(n) \text{ is even}, q(n) \text{ is odd} \},\$$

$$I_{4} = \{n: p(n), q(n) \text{ are even} \}.$$

Then at least one of I_1 , I_2 , I_3 , I_4 is infinite. Suppose first that I_1 is infinite. Let

$$d_n = d(x_{p(n)-1}, x_{q(n)})$$
, $n = 0, 1, 2, \cdots$.

Since $\{c_n\}$ converges to ε and $\{b_n\}$ converges to 0, we conclude from (22) that $\{d_n\}$ converges to ε from the left. Thus

$$J_1 = \{n \in I_1: x_{p(n)-1} \neq x_{q(n)}\}$$

is infinite. Let $n \in J_1$, $u_n = d(x_{p(n)-1}, x_{q(n)+1})$. Then

(24)
$$c_n = d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)}) \\ \leq d(S(x_{p(n)-1}), T(x_{q(n)})) + b_{q(n)}.$$

From (d),

(25)
$$d(S(x_{p(n)-1}), T(x_{q(n)})) \leq \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n + \alpha_4(d_n)c_n + \alpha_5(d_n)d_n .$$

From (24) and (25),

(26)
$$c_n \leq \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n + \alpha_4(d_n)c_n + \alpha_5(d_n)d_n + b_{q(n)}.$$

Without loss of generality, we may assume that each α_i is continuous from the left, for we can replace the α_i 's by

$$eta_i(t) = \lim_{s \, ot \, t} lpha_i(s)$$
 , $t > 0$, $i = 1, 2, 3, 4, 5$

and conditions (a), (b), (c), and (d) still hold. Thus

$$\lim_{n\to\infty}\alpha_i(d_n)=\alpha_i(\varepsilon), \qquad i=1,2,3,4,5.$$

So from (26),

$$arepsilon \leq (lpha_{ extsf{s}}(arepsilon)+lpha_{ extsf{s}}(arepsilon))arepsilon < arepsilon$$
 ,

306

a contradiction. Now suppose that I_2 is infinite. By a similar argument, $J_2 = \{n \in I_2: x_{p(n)-1} \neq x_{q(n)-1}\}$ is infinite. Let $n \in J_2$,

$$v_n = d(x_{p(n)-1}, x_{q(n)-1})$$
, $w_n = d(x_{p(n)}, x_{q(n)-1})$.

Then

(27)
$$c_n = d(S(x_{p(n)-1}), T(x_{q(n)-1})) \\ \leq \alpha_1(v_n)b_{p(n)-1} + \alpha_2(v_n)b_{q(n)-1} + \alpha_3(v_n)d_n + \alpha_4(v_n)w_n + \alpha_5(v_n)v_n .$$

Since $\{v_n\}$ converges to ε (not necessarily from the left or right), we obtain the same contradiction from (27). The other two cases are similar to the above two except the roles of S, T interchange. Hence $\{x_n\}$ is Cauchy. By completeness, $\{x_n\}$ converges to a point x in X. Since $b_n > 0$ for each $n, J = \{n: x \neq x_{2n+1}\}$ or $K = \{n: x \neq x_{2n}\}$ is infinite. Suppose that K is infinite. Let $n \in K$,

$$l_n = d(x, x_{2n})$$
 , $h_n = d(x, x_{2n+1})$.

Then

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_{2n+1}) + d(x_{2n+1}, T(x)) \\ &= h_n + d(S(x_{2n}), T(x)) \\ &\leq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)d(x_{2n}, T(x)) \\ &+ \alpha_4(l_n)h_n + \alpha_5(l_n)l_n \\ &\leq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)[l_n + d(x, T(x))] \\ &+ \alpha_4(l_n)h_n + \alpha_5(l_n)l_n . \end{aligned}$$

So

(28)
$$d(x, T(x)) \leq \frac{1 + \alpha_4(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} h_n + \frac{\alpha_3(l_n) + \alpha_5(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} l_n + \frac{\alpha_1(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} b_{2n}.$$

From (a) and (c), the sequences

$$rac{1+lpha_4(l_n)}{1-lpha_2(l_n)-lpha_3(l_n)}$$
 , $rac{lpha_3(l_n)+lpha_5(l_n)}{1-lpha_2(l_n)-lpha_3(l_n)}$, $rac{lpha_1(l_n)}{1-lpha_2(l_n)-lpha_3(l_n)}$

are bounded. So from (28), T(x) = x. Similarly, S(x) = x if J is infinite. Hence S or T has a fixed point.

The following result follows easily from Theorem 2.

THEOREM 3. With the conditions of Theorem 2, if further,

$$d(S(x), T(x)) \leq \alpha \left[d(x, S(x)) + d(x, T(x)) \right], \quad x \in X$$

for some $\alpha \in [0, 1)$, then each of S, T has a unique fixed point and these two fixed points coincide.

We remark that the conditions of Theorem 1 imply the conditions of Theorem 3. Also, G. Hardy and T. Rogers [5, Theorem 2] gave a different proof for the case S = T. Their proof cannot be modified for the general case. To see that the conclusion of Theorem 2 is best possible, we note that if $X = \{0, 1\}$ with the usual distance and if S, T are two distinct functions of X onto X, then S, T satisfy the conditions of Theorem 2 (and Theorem 3 with $\alpha = 1$), but one has two fixed points and the other has none.

THEOREM 4. Let (X, d) be a nonempty compact metric space. Let S, T be functions of X into itself. Suppose that S or T is continuous. Suppose further that there exist nonnegative real-valued decreasing functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ on $(0, \infty)$ such that

(a) $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \leq 1$,

(b) $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$,

(c) for any distinct x, y in X,

$$egin{aligned} d(S(x),\ T(y)) &< a_1 d(x,\ S(x)) + a_2 d(y,\ T(y)) + a_3 d(x,\ T(y)) + a_4 d(y,\ S(x)) + a_5 d(x,\ y) \ , \end{aligned}$$

where $a_i = \alpha_i(d(x, y))$.

Then S or T has a fixed point. If both S and T have fixed points, then each of S and T has a unique fixed point and these two fixed points coincide.

Proof. By symmetry, we may assume that S is continuous. Let f be the function on X such that

$$f(x) = d(x, S(x))$$
, $x \in X$.

Then f is continuous (we merely need the fact that f is lower semicontinuous) on X. So f takes its minimum value at some x_0 in X. We claim that x_0 is a fixed point of S or $S(x_0)$ is a fixed point of T. Suppose not. Let

Then $b_0 > 0$, $b_1 > 0$. From (c), we can prove that

(29)
$$(1 - \alpha_2(b_0) - \alpha_3(b_0))b_1 < (\alpha_1(b_0) + \alpha_3(b_0) + \alpha_5(b_0))b_0$$
.

Let

 $p(t) = 1 - lpha_{_2}(t) - lpha_{_3}(t)$, $q(t) = lpha_{_1}(t) + lpha_{_3}(t) + lpha_{_5}(t)$, t > 0 .

From (a) and (b), $p(b_0) > 0$. So

$$(30) b_{\scriptscriptstyle 1} < \frac{q(b_{\scriptscriptstyle 0})}{p(b_{\scriptscriptstyle 0})} \, b_{\scriptscriptstyle 0} \, .$$

Similarly,

(31)
$$b_2 < rac{v(b_1)}{u(b_1)} b_1$$
 ,

where

$$u(t) = 1 - lpha_{_1}(t) - lpha_{_4}(t), \, v(t) = lpha_{_2}(t) + lpha_{_4}(t) + lpha_{_5}(t) \;, \;\; t > 0 \;.$$

From (30) and (31),

(32)
$$b_2 < \frac{v(b_1)}{u(b_1)} \frac{q(b_0)}{p(b_0)} b_0$$
.

It suffices to prove that $(v(b_1)q(b_0)/u(b_1)p(b_0)) < 1$, for then, $b_2 < b_0$, a contradiction to the minimality of b_0 . Let $b = \min\{b_0, b_1\}$. Then

$$v(b_1)q(b_0) - u(b_1)p(b_0) \leq v(b)q(b) - u(b)p(b) < 0$$

if $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$. So S or T has a fixed point. Now suppose that x is a fixed point of S and y is a fixed point of T. Then x = y, otherwise, from (c),

$$d(x, y) = d(S(x), T(y)) < d(x, y)$$
,

a contradiction.

The following result is stated without proof.

THEOREM 5. Let (X, d) be complete metric space. Let $\{S_n\}$, $\{T_n\}$ be sequence of functions of X into X which converge pointwise to S, T respectively. Suppose that the pairs (S_n, T_n) satisfy the conditions of Theorem 3 with the same $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$. Then S, T have a unique common fixed point x and x is the limit of the sequence $\{x_n\}$ of the fixed points x_n of S_n .

THEOREM 6. Let (X, d) be a nonempty compact metric space. Let $\{S_n\}, \{T_n\}$ be sequences of functions of X into itself which converge pointwise to the functions S, T on X respectively. Suppose that for each n, there exist decreasing functions $\alpha_1^n, \alpha_2^n, \alpha_3^n, \alpha_4^n, \alpha_5^n$ of $(0, \infty)$ into $[0, \infty)$ such that

$$\begin{array}{ll} (a) & \alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \alpha_5^n \leq 1, \\ (b) & \alpha_1^n = \alpha_2^n \ and \ \alpha_3^n = \alpha_4^n, \\ (c) & for \ any \ distinct \ x, \ y \ in \ X, \\ & d(S_n(x), \ T_n(y)) < a_1^n d(x, \ S_n(x)) + a_2^n d(y, \ T_n(y)) + a_3^n d(x, \ T_n(y)) \\ & \quad + a_4^n d(y, \ S_n(x)) + a_5^n d(x, \ y) \ , \end{array}$$

where

$$a_i^n = \alpha_i^n(d(x, y))$$
.

Then S or T has a fixed point. Indeed, every cluster point of a sequence $\{x_n\}$ of fixed points x_n of S_n or T_n is a fixed point of S or T.

Proof. By Theorem 4, for each n, either S_n or T_n has a fixed point. By symmetry, we may assume that S_n has a fixed point for infinitely many of n's. So there is a subsequence $\{S_{n(k)}\}$ of $\{S_n\}$ such that each $S_{n(k)}$ has a fixed point, say x_k . By compactness, we may (by taking a subsequence) assume that $\{x_k\}$ converges to some x in X. We shall prove that x is a fixed point of S or T. If $x_k \neq x$ for only finitely many of k's, then

$$S(x) = \lim_{k \to \infty} S_{n(k)}(x)$$
$$= \lim_{k \to \infty} S_{n(k)}(x_k)$$
$$= \lim_{k \to \infty} x_k$$
$$= x .$$

So we may assume that $x_k \neq x$ for infinitely many of k's. By taking a subsequence, we may assume that $x_k \neq x$ for each k. Let $k \ge 1$ and $b_k = d(x, x_k)$. Then

(33)
$$\begin{aligned} d(x, T(x)) &\leq d(x, x_k) + d(x_k, T_{n(k)}(x)) + d(T_{n(k)}(x), T(x)) \\ &= d(x, x_k) + d(S_{n(k)}(x_k), T_{n(k)}(x)) + d(T_{n(k)}(x), T(x)) . \end{aligned}$$

From (c),

(34)
$$d(S_{n(k)}(x_k), T_{n(k)}(x)) < \alpha_2^k(b_k)d(x, T_{n(k)}(x)) + \alpha_3^k(b_k)d(x_k, T_{n(k)}(x)) \\ + \alpha_4^k(b_k)d(x, x_k) + \alpha_5^k(b_k)b_k .$$

Combining (33) and (34) and letting k tend to the infinity, we have

(35)
$$d(x, T(x)) \leq \limsup_{k \to \infty} (\alpha_2^k(b_k) + \alpha_3^k(b_k))d(x, T(x)) \\\leq \limsup_{k \to \infty} \lim_{t \downarrow 0} (\alpha_2^k(t) + \alpha_3^k(t))d(x, T(x)) .$$

From (b), $\alpha_2^k(t) + \alpha_3^k(t) \leq 1/2$ for each t > 0, $k = 1, 2, \cdots$. So

(36)
$$\limsup_{k \to \infty} \lim_{t \downarrow 0} \left(\alpha_2^k(t) + \alpha_3^k(t) \right) \leq \frac{1}{2} .$$

From (35) and (36), we conclude that T(x) = x.

From the proof, we know that the same conclusion holds if in Theorem 6, we replace (b) by the following weaker conditions:

$$lpha_1^n=lpha_2^n$$
 or $lpha_3^n=lpha_4^n$, $\limsup_{k o\infty}\lim_{t\downarrow 0} \left(lpha_2^k(t)+lpha_3^k(t)
ight)<1$

and

$$\limsup_{k\to\infty} \lim_{t\downarrow 0} \left(\alpha_{\iota}^n(t) + \alpha_{\bullet}^n(t)\right) < 1.$$

We note that, unlike Theorem 5, S, T in Theorem 6 need not satisfy the condition required for the pairs (S_n, T_n) .

THEOREM 7. Let (X, d) be a nonempty compact metric space. Let $\{S_n\}$ be a sequence of functions of X into itself which converges pointwise to some function S on X. Suppose that for each n, there exist decreasing functions α_1^n , α_2^n , α_3^n , α_4^n , α_5^n of $(0, \infty)$ into $[0, \infty)$ such that

where

$$a_i = \alpha_i(d(x, y))$$
.

Then S has a fixed point. Indeed, every cluster point of the sequence of fixed points of S_n is a fixed point of S.

The above result follows from Theorem 6 by averaging two applications of condition (b).

We shall now give a simple example to show that the conclusion of Theorem 7 is best possible. Let X be a star-shaped [4] compact subset of a normed linear space B. Then there exists a point z in X such that for any y in X, the line segment

$$\{tz + (1 - t)y: t \in [0, 1]\}$$

is contained in X. For each n, let

$$S_n(x) = rac{1}{n}z + \left(1-rac{1}{n}
ight)x$$
, $x \in X$.

 $a_{3}d(x, S_{n}(y))$

Then $\{S_n\}$ is a sequence of mappings of X into X which satisfy the conditions of Theorem 7. $\{S_n\}$ converges pointwise to the identity function S on X. Every point of X is a fixed point of S. So unlike Theorem 5, it is too much to ask that S in Theorem 7 has a unique fixed point.

References

1. W. M. Boyce, Commuting functions with no common fixed point, Trans. Amer. Math. Soc., **137** (1969), 77-92.

2. ____, Γ-compact mappings on an interval and fixed points, Trans. Amer. Math. Soc., 160 (1971), 87-102.

3. M. Edelstein, On fixed and periodic points under contractive mappings, J. London Math. Soc., **37** (1962), 74-79.

4. B. Halpern, The kernel of a starshaped subset of the plane, Proc. Amer. Math. Soc., 23 (1969), 692-696.

5. G. Hardy and T. Rogers, A generalization of a fixed point theorem of Reich, (to appear).

6. J. P. Huneke, On common fixed point of commuting continuous functions on an interval, Trans. Amer. Math. Soc., **139** (1969), 371-381.

7. R. Kannan, Some results on fixed points-II, Amer. Math. Monthly, 76 (1969), 405-408.

8. E. Rakotch, A note on contractive mappings, Proc. Amer. Math. Soc., 13 (1962), 469-465.

9. S. Reich, Some remarks concerning contraction mappings, Canad. Math. Bull. 14 (1971), 121-124.

10. P. Srivastava and V. K. Gupta, A note on common fixed points, Yokohama Math. J., XIX (1971), 91-95.

11. Chi Song Wong, Fixed point theorems for nonexpansive mappings, J. Math. Anal. Appl., **37** (1972), 142-150.

Received June 13, 1972. This research was partially supported by the National Research Council of Canada Grant A8518 and a grant from the Canadian Mathematical Congress. It was prepared while the author was at the Summer Research Institute, University of Alberta.

UNIVERSITY OF WINDSOR

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor) University of California Los Angeles, California 90024

R. A. BEAUMONT University of Washington Seattle, Washington 98105 J. DUGUNDJI* Department of Mathematics University of Southern California Los Angeles, California 90007

D. GILBARG AND J. MILGRAM Stanford University Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA CALIFORNIA INSTITUTE OF TECHNOLOGY UNIVERSITY OF CALIFORNIA MONTANA STATE UNIVERSITY UNIVERSITY OF NEVADA NEW MEXICO STATE UNIVERSITY OREGON STATE UNIVERSITY UNIVERSITY OF OREGON OSAKA UNIVERSITY UNIVERSITY OF SOUTHERN CALIFORNIA STANFORD UNIVERSITY UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON * * * AMERICAN MATHEMATICAL SOCIETY

NAVAL WEAPONS CENTER

K. YOSHIDA

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics Vol. 48, No. 1 March, 1973

Jan Aarts and David John Lutzer, <i>Pseudo-completeness and the product of Baire</i>	
spaces	1
Gordon Owen Berg, <i>Metric characterizations of Euclidean spaces</i>	11
Ajit Kaur Chilana, <i>The space of bounded sequences with the mixed topology</i>	29
Philip Throop Church and James Timourian, <i>Differentiable open maps of</i>	25
(p+1)-manifold to p-manifold	35
P. D. T. A. Elliott, On additive functions whose limiting distributions possess a finite	47
mean and variance	47 57
M. Solveig Espelie, <i>Multiplicative and extreme positive operators</i>	57
Jacques A. Ferland, <i>Domains of negativity and application to generalized convexity</i>	67
on a real topological vector space	07
Michael Benton Freeman and Reese Harvey, A compact set that is locally holomorphically convex but not holomorphically convex	77
	//
Roe William Goodman, <i>Positive-definite distributions and intertwining</i> operators	83
Elliot Charles Gootman, <i>The type of some C* and W*-algebras associated with</i>	05
transformation groups	93
David Charles Haddad, <i>Angular limits of locally finitely valent holomorphic</i>)5
functions	107
William Buhmann Johnson, <i>On quasi-complements</i>	113
William M. Kantor, <i>On 2-transitive collineation groups of finite projective</i>	115
spaces	119
Joachim Lambek and Gerhard O. Michler, <i>Completions and classical localizations</i>	117
of right Noetherian rings	133
Kenneth Lamar Lange, <i>Borel sets of probability measures</i>	141
David Lowell Lovelady, <i>Product integrals for an ordinary differential equation in a</i>	
Banach space	163
Jorge Martinez, A hom-functor for lattice-ordered groups	169
W. K. Mason, Weakly almost periodic homeomorphisms of the two sphere	185
Anthony G. Mucci, <i>Limits for martingale-like sequences</i>	197
Eugene Michael Norris, <i>Relationally induced semigroups</i>	203
Arthur E. Olson, <i>A comparison of c-density and k-density</i>	209
Donald Steven Passman, On the semisimplicity of group rings of linear groups.	20)
II	215
Charles Radin, <i>Ergodicity in von Neumann algebras</i>	235
P. Rosenthal, On the singularities of the function generated by the Bergman operator	233
of the second kind	241
Arthur Argyle Sagle and J. R. Schumi, <i>Multiplications on homogeneous spaces</i> ,	
nonassociative algebras and connections	247
Leo Sario and Cecilia Wang, <i>Existence of Dirichlet finite biharmonic functions on</i>	
the Poincaré 3-ball	267
Ramachandran Subramanian, On a generalization of martingaler due to Blake	275
Bui An Ton, On strongly nonlinear elliptic variational inequalities	279
Seth Warner, A topological characterization of complete, discretely valued	
fields	293
Chi Song Wong, Common fixed points of two mappings	299